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A NEW STUDY IN EUCLID'S METRIC SPACE CONTRACTION MAPPING AND PYTHAGOREAN RIGHT TRIANGLE RELATIONSHIP

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Abstract. Our study explores the connection between the Pythagorean theorem and the Fixed-point theorem in metric spaces. Both of which center around the concepts of distance transformations and point relationships. The Pythagorean theorem deals with right triangles in Euclidean space, emphasizing distances between points. In contrast, fixed-point theorems pertain to the points that remain unchanged under specific transformations thereby preserving distances. The article delves into the intrinsic correlation between these concepts and presents a novel study in Euclidean metric spaces, examining the relationship between contraction mapping and Pythagorean Right Triangles. Practical applications are also discussed particularly in the context of image compression. Here, the integration of the Pythagorean right triangle paradigm with contraction mappings results in efficient data representation and the preservation of visual data relationships.This illustrates the practical utility of seemingly abstract theories in addressing real-world challenges.

AMS Mathematics Subject Classification : 47H10, 54H25, 01A16, 0A25, 03F60, 37C25.
Key words and phrases : Fixed point theorem, contraction mapping, pythagorean right triangle, Euclid's metric space.

1. Introduction

The Pythagorean theorem emerged in ancient Greece, attributed to the mathematician Pythagoras or his followers. It states that in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides[1]. Mathematically, it can be written as:

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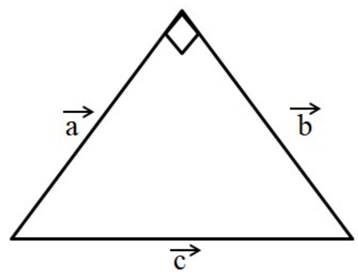

FIGURE 1

$$\overrightarrow{c}^2 = \overrightarrow{a}^2 + \overrightarrow{b}^2. \tag{1}$$

(See figure no.1). This theorem has profound implications in geometry and mathematics (see [2, 3, 8]). On the other hand, the fixed point theory originated from the work of mathematician David Brouwer in the early 20^{th} century. Brouwer's fixed point theorem proved in 1912. The statement is that any continuous mapping on a closed interval to itself must have at least one fixed point. This concept is a fundamental tool in various branches of mathematics, such as topology, functional analysis, and some other fields like economics and computer science. (see [4, 5, 9]).

Banach contraction principle 1.1 [6] *This principle states that, if* (X, d) *is a complete metric space and* $T : X \to X$ *is a contraction map,*

$$i.e.,\ d(T(x), T(y)) \leq \lambda d(x, y), \forall x, y \in X, \lambda \in [0, 1) \tag{2}$$

where λ *is a constant, then* T *has a unique fixed point.*

Definition 1.2 [7] *Let a self-map* $T : X \to X$ be defined on a metric space (X, d), if $\exists k \in (0, 1)$ such that, $\forall x, y \in X$ then, $d(T(x), T(y)) \leq kd(x, y)$.

Note: In our work, we introduced a novel idea that establishes a connection between the Pythagorean theorem and the fixed-point theorem, thereby delving into a historically unexplored topic. This conceptual link represents a significant advancement in our field. The unique synthesis of these mathematical principles in our research enhances understanding and provides new opportunities for exploration and application. Our work is groundbreaking in its exploration of this uncharted territory, emphasizing the originality and importance of our contribution to the mathematical discourse.

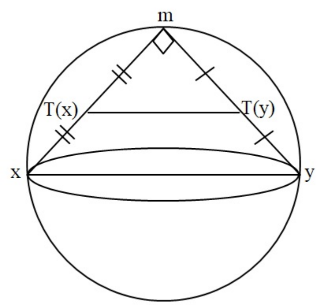

FIGURE 2

2. Main results

In our article, we present a new study on fixed point theorems for a contraction mapping on Euclid's metric spaces $X \subset \mathbb{R}$. By using Pythagorean theorem, we give some definitions, theorems and examples to support our study as follows: If we draw a right angled triangle $\Delta(x, y, m)$ at $\angle m$ on a hemisphere in Euclidean metric spaces (X, d) such that, $x, y, m \in X$ be vertices the triangle, (see figure no.2). Let $T(x)$ on $|\overrightarrow{mx}|$ and $T(y)$ on $|\overrightarrow{my}|$. We draw the line $|\overrightarrow{T(x)T(y)}|$, the $\Delta(x, m, y)$ satisfies a Pythagorean theorem, i.e., $|\overrightarrow{xy}|^2 = |\overrightarrow{xm}|^2 + |\overrightarrow{ym}|^2$.
Now, if we denote the side $|\overrightarrow{xy}|$ by $d(x, y)$ such that $d(x, y)$ is the distance between x and y. Similarly, we denote the side $|\overrightarrow{T(x)T(y)}|$ by $d(T(x), T(y))$. Suppose that, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $T(x) = (Tx_1, Tx_2)$, $T(y) = (Ty_1, Ty_2)$, so that

$$d(x, y) = |\overrightarrow{xy}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{3}$$

and

$$d(T(x), T(y)) = |\overrightarrow{T(x)T(y)}| = \sqrt{(Tx_1 - Ty_1)^2 + (Tx_2 - Ty_2)^2}. \tag{4}$$

Also, if we denote the square of distances $d^2(x, y)$, $d^2(T(x), T(y))$ as follows:

$$d^2(x, y) = d(x, y)^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 \tag{5}$$

and

$$d^2(T(x), T(y)) = d(T(x), T(y))^2 = (Tx_1 - Ty_1)^2 + (Tx_2 - Ty_2)^2. \tag{6}$$

Suppose that, $\forall X \subset \mathbb{R}$, $T : X \to X$ be a self map, then the triangle $\Delta(x, m, y)$ satisfies the Pythagorean theorem

$$d^2(x, y) = d^2(x, m) + d^2(y, m). \tag{7}$$

Note: From figure no.3, we get the triangle $\Delta(x, m, y)$ which also satisfies the Pythagorean properties as follows:

$$\frac{d(x, T(x))}{d(x, m)} = \frac{d(y, T(y))}{d(y, m)} = \frac{d(T(x), T(y))}{d(x, y)}. \tag{8}$$

Theorem 2.1. *Let's assume that the triangle $\Delta(x, m, y)$ is right-angled at m and the self-map $T : X \to X$ is defined on Euclid's metric space (X, d). If $T(x)$, $T(y)$ on the sides $|\overrightarrow{xm}|, |\overrightarrow{ym}|$ respectively, if the triangle $\Delta(x, m, y)$ satisfies the Pythagorean theorem, then T is a contraction mapping such that, $d(T(x), T(y)) \leq \lambda d(x, y), \forall x, y \in X, \lambda \in (0, 1)$.*

Proof. There are several cases, we have only explain three cases until the idea becomes clear.

Case (1): When $d(x, T(x)) = d(T(x), m)$, $d(y, T(y)) = d(T(y), m)$. Now when the the triangle $\Delta(x, m, y)$ satisfies the Pythagorean theorem, we get

$$d^2(x, y) = d^2(x, m) + d^2(y, m), \tag{9}$$

but

$$d(x, m) = d(x, T(x)) + d(T(x), m) \tag{10}$$

and

$$d(y, m) = d(y, T(y)) + d(T(y), m), \tag{11}$$

put equations (10), (11) in (9), we get

$$\begin{aligned} d^2(x, y) &= [d(x, T(x)) + d(T(x), m)]^2 + [d(y, T(y)) + d(T(y), m)]^2 \qquad (12) \\ &= d^2(x, T(x)) + 2d(x, T(x))d(T(x), m) + d^2(T(x), m) + d^2(y, T(y)) \\ &+ 2d(y, T(y))d(T(y), m) + d^2(T(y), m), \end{aligned}$$

but $d^2(T(x), m) + d^2(T(y), m) = d^2(T(x), T(y))$ in (12), we get

$$\begin{aligned} d^2(x, y) = d^2(x, T(x)) &+ 2d(x, T(x))d(T(x), m) + d^2(y, T(y)) \qquad (13) \\ &+ 2d(y, T(y))d(T(y), m) + d^2(T(x), T(y)). \end{aligned}$$

From (13), we get either

$$2d(x, T(x))d(T(x), m) + 2d(y, T(y))d(T(y), m) = \frac{1}{2}d^2(x, y) \tag{14}$$

and

$$d^2(x, T(x)) + d^2(y, T(y)) = \frac{1}{4}d^2(x, y). \tag{15}$$

Or

$$2d(x, T(x))d(T(x), m) + 2d(y, T(y))d(T(y), m) = 2d^2(T(x), T(y)) \tag{16}$$

and

$$d^2(x, T(x)) + d^2(y, T(y)) = d^2(T(x), T(y)). \tag{17}$$

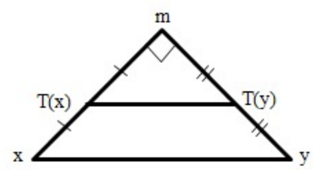

FIGURE 3

There are two sub-cases:
Sub-case (1-a): Put equations (14), (15) in (13), we get

$$\begin{aligned} d^2(x,y) &= \frac{1}{2}d^2(x,y) + \frac{1}{4}d^2(x,y) + d^2(T(x),T(y)) \quad (18)\\ &= \frac{3}{4}d^2(x,y) + d^2(T(x),T(y)) \end{aligned}$$

$$\Rightarrow d(T(x),T(y)) = \frac{1}{2}d(x,y). \quad (19)$$

Sub-case (1-b): Put equations (16), (17) in (13), we get

$$\begin{aligned} d^2(x,y) &= 2d^2(T(x),T(y)) + d^2(T(x),T(y)) + d^2(T(x),T(y)) \quad (20)\\ &= 4d^2(T(x),T(y)) \end{aligned}$$

$$\Rightarrow d(T(x),T(y)) = \frac{1}{2}d(x,y). \quad (21)$$

So that (19), (21) are equal when $d(x,T(x)) = d(T(x),m)$, $d(y,T(y)) = d(T(y),m)$. To verify the validity of case (1) and the two sub-cases related to it, see example (2.3)
Case (2): When, $d(x,T(x)) = \frac{1}{2}d(T(x),m)$, $d(y,T(y)) = \frac{1}{2}d(T(y),m)$.
Now, the triangle $\Delta(x,m,y)$ satisfies the Pythagorean theorem, by using the similar last process in first case (1), we get:

$$\begin{aligned} d^2(x,y) = d^2(x,T(x)) &+ 2d(x,T(x))d(T(x),m) + d^2(y,T(y)) \quad (22)\\ &+ 2d(y,T(y))d(T(y),m) + d^2(T(x),T(y)). \end{aligned}$$

Now, either

$$2d(x,T(x))d(T(x),m) + 2d(y,T(y))d(T(y),m) = \frac{4}{9}d^2(x,y) \quad (23)$$

and

$$d^2(x,T(x)) + d^2(y,T(y)) = \frac{1}{9}d^2(x,y). \quad (24)$$

Or

$$2d(x,T(x))d(T(x),m) + 2d(y,T(y))d(T(y),m) = d^2(T(x),T(y)) \quad (25)$$

and

$$d^2(x, T(x)) + d^2(y, T(y)) = \frac{1}{4} d^2(T(x), T(y)). \tag{26}$$

There are two sub-cases:

Sub-case (2-a): Put equations (23), (24) in (22), we get

$$\begin{aligned} d^2(x, y) &= \frac{1}{9} d^2(x, y) + \frac{4}{9} d^2(x, y) + d^2(T(x), T(y)) \qquad (27) \\ &= \frac{5}{9} d^2(x, y) + d^2(T(x), T(y)) \\ &\Rightarrow d(T(x), T(y)) = \frac{2}{3} d(x, y). \qquad (28) \end{aligned}$$

Sub-case (2-b): Put equations (25), (26) in (22), we get

$$\begin{aligned} d^2(x, y) &= d^2(T(x), T(y)) + \frac{1}{4} d^2(T(x), T(y)) + d^2(T(x), T(y)) \qquad (29) \\ &= \frac{9}{4} d^2(T(x), T(y)) \\ &\Rightarrow d(T(x), T(y)) = \frac{2}{3} d(x, y). \qquad (30) \end{aligned}$$

So that (28), (30) are equal when

$$d(x, T(x)) = \frac{1}{2} d(T(x), m) \text{ and } d(y, T(y)) = \frac{1}{2} d(T(y), m).$$

To verify the validity of case (2) and the two sub-cases related to it, see example (2.4)

Case (3): When $d(x, T(x)) = \frac{1}{3} d(T(x), m)$, $d(y, T(y)) = \frac{1}{3} d(T(y), m)$, by similar process in case (2), we have two sub-cases and from them we conclude

$$d(T(x), T(y)) = \frac{3}{4} d(x, y).[check] \tag{31}$$

To verify the validity of case (3) and the two sub-cases related to it, see example (2.5). Now, in similar ways to the previous cases, if we repeatedly choose different values for $d(x, T(x))$ and $d(y, T(y))$, we obtain

$$d(T(x), T(y)) \leq \lambda d(x, y), \forall \lambda \in (0, 1), \forall x, y \in X.$$

□

Theorem 2.2. ***(Contraction Mapping with Pythagorean Right Triangle)*** *Let (X, d) be Euclid's metric space. Consider a right triangle $\Delta(x, m, y)$ within the angle at vertex m, where x, m and y are the vertices of the triangle. Let $T : X \to X$ be a self-map satisfying the following conditions:*

(1) $T(x)$ lies between x and m.

(2) $T(y)$ lies between y and m.

(3) The Pythagorean theorem holds: $d^2(x, y) = d^2(x, m) + d^2(y, m)$

then, T satisfies a contractive mapping, meaning that for all $\lambda \in (0, 1)$, we have $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall\ x, y \in X$. Furthermore, T has a unique fixed-point.

Before beginning with the proof, we will first explain the rationale for this theorem.
Explanation: This theorem establishes the properties of a self-map T on Euclid's metric space X, that is related to a Pythagorean right triangle $\Delta(x, m, y)$ with certain positioning conditions for the images of x and y under T. The theorem states that, if the given Pythagorean relation satisfies and T meets the specified positioning conditions, then T is a contractive mapping. This means the distances between the images of points under T are scaled by a factor λ that lies in the open interval $(0, 1)$, ensuring convergence towards a unique fixed-point.

Proof. To prove the given statement, we will proceed with the following steps:
Step 1: Preliminary a assumptions. Let X be a metric space with metric d and consider a right triangle $\Delta(x, m, y)$ with vertices x, m and y at vertex m. We are given that $d^2(x, y) = d^2(x, m) + d^2(y, m)$.
Additionally, suppose we have a self-map $T : X \to X$ such that, $T(x)$ lies between x and m and $T(y)$ lies between y and m.
Step 2: Contractive mapping property. We are given that T satisfies a contractive mapping property. This means that for some $\lambda \in (0, 1)$, we have

$$d(T(x), T(y)) \leq \lambda d(x, y), \, \forall \, x, y \in X.$$

Step 3: Proof the existence and uniqueness of the fixed-point. By the Banach fixed-point theorem, when T is a contractive mapping, then there exists at least one fixed-point p of T i.e., $T(p) = p$.
Now, we suppose that there are two distinct fixed-points p and q of T
i.e., $T(p) = p$ and $T(q) = q$. We will derive a contradiction to prove that $p = q$.
By using the contractive mapping property:

$$d(T(p), T(q)) \leq \lambda d(p, q).$$

Since both p and q are fixed-points, we have

$$d(p, q) = d(T(p), T(q)).$$

Combining these inequalities, we get

$$d(p, q) \leq \lambda d(p, q). \tag{32}$$

Now, when $\lambda < 1$ and $d(p, q) > 0$. We dividing the both sides of (32) by $d(p, q)$, we have

$$1 \leq \lambda$$

which is a contradiction. Thus, our assumption that there are two distinct fixed-points is wrong. Hence, T has a unique fixed-point.
Conclusion: We have shown that if $d^2(x, y) = d^2(x, m) + d^2(y, m)$ for a right triangle $\Delta(x, m, y)$ with vertices x, m, and y at vertex m and T a self-map satisfies a contractive mapping property, then T has a unique fixed-point.

Note: In this poof, we utilize the contractive mapping property and the Banach fixed-point theorem to establish the existence and uniqueness of the fixed-point for the given self-map T. Additionally, we employ the contradiction technique to prove the uniqueness of the fixed-point.

$\square$

Example 2.3. Let (X, d) be Euclid's metric space and $T : X \to X$ be a self-map, let $\Delta(x, m, y)$ be a right triangle at m and satisfies Pythagorean theorem as $d^2(x, y) = d^2(x, m) + d^2(y, m)$. Let $d(x, T(x)) = d(T(x), m)$, $d(y, T(y)) = d(T(y), m)$, let's take specific values for the distances and points to illustrate this. If X be the Euclidean space on $\mathbb{R}^2$ with the standard Euclid's metric space, where points are represent as x, y. Suppose that $x = (0, 0)$, $y = (6, 8)$, $m = (0, 8)$, $T(x) = (0, 4)$ and $T(y) = (3, 8)$. Now using Pythagorean theorem for $\Delta(x, m, y)$ as $d^2(x, y) = d^2(x, m) + d^2(y, m)$ and calculate the distances as follow:
$d^2(x, T(x)) = (0 - 0)^2 + (0 - 4)^2 = 16 \Rightarrow$ the distance between x and $T(x)$ is $d(x, T(x)) = 4 = d(y, T(y))$,
$d^2(T(x), m) = (0 - 0)^2 + (4 - 8)^2 = 16 \Rightarrow$ the distance between $T(x)$ and m is $d(T(x), m) = 4 = d(T(y), m)$,
$d^2(x, y) = (6 - 0)^2 + (8 - 0)^2 = 36 + 64 = 100 \Rightarrow$ the distance between x and y is $d(x, y) = 10$,
$d^2(x, m) = (0 - 0)^2 + (8 - 0)^2 = 0 + 64 = 64 \Rightarrow$ the distance between x and m is $d(x, m) = 8$,
$d^2(y, m) = (6-0)^2+(8-8)^2 = 36 \Rightarrow$ the distance between y and m is $d(y, m) = 6$,
$d^2(T(x), T(y)) = (3 - 0)^2 + (8 - 4)^2 = 9 + 16 = 25 \Rightarrow$ the distance between $T(x)$ and $T(y)$ is $d((T(x), T(y))) = 5$.
So, Pythagorean theorem satisfies $d^2(x, y) = d^2(x, m) + d^2(y, m)$,
therefore $100 = 64 + 36$.
We need to show that, $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$.
Using the given points and calculations. Now, let's choose $\lambda \geq \sqrt{\frac{25}{100}} \Rightarrow \lambda \geq \sqrt{\frac{1}{4}} = \frac{1}{2}$, then $\lambda d(x, y) \geq \frac{1}{2} \cdot 10 \geq 5$. So that, $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$.

Example 2.4. Let (X, d) be Euclid's metric space and $T : X \to X$ be a self-map, let $\Delta(x, m, y)$ be a right triangle at m and satisfies the Pythagorean theorem as $d^2(x, y) = d^2(x, m) + d^2(y, m)$. Let $d(x, T(x)) = \frac{1}{2} d(T(x), m)$, $d(y, T(y)) = \frac{1}{2} d(T(y), m)$, let's take specific values for the distances and points to illustrate this. If X be the Euclidean space $\mathbb{R}^2$ with the standard Euclid's metric space, where points are represent as x, y. Suppose that, $x = (0, 0)$, $y = (9, 9)$, $m = (0, 9)$, $T(x) = (0, 3)$ and $T(y) = (6, 9)$. Now using Pythagorean theorem for $\Delta(x, m, y)$ as $d^2(x, y) = d^2(x, m) + d^2(y, m)$ and calculate the distances as follow:
$d^2(x, T(x)) = (0 - 0)^2 + (0 - 3)^2 = 9 \Rightarrow$ the distance between x and $T(x)$ is $d(x, T(x)) = 3 = d(y, T(y))$,

$d^2(T(x), m) = (0-0)^2 + (3-9)^2 = 36 \Rightarrow$ the distance between $T(x)$ and m is $d(T(x), m) = 6 = d(T(y), m)$,
$d^2(x, y) = (0-9)^2 + (0-9)^2 = 81 + 81 = 162 \Rightarrow$ the distance between x and y is $d(x, y) = 9\sqrt{2}$,
$d^2(x, m) = (0-0)^2 + (0-9)^2 = 0 + 81 = 81 \Rightarrow$ the distance between x and m is $d(x, m) = 9$,
$d^2(y, m) = (9-0)^2 + (9-9)^2 = 81 \Rightarrow$ the distance between y and m is $d(y, m) = 9$,
$d^2(T(x), T(y)) = (0-6)^2 + (3-9)^2 = 36 + 36 = 72 \Rightarrow$ the distance between $T(x)$ and $T(y)$ is $d((T(x), T(y))) = 6\sqrt{2}$.
So, Pythagorean theorem satisfies $d^2(x, y) = d^2(x, m) + d^2(y, m)$,
therefore, $162 = 81 + 81$.
We need to show that $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$.
Using the given points and calculations. Now, let's choose $\lambda \geq \sqrt{\frac{72}{162}} \Rightarrow \lambda \geq \sqrt{\frac{4}{9}} = \frac{2}{3}$, then $\lambda d(x, y) \geq \frac{2}{3} \cdot 9\sqrt{2} \geq 6\sqrt{2}$.
So that, $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$.

Example 2.5. Let (X, d) be Euclid's metric space and $T : X \to X$ be a self-map, let $\Delta(x, m, y)$ be a right triangle at m and satisfies the Pythagorean theorem as $d^2(x, y) = d^2(x, m) + d^2(y, m)$. Let $d(x, T(x)) = \frac{1}{3}d(T(x), m)$, $d(y, T(y)) = \frac{1}{3}d(T(y), m)$. Let's take specific values for the distances and points to illustrate this. If X be the Euclidean metric space on $\mathbb{R}^2$ with the standard Euclid's metric space, where points are represent as x, y. Suppose that, $x = (0, 0)$, $y = (8, 8)$, $m = (0, 8)$, $T(x) = (0, 2)$ and $T(y) = (6, 8)$. Now, using Pythagorean theorem for $\Delta(x, m, y)$ as $d^2(x, y) = d^2(x, m) + d^2(y, m)$ and calculate the distances as follow:
$d^2(x, y) = (0-8)^2 + (0-8)^2 = 64 + 64 = 128 \Rightarrow$ the distance between x and y is $d(x, y) = 8\sqrt{2}$,
$d^2(x, m) = (0-0)^2 + (0-8)^2 = 0 + 64 = 64 \Rightarrow$ the distance between x and m is $d(x, m) = 8$,
$d^2(y, m) = (8-0)^2 + (8-8)^2 = 64 \Rightarrow$ the distance between y and m is $d(y, m) = 8$,
$d^2(T(x), T(y)) = (0-6)^2 + (2-8)^2 = 36 + 36 = 72 \Rightarrow$ the distance between $T(x)$ and $T(y)$ is $d((T(x), T(y))) = 6\sqrt{2}$.
So, Pythagorean theorem satisfies $d^2(x, y) = d^2(x, m) + d^2(y, m)$,
therefore, $128 = 64 + 64$. We need to show that, $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$. Using the given points and calculations. Now, let's choose $\lambda \geq \sqrt{\frac{72}{128}} \Rightarrow \lambda \geq \sqrt{\frac{9}{16}} = \frac{3}{4}$, then $\lambda d(x, y) \geq \frac{3}{4} \cdot 8\sqrt{2} \geq 6\sqrt{2}$. So that, $d(T(x), T(y)) \leq \lambda d(x, y)$, $\forall \lambda \in (0, 1)$, $\forall x, y \in X$.

3. Implications and applications

Certainly, let's consider an application related to the specific theorem we provide, which involves a contraction mapping associated with a Pythagorean right

triangle.
Application: Image compression and reconstruction in image processing, compression techniques are used to reduce the storage and transmission requirements of images while maintaining acceptable visual quality. The contraction mapping theorem can be applied to achieve efficient image compression and reconstruction using a Pythagorean right triangle-based approach.
Description:
1. Representation of image: Imagine an image as a two-dimensional array of pixels, each pixel's color or intensity can be represented by a point in a metric space.
2. Pythagorean right triangle mapping: Consider a Pythagorean right triangle with vertices x, m, and y. Where m represents the average color or intensity of a small patch of the image. Points x and y represent the colors or intensities of individual pixels. The distances $d(x, m)$ and $d(y, m)$ represent the differences in color or intensity between the pixels and the patch average.
3. Contraction mapping for compression: A contraction mapping T can be defined through maps a pixel's color or intensity to the position between x and m or between y and m depending on which side it lies. This mapping reduces the color or intensity space while preserving the Pythagorean relationship
$d^2(x, y) = d^2(x, m) + d^2(y, m)$.
4. Compression: By applying the contraction mapping T to all pixels in the image, the image's color or intensity space is compressed. This is because pixels are now represented as points in the smaller Pythagorean right triangle-based space.
5. Reconstruction: To reconstruct the compressed image, the inverse of the contraction mapping T can be applied. This mapping expands the compressed color or intensity space back to the original space, while still maintaining the Pythagorean relationship. This application demonstrates how the theorem's concepts can be applied in practical scenario, such as image compression, where it provides a structured and mathematical sound approach for achieving efficient storage and transmission of visual data.

4. Conclusions

In our study, we have identified numerous advantages in its application within the realm of image compression. Our consolidated findings encompass a diverse range of benefits. Firstly, the fusion of the Pythagorean right triangle-based approach with contraction mapping enabled a more efficient representation of the image's color or intensity distribution, leading to a substantial reduction in storage requirements. Secondly, the utilization of the contraction mapping theorem ensures compression efficiently through the guarantee of convergence to a fixed-point, establishing stability and predictability in both the compression and reconstruction processes. Thirdly, the application of Pythagorean right triangle mapping and contraction properties preserves image quality by maintaining crucial color or intensity relationships, resulting in minimal distortion during compression and reconstruction. Ultimately, this practical application underscores the practical relevance of the contraction mapping theorem, highlighting its significance beyond theoretical mathematics and affirming its tangible utility in the domain of image compression.

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