

**VARIOUS PROPERTIES OF HIGH-ORDER
(p, q)-POLY-TANGENT POLYNOMIALS AND THE
PHENOMENA OF THEIR ROOTS**

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ABSTRACT. In this paper, we construct higher-order (p, q) -poly-tangent numbers and polynomials and give several properties, including addition formula and multiplication formula. Finally, we explore the distribution of roots of higher-order (p, q) -poly-tangent polynomials.

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1. Introduction

In [7], we defined the tangent numbers and polynomials. The tangent polynomials are defined as the following generating function

$$\left(\frac{2}{e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} \mathbf{T}_n(x) \frac{t^n}{n!}.$$

In [8], we constructed the poly-tangent numbers and polynomials. A modified poly-tangent numbers and polynomials different from the poly-tangent numbers and polynomials defined in [8] was introduced. In [9], we introduced tangent numbers and tangent polynomials of higher order. We also obtain interesting properties of these numbers and polynomials. For a nonnegative integer r , tangent polynomials of higher order are defined as the following generating function

$$\left(\frac{2}{e^{2t} + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{T}_n^{(r)}(x) \frac{t^n}{n!}.$$

Definition 1.1. For any integer k , the modified poly-tangent polynomials are defined as the following generating function

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$$\left(\frac{2Li_k(1 - e^{-t})}{t(e^{2t} + 1)}\right) e^{xt} = \sum_{n=0}^{\infty} T_n^{(k)}(x) \frac{t^n}{n!},$$

where $Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$ is polylogarithm function.

$T_n^{(k)} = T_n^{(k)}(0)$ are the called poly-tangent numbers when $x = 0$. If we set $k = 1$ in Definition 1.1, then the poly-tangent polynomials are reduced to classical tangent polynomials because of $Li_1(1 - e^{-t}) = t$. That is, $T_n^{(1)}(x) = \mathbf{T}_n(x)$.

In [11], we constructed the higher-order poly-tangent numbers and polynomials.

Definition 1.2. For any integer k , a nonnegative integer r , higher-order poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_k(1 - e^{-t})}{t(e^{2t} + 1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_n^{(k,r)}(x) \frac{t^n}{n!}.$$

$T_n^{(k,r)} = T_n^{(k,r)}(0)$ are called higher-order poly-tangent numbers when $x = 0$. If we set $k = 1$ and $r = 1$ in Definition 1.2, then the higher-order poly-tangent polynomials are reduced to classical tangent polynomials because of $Li_1(1 - e^{-t}) = t$. That is, $T_n^{(1,1)}(x) = \mathbf{T}_n(x)$. Especially when $k = 1$, we get $T_n^{(1,r)}(x) = \mathbf{T}_n^{(r)}(x)$.

In [3], [2], [8], the q -number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (q \neq 1).$$

We note that $\lim_{q \rightarrow 1} [x]_q = x$. For $k \in \mathbb{Z}$, the q -analogue of polylogarithm function $Li_{k,q}$ is known by

$$Li_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k}.$$

Definition 1.3. For any integer k , a nonnegative integer r , higher-order (p, q) -poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_{k,q}(1 - e^{-t})}{t(e^{2t} + 1)}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(k,r)}(x) \frac{t^n}{n!}.$$

$T_{n,q}^{(k,r)} = T_{n,q}^{(k,r)}(0)$ are called higher-order q -poly-tangent numbers when $x = 0$. If we set $q \rightarrow 1$ in Definition 1.3, then the higher-order q -poly-tangent polynomials are reduced to higher-order poly-tangent polynomials.

2. Some properties of the higher-order (p, q) -poly-tangent numbers and polynomials

In this section, we define higher-order (p, q) -poly-tangent polynomials and study several properties, including addition formula and multiplication formula.

In [3], [2], [8], For $0 < q < p \leq 1$, the (p, q) -number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

The (p, q) -analogue of polylogarithm function $Li_{k,p,q}$ is known by

$$Li_{k,p,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^k}, (k \in \mathbb{Z}).$$

Definition 2.1. For any integer k , a nonnegative integer r , and $0 < q < p \leq 1$, higher-order (p, q) -poly-tangent polynomials are defined as the following generating function

$$\left(\frac{2Li_{k,p,q}(1 - e^{-t})}{t(e^{2t} + 1)} \right)^r e^{xt} = \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!}.$$

$T_{n,p,q}^{(k,r)} = T_{n,p,q}^{(k,r)}(0)$ are called higher-order q -poly-tangent numbers when $x = 0$. If we set $p = 1, q \rightarrow 1$ in Definition 2.1, then the higher-order (p, q) -poly-tangent polynomials are reduced to higher-order poly-tangent polynomials.

Theorem 2.2. For any integer k and a nonnegative integer r, n , and m , we get

$$T_{n,p,q}^{(k,r)}(mx) = \sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)} m^{n-l} x^{n-l}.$$

Proof. From Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(mx) \frac{t^n}{n!} &= \left(\frac{2Li_{k,p,q}(1 - e^{-t})}{t(e^{2t} + 1)} \right)^r e^{mxt} \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (mx)^n \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)} m^{n-l} x^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{1}$$

Therefore, we finish the proof of Theorem 2.2 by comparing the coefficients of $\frac{t^n}{n!}$. □

If $m = 1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.3. For any integer k and a nonnegative integer r and n , we have

$$T_{n,p,q}^{(k,r)}(x) = \sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)} x^{n-l}.$$

Theorem 2.4. For any integer k and a nonnegative integer r and n , we get

$$(1) T_{n,p,q}^{(k,r)}(x+y) = \sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)}(x) y^{n-l}.$$

$$(2) T_{n,p,q}^{(k,r+s)}(x+y) = \sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)}(x) T_{n-l,p,q}^{(k,s)}(y).$$

Proof. (1) Proof is omitted since it is a similar method of Theorem 2.2.

(2) From Definition 1.1, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r+s)}(x+y) \frac{t^n}{n!} \\ &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^{r+s} e^{(x+y)t} \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,s)}(y) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)}(x) T_{n-l,p,q}^{(k,s)}(y) \right) \frac{t^n}{n!}. \end{aligned} \tag{2}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (2). \square

Theorem 2.5. For any integer k and a nonnegative integer r , n , and m , we obtain

$$T_{n,p,q}^{(k,r)}(mx) = \sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)}(x) (m-1)^{n-l} x^{n-l}.$$

Proof. By utilizing Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(mx) \frac{t^n}{n!} &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{xt} e^{(m-1)xt} \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (m-1)^n x^n \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} T_{l,p,q}^{(k,r)}(x) (m-1)^{n-l} x^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{3}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (3). \square

Theorem 2.6. For any integer k , a nonnegative integer r , and a positive integer n , we have

$$T_{n,p,q}^{(k,r)}(x+1) - T_{n,p,q}^{(k,r)}(x) = \sum_{l=0}^{n-1} \binom{n}{l} T_{l,p,q}^{(k,r)}(x).$$

Proof. By using Definition 2.1, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x+1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} \\ &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{(x+1)t} - \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{xt} \\ &= \left(\frac{2Li_{k,p,q}(1-e^{-t})}{t(e^{2t}+1)} \right)^r e^{xt} (e^t - 1) \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1 \right) \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=1}^{\infty} \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n+1}{l} T_{l,p,q}^{(k,r)}(x) \frac{t^{n+1}}{(n+1)!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \binom{n}{l} T_{l,p,q}^{(k,r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{4}$$

Then we compare the coefficients of $\frac{t^n}{n!}$ for $n \geq 1$. The reason both sides of the above equation (4) can be compared the coefficients is that $T_{0,q}^{(k,r)}(x+1) - T_{0,q}^{(k,r)}(x) = 0$. Thus, the proof is done. \square

3. Polynomials and numbers related to higher-order (p, q) -poly-tangent polynomials and its symmetric property

In this section, we examine the association between higher-order poly-tangent polynomials and poly-tangent polynomials using Cauchy product. We also explore relation of higher-order poly-tangent polynomials and Stirling numbers of the second kind. Furthermore, we study the symmetry properties of higher-order poly-tangent polynomials.

Generating function of the Stirling numbers of the second kind $S_2(n, k)$ is defined as follows:

$$\sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

Theorem 3.1. For any integer k , a nonnegative integer r and a positive integer n , we obtain

$$T_{n,p,q}^{(k,r)}(x) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} (x)_m S_2(l, m) T_{n-l,p,q}^{(k,r)},$$

where $(x)_m = x(x - 1) \cdots (x - m + 1)$ is falling factorial.

Proof. From Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)}(x) \frac{t^n}{n!} &= \left(\frac{2Li_{k,p,q}(1 - e^{-t})}{t(e^{2t} + 1)} \right)^r e^{xt} \\ &= \left(\frac{2Li_{k,p,q}(1 - e^{-t})}{t(e^{2t} + 1)} \right)^r \sum_{m=0}^{\infty} (x)_m \frac{(e^t - 1)^m}{m!} \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (x)_m S_2(n, m) \frac{t^n}{n!} \right) \tag{5} \\ &= \left(\sum_{n=0}^{\infty} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^n (x)_m S_2(n, m) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} (x)_m S_2(l, m) T_{n-l,p,q}^{(k,r)} \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, we finish the proof by comparing the coefficients of $\frac{t^n}{n!}$. □

We recall a multinomial coefficient, which is

$$\binom{n}{m_1, m_2, \dots, m_l} = \frac{n!}{m_1! m_2! \cdots m_l!}. \tag{6}$$

Let us consider the following equation using the equation (6) above. This equation is an equation expressed by applying Cauchy product continuously.

Theorem 3.2. For any integer k , a nonnegative integer n , and $r \geq 3$, we get

$$\begin{aligned} T_{n,p,q}^{(k,r)}(rx) &= \sum_{m_{r-1}=0}^n \sum_{m_{r-2}=0}^{m_{r-1}} \cdots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \\ &\quad \times \binom{n}{m_1, m_2 - m_1, \dots, m_{r-1} - m_{r-2}, n - m_{r-1}} T_{m_1,p,q}^{(k)}(x) \\ &\quad \times T_{m_2 - m_1,p,q}^{(k)}(x) \cdots T_{m_{r-1} - m_{r-2},p,q}^{(k)}(x) T_{n - m_{r-1},p,q}^{(k)}(x), \end{aligned}$$

where $\binom{n}{m_1, m_2, \dots, m_l}$ is multinomial coefficient.

Theorem 3.3. Let r and n be a nonnegative integer and $w_1, w_2 > 0 (w_1 \neq w_2)$. Then we have

$$\begin{aligned} & \sum_{l=0}^n \binom{n}{l} w_1^l w_2^{n-l} T_{l,p,q}^{(k,r)}(w_2 x) T_{n-l,p,q}^{(k,r)}(w_1 x) \\ &= \sum_{l=0}^n \binom{n}{l} w_2^l w_1^{n-l} T_{l,p,q}^{(k,r)}(w_1 x) T_{n-l,p,q}^{(k,r)}(w_2 x). \end{aligned}$$

Proof. Let us consider the function

$$F(t) = \left(\frac{4Li_{k,p,q}(1 - e^{-w_1 t}) Li_{k,p,q}(1 - e^{-w_2 t})}{t^2(e^{2w_1 t} + 1)(e^{2w_2 t} + 1)} \right)^r e^{2w_1 w_2 x t}. \tag{7}$$

Then we obtain

$$\begin{aligned} F(t) &= \left(\frac{2Li_{k,p,q}(1 - e^{-w_1 t})}{t(e^{2w_1 t} + 1)} \right)^r e^{w_1 w_2 x t} \left(\frac{2Li_{k,p,q}(1 - e^{-w_2 t})}{t(e^{2w_2 t} + 1)} \right)^r e^{w_1 w_2 x t} \\ &= \left(\sum_{n=0}^{\infty} w_1^{n+r} T_{n,p,q}^{(k,r)}(w_2 x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} w_2^{n+r} T_{n,p,q}^{(k,r)}(w_1 x) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} w_1^{l+r} w_2^{n-l+r} T_{l,p,q}^{(k,r)}(w_2 x) T_{n-l,p,q}^{(k,r)}(w_1 x) \right) \frac{t^n}{n!}. \end{aligned} \tag{8}$$

By calculating in the same way as the above equation (8), we can get

$$F(t) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} w_2^{l+r} w_1^{n-l+r} T_{l,p,q}^{(k,r)}(w_1 x) T_{n-l,p,q}^{(k,r)}(w_2 x) \right) \frac{t^n}{n!}. \tag{9}$$

The proof is complete as a result of the equations (8) and (9). □

Let w is an odd number. Then we can easily see

$$\sum_{n=0}^{\infty} \tilde{A}_n(w) \frac{t^n}{n!} = \frac{e^{wt} + 1}{e^t + 1}, \tag{10}$$

where $\tilde{A}_n(w) = \sum_{l=0}^{w-1} (-1)^l l^n$ is called alternating power sum.

Theorem 3.4. Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we have

$$\begin{aligned} & \sum_{j=0}^n \sum_{i=0}^j \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} 2^{n-j-l} w_1^{i+l+r} w_2^{2n-2j-i-l+r} T_{i,p,q}^{(k,r)} \\ & \quad \times T_{n-j-i,p,q}^{(k,r)} \mathbf{T}_l(w_2 x) \tilde{A}_{n-j-l}(w_1) \\ &= \sum_{j=0}^n \sum_{i=0}^j \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} 2^{n-j-l} w_2^{i+l+r} w_1^{2n-2j-i-l+r} T_{i,p,q}^{(k,r)} \\ & \quad \times T_{n-j-i,p,q}^{(k,r)} \mathbf{T}_l(w_1 x) \tilde{A}_{n-j-l}(w_2). \end{aligned}$$

Proof. First, let us assume that

$$G(t) = 2 \frac{4^r (Li_{k,p,q}(1 - e^{-w_1 t}))^r (Li_{k,p,q}(1 - e^{-w_2 t}))^r (e^{2w_1 w_2 t} + 1)}{t^{2r} (e^{2w_1 t} + 1)^r (e^{2w_2 t} + 1)^r (e^{2w_1 t} + 1)(e^{2w_2 t} + 1)} e^{2w_1 w_2 x t}. \tag{11}$$

Then we calculate

$$\begin{aligned} G(t) &= 2 \left(\frac{2Li_{k,p,q}(1 - e^{-w_1 t})}{t(e^{2w_1 t} + 1)} \right)^r \left(\frac{2Li_{k,p,q}(1 - e^{-w_2 t})}{t(e^{2w_2 t} + 1)} \right)^r \\ &\quad \times \frac{2}{(e^{2w_1 t} + 1)} e^{2w_1 w_2 x t} \frac{e^{2w_1 w_2 t} + 1}{e^{2w_2 t} + 1} \\ &= \left(\sum_{n=0}^{\infty} w_1^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} w_2^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \\ &\quad \times \left(\sum_{n=0}^{\infty} w_1^n T_{n,p,q}(w_2 x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} 2^n w_2^n \tilde{A}_n(w_1) \frac{t^n}{n!} \right) \\ &= \left(\sum_{n=0}^{\infty} w_1^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} w_2^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \\ &\quad \times \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} 2^{n-l} w_1^l w_2^{n-l} \mathbf{T}_l(w_2 x) \tilde{A}_{n-l}(w_1) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \sum_{i=0}^j \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} 2^{n-j-l} w_1^{i+l+r} w_2^{2n-2j-i-l+r} \right. \\ &\quad \left. \times T_{i,p,q}^{(k,r)} T_{n-j-i,p,q}^{(k,r)} \mathbf{T}_l(w_2 x) \tilde{A}_{n-j-l}(w_1) \right) \frac{t^n}{n!}. \end{aligned} \tag{12}$$

In a similar way to the above equation (12), we get

$$\begin{aligned} G(t) &= \left(\sum_{n=0}^{\infty} w_1^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} w_2^{n+r} T_{n,p,q}^{(k,r)} \frac{t^n}{n!} \right) \\ &\quad \times \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} 2^{n-l} w_2^l w_1^{n-l} \mathbf{T}_l(w_1 x) \tilde{A}_{n-l}(w_2) \frac{t^n}{n!}. \end{aligned} \tag{13}$$

Hence, by using Cauchy product, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equations (12) and (13). \square

4. Distribution of zeros of the higher-order (p, q) -poly-tangent polynomials

Using generating functions, the generalized forms of known polynomials such as the Bernoulli, Euler, falling factorial and tangent polynomials are studied. In particular, various properties of these polynomials were investigated through numerical experiments, see for example [1], [4], [6], [7], [8], [9], [10], [12], [13].

In this section, we discover new interesting pattern of the zeros of the higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,3)}(x)$. We propose some conjectures by numerical experiments. The higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,3)}(x)$ can be determined explicitly.

A few of them are

$$\begin{aligned}
 T_{0,p,q}^{(k,3)}(x) &= 1, \\
 T_{1,p,q}^{(k,3)}(x) &= -\frac{9}{2} + 3 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} + x, \\
 T_{2,p,q}^{(k,3)}(x) &= \frac{35}{2} + 6 \left(\frac{p^2 - q^2}{p - q}\right)^{-2k} - 30 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} + 6 \left(\frac{p^3 - q^3}{p - q}\right)^{-k} - 9x \\
 &\quad + 6 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} x + x^2, \\
 T_{3,p,q}^{(k,3)}(x) &= -54 + 6 \left(\frac{p^2 - q^2}{p - q}\right)^{-3k} - 99 \left(\frac{p^2 - q^2}{p - q}\right)^{-2k} + 201 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} \\
 &\quad - 99 \left(\frac{p^3 - q^3}{p - q}\right)^{-k} + 36 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} \left(\frac{p^3 - q^3}{p - q}\right)^{-k} + 18 \left(\frac{p^4 - q^4}{p - q}\right)^{-k} \\
 &\quad + \frac{105x}{2} + 18 \left(\frac{p^2 - q^2}{p - q}\right)^{-2k} x - 90 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} x \\
 &\quad + 18 \left(\frac{p^3 - q^3}{p - q}\right)^{-k} x - \frac{27x^2}{2} + 9 \left(\frac{p^2 - q^2}{p - q}\right)^{-k} x^2 + x^3,
 \end{aligned}$$

We investigate the beautiful zeros of the higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,r)}(x)$ by using a computer. We plot the zeros of higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,r)}(x)$ for $n = 30, r = 3$ and $x \in \mathbb{C}$ (Figure 1).

In Figure 1(top-left), we choose $n = 30, p = \frac{9}{10}, q = \frac{1}{10}$ and $k = -4$. In Figure 1(top-right), we choose $n = 30, p = \frac{7}{10}, q = \frac{3}{10}$ and $k = -4$. In Figure 1(bottom-left), we choose $n = 30, p = \frac{9}{10}, q = \frac{1}{10}$, and $k = 4$. In Figure 1(bottom-right), we choose $n = 30, p = \frac{7}{10}, q = \frac{3}{10}$ and $k = 4$.

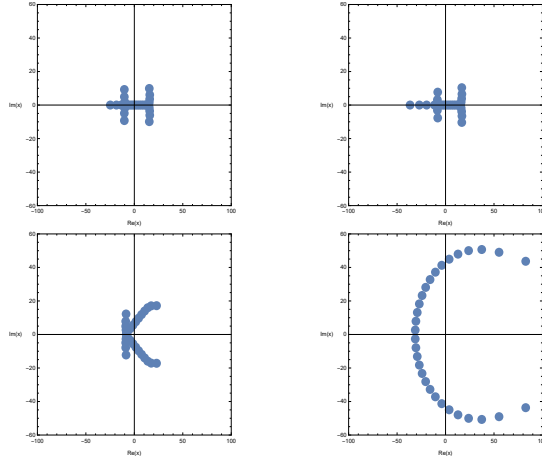


FIGURE 1. Zeros of $T_{n,p,q}^{(k,r)}(x)$

Stacks of zeros of $T_{n,p,q}^{(k,r)}(x)$ for $1 \leq n \leq 30, r = 3$ from a 3-D structure are presented(Figure 2).

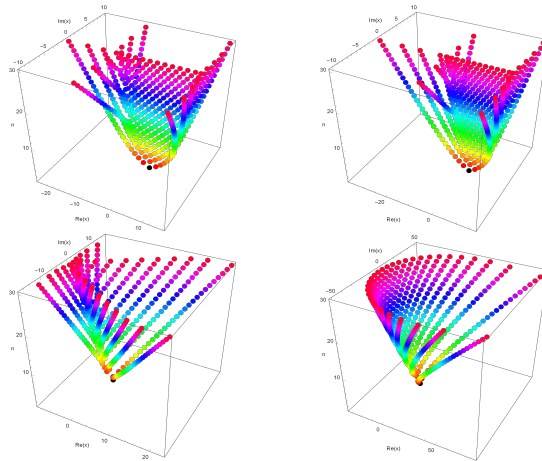


FIGURE 2. Stacks of zeros of $T_{n,p,q}^{(k,r)}(x)$ for $1 \leq n \leq 30$

In Figure 2(top-left), we choose $p = \frac{9}{10}, q = \frac{1}{10}$ and $k = -4$. In Figure 2(top-right), we choose $p = \frac{7}{10}, q = \frac{3}{10}$ and $k = -4$. In Figure 2(bottom-left), we choose $p = \frac{9}{10}, q = \frac{1}{10}$, and $k = 4$. In Figure 2(bottom-right), we choose $p = \frac{7}{10}, q = \frac{3}{10}$ and $k = 4$.

We plot the real zeros of the higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,r)}(x)$, $r = 3$, and $x \in \mathbb{C}$ (Figure 3).

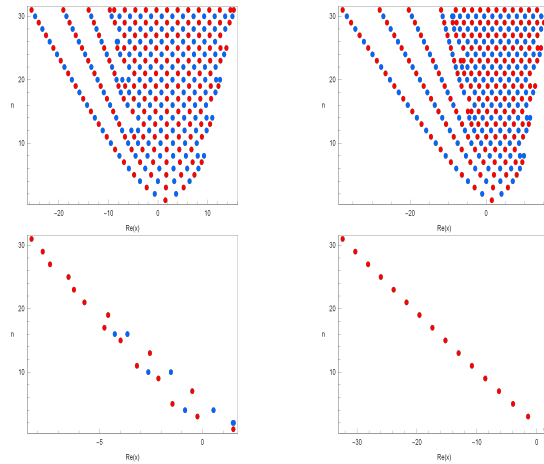


FIGURE 3. Real zeros of $T_{n,p,q}^{(k,r)}(x)$ for $1 \leq n \leq 30$

In Figure 3(top-left), we choose $p = \frac{9}{10}, q = \frac{1}{10}$ and $k = -4$. In Figure 3(top-right), we choose $p = \frac{7}{10}, q = \frac{3}{10}$ and $k = -4$. In Figure 3(bottom-left), we choose $p = \frac{9}{10}, q = \frac{1}{10}$, and $k = 4$. In Figure 3(bottom-right), we choose $p = \frac{7}{10}, q = \frac{3}{10}$ and $k = 4$.

Next, we calculated an approximate solution satisfying higher-order (p, q) -poly-tangent polynomials $T_{n,p,q}^{(k,r)}(x)$ for $x \in \mathbb{R}$. The results are given in Table 1 and Table 2.

Table 1. Approximate solutions of $T_{n,p,q}^{(k,r)}(x) = 0, k = -4, r = 3, p = \frac{9}{10}, q = \frac{1}{10}$

degree n	x
1	1.5000
2	-0.65302, 3.6530
3	-2.2803, 1.6044, 5.1758
4	-3.6357, -0.019855, 3.2797, 6.3758
5	-4.8193, -1.3903, 1.6763v4.6764, 7.3568
6	-5.8848, -2.5765, 0.26970, 3.1315, 5.8958, 8.1643
7	-6.8665, -3.6147, -0.98636, 1.7233, 4.4364, 6.9935, 8.8144
8	-7.7883, -4.5265, -2.1167, 0.42719, 3.0557, 5.6342, 8.0235, 9.2909

Table 2. Approximate solutions of $T_{n,p,q}^{(k,r)}(x) = 0$, $k = -4$, $r = 3$, $p = \frac{7}{10}$, $q = \frac{3}{10}$

degree n	x
1	1.5000
2	-1.0324, 4.0324
3	-3.0349, 1.8143, 5.7206
4	-4.7782, 0.027879, 3.7411, 7.0092
5	-6.3663, -1.5099, 2.0638, 5.2726, 8.0399
6	-7.8524, -2.8817, 0.59904, 3.6833, 6.5814, 8.8704
7	-9.2671, -4.1360, -0.70177, 2.2477, 5.0864, 7.7543, 9.5165
8	-10.630, -5.3059, -1.8680, 0.94474, 3.6839, 6.3544, 8.8869, 9.9336

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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