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ON THE GENERALIZED PRINCIPALLY INJECTIVE MODULES

Fatemeh Gholami, Zohreh Habibi, and Alireza Najafizadeh

Abstract. Some results are generalized from principally injective rings to principally injective modules. Moreover, it is proved that the results are valid to some other extended injectivity conditions which may be defined over modules. The influence of such injectivity conditions are studied for both the trace and the reject submodules of some modules over commutative rings. Finally, a correction is given to a paper related to the subject.

1. Introduction

Throughout this paper all rings R are associative with unity and all modules are unitary right modules unless otherwise stated. Moreover, the term homomorphism refers to an R-homomorphism. Recall that a ring R is called right principally injective [13] (or right p-injective for short) if every homomorphism from a principally right ideal of R to R can be extended to an endomorphism of R. The concept of p-injective modules was introduced in 1974 to study von Neumann regular rings, V-rings, self-injective rings and their generalizations (see [19, 20]). This concept has been generalized to modules in various ways (see [12, 15]). Following [12], a module M is called principally quasi-injective (or PQ-injective for short) if each homomorphism from a principal submodule of M_R to M_R can be extended to an endomorphism of M_R. Given the right R-modules M and N, we say that M is N-injective if every homomorphism from a submodule of N to M can be extended to a homomorphism from N to M. Moreover, we say that M is principally N-injective if every homomorphism from a cyclic submodule of N to M can be extended to a homomorphism of N to M. An R-module M is called quasi-injective if it is M-injective. Over the past decades, there have been several achievements in relation to N-injective and principally N-injective modules specially in the case of p-injective rings and PQ-injective modules; for instance see [2, 4, 7, 9, 13, 18–21] for the case of

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general rings and [3, 5, 6, 11] for commutative rings. Motivated by the results of [9, 13], we give some results to the case of principally N-injective and PQ-injective modules. Finally, some corrections to some of the results in [14] are given.

2. Preliminaries and notations

If M_R is a module, we write $l_M(r) = \{m \in M : mr = 0\}$ for all $r \in R$, and $r_R(m) = \{r \in R : mr = 0\}$ for all $m \in M$. Moreover, the symbols $Rad(M_R)$ and $Z(M_R)$ (or $Z_r(M)$) stand for the radical and the singular submodule of M_R, respectively. The module M is called a singular module if $Z(M_R) = M$. Moreover, M is called a non-singular module if $Z(M_R) = 0$. For a module M, the symbols $M^{(I)}$ and M^I denote the direct sum and direct product of $|I|$ copies of M, respectively, in which $|I|$ is the cardinality of the index set I. Given the R-modules M and N, the trace of M in N denoted by $Tr_N(M)$ is defined as $Tr_N(M) = \sum\{\mathrm{Im}(\varphi) \mid \varphi \in \mathrm{Hom}_R(M, N)\}$. The reject of M in N denoted by $Rej_N(M)$ is defined as $Rej_N(M) = \cap\{\ker(\varphi) \mid \varphi \in \mathrm{Hom}_R(N, M)\}$. We say that M is (semi-)N-injective if every homomorphism $f : K \to M$ with K a (N-cyclic) submodule of N can be extended to a homomorphism $g : N \to M$. The module M is called weakly N-injective if for every finitely generated submodule $K \subset N^{(\mathbb{N})}$ all homomorphisms $f : K \to M$ can be extended to $g : N^{(\mathbb{N})} \to M$. Let X be a class of R-modules. M is said to be (semi-,weakly-)X-injective if M is (semi-,weakly-)N-injective for every $N \in X$. For $X = \{M\}$ we obtain the notions of (semi-,weakly-)self-injectivity. If X is the class of all (finitely presented) modules over R, then the X-(weakly-)injective modules are called (fp-)injective. The semi-R-injective modules are called p-injective. In a similar way, we may define the notions as N-projective, principally N-projective, etc. An R-module M is called a quasi-Frobenius module or a QF module if M is weakly M-injective and a weak cogenerator in $\sigma[M]$. A good reference about the notions related to this subject is [17].

In Section 3, we give some results to principally N-injective and PQ-injective modules. In Section 4, some results related to principally N-injective and PQ-injective modules in the case that the ring under investigation is commutative are given. In Section 5, some corrections to the results of Section 2 in [14] are given.

3. Injectivity

In this section, we give some results about principally N-injective and PQ-injective modules. We recall that an R-module M is said to be torsion if every nonzero element is annihilated by a non zero-divisor in R.

Let M_R be an R-module. We say that an element $m \in M$ is faithful if $r_R(m) = 0$. An R-module M is called completely faithful if it contains a faithful element.

We observe that if R is a commutative domain, then M is completely faithful exactly if M is non-torsion.

Example 3.1. Every generator in the category of right R-modules is a completely faithful R-module. In particular, every unitary ring is completely faithful as a right module over itself.

Example 3.2. If R is a domain, then R_R is a completely faithful R-module since every non-zero element of R is faithful.

Example 3.3. The $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ is not completely faithful since for any arbitrary element $m \in M$, we have $r_R(m) \neq 0$.

Proposition 3.4. *Every right simple module over a right duo ring is completely faithful.*

Proof. Let M be a right simple module over a right duo ring R. Then, there exists a maximal ideal I of R such that $M \cong R/I$. Now the hypothesis that R is a right duo ring implies that I is a two-sided ideal of R. Therefore, we can consider R/I as a module over R/I. Consequently, R/I is a unitary ring. We conclude the result. $\square$

Proposition 3.5. *Let M_R be a faithful right module over a ring R with $S = \mathrm{End}(M_R)$. If M is finitely generated as S-module, then M_R^k is completely faithful for some positive integer k.*

Proof. Follows from [17, page 161]. In fact, the hypothesis that M_R is faithful and ${}_SM$ is finitely generated implies that $M_R^k \cong R_R$ for some positive integer k. We conclude the result. $\square$

Lemma 3.6. *Let M and N be right modules over a ring R. Then the following statements are equivalent.*

(1) *M_R is principally N-injective.*
(2) *$l_M r_R(n) = \mathrm{Hom}_R(N, M).n$ for all $n \in N$.*

Proof. (1)⇒(2) Clearly, $\mathrm{Hom}_R(N, M).n \subseteq l_M r_R(n)$. Now, suppose $m \in l_M r_R(n)$. This implies that the map $f : nR \to M$ defined as $f(nt) = mt$ is correctly defined. Hence, the hypothesis gives a homomorphism $g : N \to M$ such that $g(n) = f(n)$. Consequently, $m = f(n) = g(n) = g.n \in \mathrm{Hom}_R(N, M).n$.

(2)⇒(1) Let $f : nR \to M$ be a homomorphism for some $n \in N$. We have $f(n) \in l_M r_R(n) = \mathrm{Hom}_R(N, M).n$. Then, there exists a homomorphism $g : N \to M$ such that $f(n) = g(n)$, that is f extends to g. $\square$

Lemma 3.7. *Let M and N be right modules over a ring R. If M is a submodule of N_R, then the following statements are equivalent.*

(1) *M_R is principally N-injective.*
(2) *$l_M r_R(n) = \mathrm{Hom}_R(N, M).n$ for all $n \in N$.*
(3) *If $r_R(n_1) \subseteq r_R(n_2)$ where $n_1, n_2 \in N$, then we have $\mathrm{Hom}_R(N, M).n_2 \subseteq \mathrm{Hom}_R(N, M).n_1$.*

(4) *$l_M(bR \cap r_R(n)) = l_M(b) + \mathrm{Hom}_R(N, M).n$ for all $b \in R$ and $n \in N$.*

Proof. In view of Lemma 3.6, we need to prove the equivalences of (2) with (3) and (4).

(2)⇒(3) We prove $l_M r_R(n_2) \subseteq l_M r_R(n_1)$, where $n_1, n_2 \in N$. This is done by some easy calculations.

(3)⇒(4) Let $m = m_1 + m_2$, where $m_1 \in l_M(b)$ and $m_2 \in \mathrm{Hom}_R(N, M).n$. Then $m = m_1 + f(n)$ for some homomorphism $f : N \to M$ and $n \in N$. Now, if $s = br \in bR \cap r_R(n)$, then $m.s = (m_1 b)r + f(ns) = 0$. Conversely, suppose that $m \in l_M(bR \cap r_R(n))$. Then, $m.(bR \cap r_R(n)) = 0$. Now an easy calculation shows that $r_R(nb) \subseteq r_R(mb)$. Hence, the hypothesis implies that $\mathrm{Hom}_R(N, M).mb \subseteq \mathrm{Hom}_R(N, M).nb$. Now, in view of the fact that $M \leq N_R$ we get $\mathrm{Hom}_R(N, N)mb \subseteq \mathrm{Hom}_R(N, M)mb \subseteq \mathrm{Hom}_R(N, M)nb$. Since $1_N \in \mathrm{Hom}_R(N, N)$, there exists a homomorphism $\alpha : N \to M$ such that $mb = 1_N(mb) = \alpha(nb) = \alpha(n)b$. Therefore, $m - \alpha(n) \in l_M(b)$. Finally, we have $m = m - \alpha(n) + \alpha(n) \in l_M(b) + \mathrm{Hom}_R(N, M).n$.

(4)⇒(2) Set $b = 1$. □

The equivalence of assertions (1), (2) and (3) in the following corollary follows from [12, Lemma 1.1].

Corollary 3.8. *Given a module M_R with $S = \mathrm{End}(M_R)$, the following are equivalent.*

(1) *M_R is PQ-injective.*
(2) *$l_M r_R(m) = Sm$.*
(3) *If $r_R(m) \subseteq r_R(n)$ where $m, n \in M$, then $Sn \subseteq Sm$.*
(4) *$l_M(bR \cap r_R(m)) = l_M(b) + Sm$ for all $b \in R$ and $m \in M$.*

Proof. Set $N = M$ in Lemma 3.7.

(3)⇒(4) Let $n = m_1 + m_2$, where $m_1 \in l_M(b)$ and $m_2 \in Sm$. Then $m = m_1 + f(n)$ for some $m \in M$ and $f \in S$. If $s = br \in bR \cap r_R(m)$, then $ns = m_1(br) + f(m) = (m_1 b)r + f(ms) = 0$. Conversely, suppose that $n \in l_M(bR \cap r_R(m))$. Then, $n(bR \cap r_R(m)) = 0$. Now an easy calculation shows that $r_R(mb) \subseteq r_R(nb)$. Consequently, there exists an $f \in S$ such that $nb = f(mb) = f(m)b$. This yields $n - fm \in l_M(b)$, hence $n \in l_M(b) + Sm$.

(4)⇒(2) Set $b = 1$. □

Proposition 3.9. *Let M_R be a module over a ring R with $S = \mathrm{End}(M_R)$. Then the following statements are equivalent.*

(1) *If $r_R(m) \subseteq r_R(n)$ where $m, n \in M$, then $Sn \subseteq Sm$.*
(2) *Let N and K be submodules of M_R and let $f : N \to K$ be a homomorphism. If $y = f(x)$ for some $x \in N$ and $y \in K$, then $Sy \subseteq Sx$.*
(3) *Let N and K be submodules of M_R and let $f : N \to K$ be a homomorphism. If $y = f(x)$ for some $x \in N$ and $y \in K$, then there exists some $\beta : M_R \to M_R$ such that $y = \beta(x)$.*

Proof. (1)⇒(2) Let $f : N \to K$ be such that $y = f(x)$ for some $x \in N$. This implies that $r_R(x) \subseteq r_R(f(x))$. Now in view of the hypothesis we conclude the result.

(2)⇒(1) Let m and n belong to M_R such that $r_R(m) \subseteq r_R(n)$. This implies that the map $f : mR \to nR$ defined as $f(mr) = nr$ is correctly defined. Moreover, we have $n = f(m)$. Now, the hypothesis implies that $Sn \subseteq Sm$.

(2)⇒(3) It is obvious. □

Proposition 3.10. *Let $\{R_i\}_{i\in I}$ be a collection of rings. Suppose that $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ be collections of modules such that M_i and N_i are R_i-modules for every $i \in I$. Let $R = \prod_{i\in I} R_i$, $M = \prod_{i\in I} M_i$ and $N = \prod_{i\in I} N_i$. Then, M is principally N-injective as an R-module exactly if each M_i is principally N_i-injective as R_i-module for all $i \in I$.*

Proof. (⇒) It is clear.

(⇐) We show that $M = \prod_{i\in I} M_i$ is principally $(\prod_{i\in I} N_i)$-injective. In view of Lemma 3.7, we need to show that $l_M(r_R(n)) = \mathrm{Hom}_R(\prod_{i\in I} M_i, \prod_{i\in I} N_i)n$ for all $n = \{n_i\}_{i\in I} \in N$. Suppose that $m = \{m_i\}_{i\in I} \in l_M(r_R(n))$, then $m.r_R(n) = 0$. Clearly, $m_i \in l_{M_i} r_{R_i}(n_i)$ for all $i \in I$. Now in view of the fact that each M_i is principally N_i-injective as R_i-module, we have $l_{M_i} r_{R_i}(n_i) = \mathrm{Hom}_{R_i}(M_i, N_i)n_i$. Therefore, $m_i = \alpha_i n_i$ for some $\alpha_i \in \mathrm{Hom}_{R_i}(M_i, N_i)$ and $i \in I$. Clearly, for $\alpha = \{\alpha_i\}_{i\in I}$ we have $m = \{m_i\}_{i\in I} = \{\alpha_i n_i\}_{i\in I} = \alpha\{n_i\}_{i\in I} = \alpha(n)$, which yields the result. □

The next corollary generalizes Example 2 in [13] from p-injective rings to the case of PQ-injective modules.

Corollary 3.11. *Let $\{R_i\}_{i\in I}$ be a collection of rings and $\{M_i\}_{i\in I}$ be a collection of modules such that M_i is an R_i-module for all $i \in I$. Let $R = \prod_{i\in I} R_i$ and $M = \prod_{i\in I} M_i$. Then, M is PQ-injective as an R-module exactly if each M_i is PQ-injective as R_i-module.*

Proof. Set $N = M$ in Proposition 3.10. □

Lemma 3.12. *Let M_R be a module over a ring R with $S_0 = End(M_R)$ and $B = \mathrm{End}({}_{S_0}M)$. Then, every element t of R may be viewed as an element of B as $t :_{S_0} M \to_{S_0} M$ defined by $(m)t = mt$. Moreover, if S is a left denominator set in S_0, then t induces an element of $\mathrm{End}({}_{S^{-1}S_0}S^{-1}M)$.*

Proof. It is straightforward. □

Lemma 3.13. *Let M_R be a module over a ring R with $S_0 = \mathrm{End}(M_R)$ and $B = \mathrm{End}({}_{S_0}M)$. If S is a left denominator set in S_0, then every left $S^{-1}S_0$-module may be considered as S_0-module.*

Proof. It is straightforward. □

The next result generalizes Example 3 in [13]. A good reference for the notions related to the rings and modules of quotients is [16].

Proposition 3.14. *Let M_R be a PQ-injective module over a ring R with $S_0 = \mathrm{End}(M_R)$. Let $S \subseteq S_0$ be a left denominator set and $T = \mathrm{End}({}_{S^{-1}S_0}S^{-1}M)$. Then $S^{-1}M$ is PQ-injective as a right T-module.*

Proof. We consider $S^{-1}M$ as a left module over $S^{-1}S_0$. Let $m \in M$ and $s \in S$ be arbitrary elements. In view of Corollary 3.8, we need to prove that $l_{(S^{-1}M)}r_T(m/s) = B(m/s)$ in which $B = \mathrm{End}(S^{-1}M_T)$. For this, suppose that $m_0/s_0 \in l_{(S^{-1}M)}r_T(m/s)$. Then, $(m_0/s_0)\alpha = 0$ for all $\alpha \in r_T(m/s)$, where $r_T(m/s) = \{f \in T : (m/s).f = 0\} = \{f \in T : (m/s)f = 0\}$. Clearly, $m_0 \in l_M r_R(m)$. Indeed, if $r \in r_R(m)$, then $mr = 0$. We shall prove that $m_0 r = 0$. We define $\bar{r} :_{S^{-1}S_0} S^{-1}M \longrightarrow_{S^{-1}S_0} S^{-1}M$ as $(n/s)\bar{r} = (nr/s)$. Clearly, $\overline{r} \in \mathrm{End}({}_{S_0S^{-1}}S^{-1}M) = T$. Now we have $(m/s)\overline{r} = mr/s = 0/s = 0$. This yields $(m_0/s_0)r = 0$ since $(m_0/s_0)\alpha = 0$ for all $\alpha \in r_T(m/s)$. Hence, $m_0r/s_0 = 0$. Therefore, $m_0r/1 = (s_0/1)(m_0r/s_0) = (s_0/1).0 = 0$. By Lemma 3.13, we have $m_0r = m_0r/1 = 0$. This means that, $m_0 \in l_M r_R(m) = S_0 m$, so we get an element $\beta \in S_0$ such that $m_0 = \beta m$. Consequently, $m_0/s_0 = \beta m/s_0 = (\beta/s_0)(m/1) = (s\beta/s_0)(m/s) \in B(m/s)$. In fact, since $\beta, s \in S_0$ and $s_0 \in S$ we get $s\beta/s_0 \in S_0$. □

Following [14], a proper submodule of a right R-module M is called completely prime if for each $r \in R$ and every $m \in M$ such that $mr \in P$, we have $m \in P$ or $Mr \subseteq P$.

Proposition 3.15. *Let M_R and N_R be modules over a ring R. Suppose that N_R is faithful and $\{0\}$ is a completely prime submodule of N_R. Then, M_R is principally N-injective exactly if $\mathrm{Hom}_R(N, M)n = M$ for all non zero $n \in N$.*

Proof. ($\Rightarrow$) Let M_R be principally N-injective and n be a non-zero element of N. We prove that $r_R(n) = 0$. To do this, suppose that $r \in r_R(n)$, then $nr = 0$. Therefore, $Nr = 0$ since $\{0\}$ is completely prime. Now in view of the fact that N is faithful we get $r = 0$. We conclude that $r_R(n) = 0$, which yields $l_M r_R(n) = M$. Therefore, $\mathrm{Hom}_R(N, M)n = M$. We are done by Lemma 3.6.

($\Leftarrow$) Let n be a non-zero element of N_R. We shall prove that $l_M r_R(n) = \mathrm{Hom}_R(N, M)n$. But in view of the hypothesis, we need to prove that $l_M r_R(n) = M$. Let $r \in r_R(n)$ be arbitrary. Then, $nr = 0$. This implies that $Nr = 0$ since $\{0\}$ is completely prime. Therefore, $r = 0$ because N is faithful. We conclude that $r_R(n) = 0$, which means $l_M r_R(n) = M$. □

The next corollary generalizes the first part of Example 4 in [13].

Corollary 3.16. *Let M_R be a faithful module over a ring R with $S = \mathrm{End}(M_R)$. Suppose that $\{0\}$ is a completely prime submodule of M. Then, M_R is PQ-injective exactly if ${}_SM$ is simple.*

Proof. Set $N = M$ in Proposition 3.15. □

Proposition 3.17. *Let N_R be a module over a commutative ring R and M_R be an R-submodule of N_R. Suppose that $\{0\}$ is a completely prime submodule of*

M_R and each finitely generated submodule of M is cyclic. Then, for any non-zero element m of M and any ideal I of R such that $I \subseteq Ann_{R/I}(M/mR)$, the quotient module M/mR is principally N/mR-injective as an R/I-module.

Proof. Let $\overline{M} = M/mR$, $\overline{N} = N/mR$, $\overline{R} = R/I$ and $\overline{S} = \mathrm{End}_{R/I}(M/mR)$ for a non-zero element $m \in M$ and any ideal I of R such that $I \subseteq Ann_{R/I}(M/mR)$. We prove that $l_{\overline{M}}r_{\overline{R}}(\overline{n}) = \mathrm{Hom}_{\overline{R}}(\overline{N}, \overline{M})\overline{n}$, where $\overline{n} = n + mR$ is an arbitrary element of $\overline{N}$. For this suppose that $\overline{y} = y + mR \in l_{\overline{M}}r_{\overline{R}}(\overline{n})$ is arbitrary. Hence $(y+mR).r_{\overline{R}}(\overline{n}) = 0$. This implies that for any $r \in R$ we have $yr \in mR$ provided that $nr \in mR$. On the other hand the hypothesis gives an element $x \in M$ such that $xR = mR + nR$. So, there exist elements $r_1, r_2 \in R$ such that $m = xr_1$, $n = xr_2$. Hence, $nr_1 + mR = xr_2r_1 + mR = xr_1r_2 + mR = mr_2 + mR = mR$. Therefore, $nr_1 \in mR$ which yields $yr_1 \in mR$. So, there exists some $t \in R$ such that $yr_1 = mt$. We have $yr_1 = xr_1t = xtr_1$ hence, $(y - xt)r_1 = 0$. Therefore, $y = xt$ since $\{0\}$ is completely prime. Finally, an easy calculation shows that $\mathrm{Hom}_{\overline{R}}(\overline{N}, \overline{M})\overline{x} = \mathrm{Hom}_{\overline{R}}(\overline{N}, \overline{M})\overline{n}$. We conclude the result by Lemma 3.6. □

The next corollary generalizes the second part of Example 4 in [13].

Corollary 3.18. *Let M_R be a module over a commutative ring R such that $\{0\}$ is a completely prime submodule of M_R. Moreover, suppose that each finitely generated submodule of M is cyclic. Then, for any non-zero element m of M and any ideal I of R such that $I \subseteq Ann(M/mR)$, the quotient module M/mR is PQ-injective as an R/I-module.*

Proof. Set $N = M$ in Proposition 3.17. □

Now we give a generalization to Theorem 2.1 in [13].

Theorem 3.19. *Let M_R be a PQ-injective module with $S = \mathrm{End}(M_R)$. If M is completely faithful, then $Z(M_R) = Rad(_SM)$.*

Proof. First suppose that $m \in Rad(_SM)$ is an arbitrary element. Moreover, suppose that I is a right ideal of R such that $r_R(m) \cap I = 0$. Now let $b \in I$ be an arbitrary element. Then, $r_R(m) \cap bR = 0$. This implies that $Sm + l_M(b) = M$ since M is PQ-injective. Now in view of the fact that Sm is a superfluous submodule of M, we get $l_M(b) = M$ which yields $mb = 0$. Therefore, $b \in r_R(m) \cap bR = 0$. We conclude that $I = 0$, hence $m \in Z(M_R)$. Conversely, let $m \in Z(M_R)$ be arbitrary. The hypothesis that M is completely faithful gives an element, say n, such that $r_R(n) = 0$. Hence, $r_R(n - \alpha m) \cap r_R(\alpha m) = 0$ for all $\alpha \in S$. This implies that $r_R(n - \alpha.m) = 0$ since $\alpha m \in Z(M_R)$. We conclude that $S(n - \alpha.m) = M$. By the way of contradiction, suppose that m does not belong to $Rad(_SM)$, hence there exists a right maximal S-submodule N of M such that m does not belong to N. Thus, $Sm + N = M$. This gives $\beta \in S$ and $n_1 \in N$ such that $n = \beta m + n_1$. Consequently, $Sn_1 = S(n - \beta.m)$ which means $Sn_1 = M$, which is a contradiction. □

Remark 3.20. Let N be a right R-module and M be a submodule of N. Let $S = \mathrm{End}(M_R)$, $T = \mathrm{End}(N_R)$ and $I = I^M = \mathrm{Hom}_R(N, M)$. We define the maps $\varphi : I \to S$ and $\psi : I \to T$ as $\varphi(\alpha) = \alpha\mid_M$ and $\psi(\beta) = \beta$. Clearly, I is a right ideal of T and ψ is a ring homomorphism. Moreover, both M_R and N_R have structures as left I-modules.

Now we generalize Theorem 3.19 as follows. We note that $\mathrm{Hom}_R(N_R, M_R)m$ is an I-submodule of ${}_IM$.

Proposition 3.21. *Let M_R be a module over a ring R with $S = \mathrm{End}(M_R)$. Moreover, let N_R be a submodule of M_R such that M_R is a principally N-injective module. Let X be the set of all $m \in M$ such that $\mathrm{Hom}_R(N_R, M_R).m$ is a superfluous I-submodule of ${}_IM$, where $I = \mathrm{Hom}(N_R, M_R)$. Then $X \subseteq Z(M_R)$. The equality holds provided that M is completely faithful.*

Proof. Let $m \in X$ and I be a right ideal of R such that $r_R(m) \cap I = \{0\}$. Clearly, for any $b \in I$ we have $r_R(m) \cap bR = 0$. Now by Lemma 3.7, $\mathrm{Hom}_R(N, M).m + l_M(b) = M$. But $\mathrm{Hom}_R(N, M).m$ is a superfluous I-submodule of M, hence $l_M(b) = M$. Since $m \in M$, $mb = 0$. Hence, $b \in r_R(m) \cap bR = 0$ which implies $b = 0$. Consequently, $I = 0$. This means $m \in Z(M_R)$, hence $X \subseteq Z(M_R)$. Now suppose that M is completely faithful and $m \in Z(M_R)$. We prove that $m \in X$. The hypothesis that M is completely faithful, gives $x \in M$ such that $r_R(x) = 0$. Therefore, for all $\alpha : N \to M$ we have $r_R(x - \alpha m) \cap r_R(\alpha m) = 0$. But $\alpha m \in Z(M_R)$ which yields $r_R(x - \alpha m) = 0$. Hence, $l_M r_R(x - \alpha m) = M$. By Lemma 3.6 for all $\alpha : N \to M$ we have $\mathrm{Hom}_R(N, M)(x - \alpha m) = M$. By the way of contradiction, suppose that $m \notin X$. Hence, $\mathrm{Hom}_R(N, M)m$ is not a superfluous I-submodule of ${}_IM$. So, there exists a proper I-submodule M_1 of M such that $\mathrm{Hom}_R(N, M)m + M_1 = M$. We have $x \in M$, so there exist some $\beta : N \to M$ and $m_1 \in M_1$ such that $x = \beta m + m_1$. Therefore, $\mathrm{Hom}_R(N, M)m_1 = \mathrm{Hom}_R(N, M)(x - \beta m) = M$. Consequently, $\mathrm{Hom}_R(N, M).m_1 \subseteq I.M_1 \subseteq M_1$, so $M \leq M_1$, which is a contradiction. $\square$

At this point we turn our attention to [9]. The next result generalizes Theorem 1.3 in [9] from p-injective rings to the case of PQ-injective modules. We recall that a ring R is called von Neumann regular if every right principal ideal of R is a direct summand of R.

Theorem 3.22. *Let M_R be a right non-singular and PQ-injective module over a non-singular ring R. Assume that $l(I \cap J) = l(I) + l(J)$ for any non-zero right ideals I and J of R. Then, every cyclic S-submodule of ${}_SM$ is a direct summand of M.*

Proof. The hypothesis that R is non-singular implies that $r_R(m)$ is not an essential right ideal of R for any arbitrary $m \in M$. This means that there exists some non-zero right ideal L of R such that $r_R(m) \cap L = 0$. We can assume that L is the complement of $r_R(m)$, which implies that $r_R(m) \oplus L$ is an essential right ideal of R_R. On the other hand, if x is an element of $l_M(r_R(m) + L)$, then

$x(r_R(m)+L)=0$. But $r_R(m)+L$ is essential, hence $x\in Z_r(M)=0$. We have $l_M(r_R(m))\cap l_M(L)\subseteq l_M(r_R(m)+L)=0$. Therefore, $l_M(r_R(m))\cap l_M(L)=0$. In view of the hypothesis, we get $l_M(r_R(m))+l_M(L)=l_M(r_R(m)\cap L)=l_M(0)=M$. Consequently, $M=l_M(r_R(m))+l_M(L)=Sm\oplus l_M(L)$. We conclude the result. □

Corollary 3.23. *Let M_R be a PQ-injective module over a non-singular ring R such that $Z_r(M)$ is non-singular. Assume that $l(I\cap J)=l(I)+l(J)$ for all non-zero right ideals I and J of R. Then, every cyclic S-submodule of ${}_SM$ is a direct summand of ${}_SM$. In particular, ${}_SM$ is PQ-injective as S-module.*

Proof. We shall prove that M_R is a non-singular module. In view of [8, Proposition 3.29], we deduce that $M/Z_r(M)$ is a non-singular module. On the other hand, $Z_r(M)$ is a non-singular submodule of M, hence M_R is non-singular, see [8, Proposition 3.28]. Now our assertion is clear by Theorem 3.22. □

We recall that a ring R is called semiprime if it has no nonzero nilpotent right ideals. Following [9], a ring R is called ERT if every essential right ideal of R is two-sided. Clearly, a right duo ring is an ERT ring. Moreover, by the proof of Corollary 3.24, we observe that if R is semiprime and ERT, then R is non-singular. Hence, $Z_r(R)$ is non-singular. Indeed, Corollary 3.23 generalizes the following result.

Corollary 3.24. *Let R be a semiprime* ERT *right p-injective ring. Assume that $l(I\cap J)=l(I)+l(J)$ for any non-zero right ideals I and J of R. Then, R is a von Neumann regular.*

Proof. See [9, Corollary 1.4]. □

An R-module M_R is called an essentially multiplication module if for every essential submodule N of M_R there exists an ideal I of R such that $N=MI$. Clearly, every ring R is an essentially multiplication module. The next result generalizes Proposition 1.8 in [9].

Theorem 3.25. *Let M_R be a completely faithful PQ-injective module over a non-singular ring R such that $Z_r(M)$ is a non-singular submodule of M. Assume that $Soc(M_R)=0$ and M_R is essentially a multiplication module. Then, M is a sub direct product of simple R-modules.*

Proof. Let N be a maximal submodule of M_R. We prove that N is an essential submodule of M_R. By the way of contradiction, suppose that there exists a non-zero submodule L of M_R such that $N\cap L=0$. We can assume L to be the complement of N, hence $N\oplus L$ is an essential submodule of M_R. Since $L\neq 0$, $N\oplus L=M_R$. Moreover, we observe that L is a minimal submodule of M_R. In fact, if there exists a non-zero proper submodule K of L that is minimal, then $N\oplus K=M$ which means K is a complement of N in M_R. This implies that K is maximal among the submodules S such that $N\cap S=0$. We get $L\subseteq K$, a contradiction. Hence L is minimal, which yields L is simple. We conclude that $Soc(M_R) \neq 0$, a desired contradiction. Next, we prove that any essential submodule of M_R is fully invariant. Given an essential submodule N of M_R, the hypothesis that M_R is essentially a multiplication module implies that there exists an ideal I of R such that $N = MI$. Now for $\alpha \in \mathrm{End}_R(M)$ we have $\alpha(N) = \alpha(MI) = \alpha(M)I \subseteq MI = N$. We conclude that all maximal submodules of M_R are fully invariant. Therefore, by Theorem 3.19 we have $\mathrm{Rad}(M_R) = Z(M_R) = 0$, which means M_R is a sub direct product of simple R-modules. □

We recall that a right R-module M is a duo module if every submodule of M is fully invariant.

Proposition 3.26. *Let M_R be a PQ-injective module with $S = \mathrm{End}(M_R)$. Then the following are equivalent.*

(1) *If $r(m) \subseteq r(n)$ for some m and n in M, then $nR \subseteq mR$.*
(2) *M_R is a duo module.*

Proof. (1)⇒(2) Let N be a submodule of M_R and $\alpha : M_R \to M_R$. Given an element $n \in N$, we have $r(n) \subseteq r(\alpha(n))$. The hypothesis implies that $\alpha(n) \in nR \subseteq N$.

(2)⇒(1) The hypothesis that $r(m) \subseteq r(n)$, implies that the map $\alpha : mR \to nR$ defined as $\alpha(mr) = nr$ for all $r \in R$ is correctly defined. Therefore, there exists a homomorphism $\bar{\alpha} : M \to M$ such that $\bar{\alpha}i = j\alpha$, where $i : mR \to M$ and $j : nR \to M$ are the inclusion maps, respectively. Hence, $n = \alpha(m) = \bar{\alpha}(m)$. But mR is fully invariant, so $\bar{\alpha}(m) \in mR$ which yields $n \in mR$. □

Recall from [10] that an R-module M_R is called strongly duo if $Tr_M(N) = N$ for every submodule N of M_R.

Proposition 3.27. *Let M_R be a duo module with $S = \mathrm{End}(M_R)$. If K is a direct summand of M_R such that M is principally K-injective, then K is strongly duo.*

Proof. Let N be an arbitrary submodule of K. We prove that $Tr_K(N) \subseteq N$. Let $\varphi : N \to K$ be an arbitrary homomorphism. For all $n \in N$, we have the exact sequence $nR \xrightarrow{i} N \xrightarrow{\varphi} K \xrightarrow{j} M$, where i and j are the inclusion homomorphisms. Since M is principally K-injective, there exists a homomorphism $\bar{\varphi} : K \to M$ such that $\bar{\varphi}(n) = \varphi(n)$. Consequently, in view of the hypothesis that M is a duo module we have $\varphi(n) = \bar{\varphi}(n) \in nR \leq N$. We conclude the result. □

Corollary 3.28 ([10, Proposition 2.7])**.** *Every PQ-injective duo module is strongly duo.*

Proof. Set $K = M$ in Proposition 3.27. □

4. Commutative rings

In this section we show that to what extent the injectivity conditions influence the reject and trace submodules of modules over commutative rings. The first obvious result generalizes Example 1 in [13] from p-injective rings to the case of PQ-injective modules.

Proposition 4.1. *Any QF module over a commutative ring is PQ-injective.*

Proof. It is straightforward. □

A proof for the next result which is duo to A. M. Aghdam and the third author may be found in [1, Proposition 3.10]. It is given to the comparison with the following results.

Proposition 4.2. *Let M and N be modules over a commutative ring R such that M is N-injective. If $0 \to R \to N$ is exact, then $Tr_M(N) = M$.*

Proof. The exact sequence $0 \to R \xrightarrow{f} N$ implies the following exact sequence:

$$\mathrm{Hom}_R(N, M) \xrightarrow{f^*} \mathrm{Hom}_R(R, M) \to 0.$$

Let $m \in M$ be arbitrary. Then, there exists a homomorphism $\alpha \in \mathrm{Hom}_R(R, M)$ such that $\alpha(1) = m$. The fact that f^* is epic, implies that $f^*(\theta) = \alpha$ for some homomorphism $\theta \in \mathrm{Hom}_R(N, M)$. Therefore, we have $m = \alpha(1) = f^*(\theta(1)) = \theta(f(1))$. Consequently, $M \subseteq Tr_M(N)$. We conclude the result. □

Proposition 4.3. *Let M, N and E be modules over a commutative ring R such that E is injective. If $0 \to N \xrightarrow{g} M$ is exact and $M = Im(g) + Rej_M(E)$, then $Tr_E(M) = Tr_E(N)$.*

Proof. Let $y = \varphi(n)$ be an arbitrary generator of $Tr_E(N)$, where $\varphi : N \to E$ and $n \in N$. Since E is injective, there exists an $h : M \to E$ such that $hg = \varphi$. Therefore, $y = \varphi(n) = hg(n) \in Tr_E(M)$. Conversely, suppose that $y = \varphi(m)$ be an arbitrary generator of $Tr_E(M)$, where $\varphi : M \to E$ and $m \in M$. The hypothesis implies that $m = g(n)+z$ for some $n \in N$ and $z \in Rej_M(E)$. Hence, $y = \varphi(m) = \varphi(g(n)) + \varphi(z)$. But $\varphi(z) = 0$ since $z \in Rej_M(E)$. Therefore, $y = \varphi g(n) \in Tr_E(N)$. □

Proposition 4.4. *Let N and M be modules over a commutative ring R such that $Ann_R(M) = Ann_R(K)$ for all cyclic submodules K of N. If M is N-injective, then $Tr_M(N) = M$.*

Proof. Let m be an arbitrary element of M. Moreover, suppose that n is a non-zero element of N. We define $\alpha : Rn \to M$ as $\alpha(n) = m$. The hypothesis implies that $Ann(Rn) = Ann(M)$ which yields α is well-defined. But M is N-injective, hence there exists a homomorphism $\beta \in \mathrm{Hom}_R(N, M)$ such that $\beta(n) = \alpha(n) = m$. Consequently, $m = \beta(n) \in \mathrm{Im}(\beta) \subseteq Tr_M(N)$. □

Theorem 4.5 ([17, Excercise 3, p. 116]). *Let M, N and U be modules over a commutative ring R. Let $N \xrightarrow{g} M \to 0$ be an exact sequence of modules such that* $\ker(g) \subseteq Rej_N(U)$. *Then, $Rej_M(U) = gRej_N(U)$.*

Proof. Let $x \in Rej_N(U)$ be arbitrary. We prove that $g(x) \in Rej_M(U)$. For this, suppose that $\varphi : M \to U$ is an arbitrary homomorphism. We prove $\varphi g(x) = 0$. Since $\varphi g : N \to U$ and $x \in Rej_N(U)$, we are done. Conversely, suppose that $x \in Rej_M(U)$. We shall prove $x \in gRej_N(U)$. The map $N \xrightarrow{g} M$ is onto, hence there exists an element $y \in N$ such that $x = g(y)$. We prove that $y \in Rej_N(U)$. For this suppose that $\varphi : N \to U$ is arbitrary. Since $\ker(g) \subseteq Rej_N(U)$, we get $\ker(g) \subseteq Rej_N(U) \subseteq \ker(\varphi)$. Therefore, there exists a homomorphism $\overline{\varphi} : N/\ker(g) \to U$ such that $\overline{\varphi}(\overline{n}) = \varphi(n)$ for all $n \in N$. On the other hand, we have $M \cong N/\ker(g)$, hence we may consider $\varphi : M \to U$. Consequently, since $x \in Rej_M(U)$ we conclude the result. □

Corollary 4.6. *Let M and U be modules over a commutative ring R. Let $R \xrightarrow{g} M \to 0$ be an exact sequence of modules such that* $\ker(g) \subseteq Ann_R(U)$. *Then, $Rej(M, U) = g(Ann_R(U))$.*

Proof. Set $N = R$ in Theorem 4.5. □

Theorem 4.7. *Let M and N be modules over a commutative ring R and let $g : N \to M$ be a homomorphism such that $N \xrightarrow{g} M \to 0$ is exact. Let P be an N-projective module such that* $\ker(g) \cap Tr_N(P) = \{0\}$. *Then, $Rej_P(M) = Rej_P(N)$.*

Proof. Let $x \in Rej_P(N)$ be arbitrary. We prove that $x \in Rej_P(M)$. To do this, suppose that $\varphi : P \to M$. The hypothesis that P is N-projective, implies that there exists a homomorphism $\beta : P \to N$ such that $\varphi = g\beta$. On the other hand $x \in Rej_P(N)$, hence $\beta(x) = 0$. This implies that $\varphi(x) = 0$. Therefore, $x \in Rej_P(M)$. Conversely, suppose that $x \in Rej_P(M)$. We prove that $x \in Rej_P(N)$. To do this suppose that $\varphi : P \to N$ is arbitrary. Since $x \in Rej_P(M)$ and $P \xrightarrow{\varphi} N \xrightarrow{g} M \to 0$, we get $(g\varphi)(x) = 0$. This implies that $\varphi(x) \in \ker(g)$. On the other hand, we have $\varphi(x) \in Im(\varphi) \subseteq Tr_N(P)$. Hence, $\varphi(x) \in Tr_N(P) \cap \ker(g)$. Consequently, $\varphi(x) = 0$. We conclude the result. □

5. A correction

Puninskiĭ and Wisbauer [14] show that for any (non-Noetherian) left distributive or left duo ring there exists a 1-1 correspondence between indecomposable Σ-injective left R-modules and such completely prime ideals P of R, for which the left classical localization $R_{(P)}$ exists and is a left Noetherian ring. Moreover, they clarify the structure of arbitrary Σ-injective left modules over any left distributive or left duo ring. In particular, they completely describe Σ-injective left modules over a left uniserial ring. In §2 of [14], the authors have proved some general results which we state after this paragraph. According to the proofs given to the results, it is deduced that they are not true for general cases. However, these inaccuracies have not violated the final results of the paper in §3 and §4 since the special cases of the results of §2 have been used in the arguments. In this section, we state the accurate form of the results. The basic notions and definitions are found in [14].

Remark 5.1. Let M, N be R-modules and $S = \mathrm{End}_R M$.

(1) If M is N-injective or M is weakly N-injective and N is finitely generated, then every finitely generated S-submodule of $\mathrm{Hom}(N, M)$ lies in $\mathcal{A}(N, M)$.

(2) If M is semi-N-injective, then every cyclic S-submodule of $\mathrm{Hom}(N, M)$ lies in $\mathcal{A}(N, M)$.

Proof. See [14, Remark 2.2]. □

Corollary 5.2. *Let M be an R-module and $S = \mathrm{End}_R M$.*

(1) *If M is fp-injective, then every finitely generated S-submodule of M is equal to $\cap r_i^{\perp}$ for some $r_i \in R$.*

(2) *If M is p-injective, then every cyclic S-submodule of M is equal to $\cap r_i^{\perp}$ for some $r_i \in R$.*

(3) *If M is p-injective and $t \in R$, then S-submodule tM is equal to $\cap r_i^{\perp}$ for some $r_i \in R$.*

(4) *If R is left p-injective, then $({}^{\perp}r)^{\perp} = rR$ for every $r \in R$.*

Proof. See [14, Corollary 2.3]. □

Lemma 5.3. *Let M, N be R-modules and $S = \mathrm{End}_R M$.*

(1) *If M is semi-N-injective and ${}_R N$ is distributive (uniserial), then the right S-module $\mathrm{Hom}(N, M)_S$ is distributive (uniserial).*

(2) *If $\mathrm{Hom}(N, M)_S$ is distributive (uniserial) and ${}_R M$ cogenerates all factors of N at cyclic submodules, then N is distributive (uniserial) R-module.*

Proof. See [14, Lemma 2.7]. □

Corollary 5.4. *Let M be an R-module and $S = \mathrm{End}_R M$.*

(1) *If ${}_R M$ is p-injective and R is left distributive, then M_S is distributive. If in addition R or S is a local ring, then M_S is uniserial.*

(2) *If M_S is distributive and ${}_R M$ cogenerates all modules R/Rr, $r \in R$, then R is a left distributive ring. If in addition M_S is uniserial or one of the rings S, R is local, then R is left uniserial.*

Proof. See [14, Corollary 2.9]. □

Remark 5.5. The following corrections should be done in the above-mentioned results as follows.

(1) In Remark 5.1(1), there should be "M is M-injective or M is weakly M-injective and N is finitely generated."

(2) In Remark 5.1(2), instead of "M is semi-N-injective" should be "M is injective with respect to the diagrams" as in the proof of (2), and the same in Lemma 5.3(1).

These changes leads to some changes in other results as follows.

Remark 5.6. The following corrections should be done.

(1) In Corollary 5.2(1), replace "M is fp-injective" with "M is M-injective or M is weakly M-injective and finitely generated."
(2) In Remark 5.2(2), change to "M is injective with respect to cyclic submodules" and use the same wording in Corollary 5.4(1).

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Fatemeh Gholami
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697 Tehran, Iran
Email address: f.gholami@student.pnu.ac.ir

Zohreh Habibi
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697 Tehran, Iran
Email address: zhabibi@pnu.ac.ir

Alireza Najafizadeh
Department of Mathematics
Payame Noor University
P.O.Box 19395-3697 Tehran, Iran
Email address: najafizadeh@pnu.ac.ir