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CURVATURE ESTIMATES FOR A CLASS OF FULLY NONLINEAR ELLIPTIC EQUATIONS WITH GENERAL RIGHT HAND SIDES

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Abstract. In this paper, we establish the curvature estimates for a class of curvature equations with general right hand sides depending on the gradient. We show an existence result by using the continuity method based on a priori estimates. We also derive interior curvature bounds for solutions of a class of curvature equations subject to affine Dirichlet data.

1. Introduction

One of classic problems in differential geometry is to find a closed smooth hypersurface with prescribed curvature. For example, given a positive function ψ in $\mathbb{R}^{n+1}\backslash\{0\}$, one would like to find a star-shaped hypersurface $M \subseteq \mathbb{R}^{n+1}$ with respect to the origin such that its k-th Weingarten curvature is ψ. The problem is equivalent to solve the following equation

$$\sigma_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) = \psi(X), \ \forall X \in M, \tag{1.1}$$

where σ_k is the k-th elementary symmetric function and $(\kappa_1, \kappa_2, \ldots, \kappa_n)$ are the principal curvatures of M. Eq. (1.1) has been widely studied, seeing [1,3,4, 13,16,24,25,34] for related work. Another example is the Weingarten curvature equation in general form

$$\sigma_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) = \psi(X, \nu(X)), \ \forall X \in M, \tag{1.2}$$

where $\nu(X)$ is the outer normal vector field along the hypersurface M. Eq. (1.2) is associated with many important geometry problems, such as Minkowski problem and the problem of prescribing curvature measures in convex geometry.

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The problem of C^2 estimates for admissible solutions of Eq. (1.2) is still open. One can consult [2–4, 14, 15, 17–19, 21–23, 28–30, 32] for more work.

The following Eq. (1.3) is similar to Eq. (1.2), which has also been widely discussed. Let H be the mean curvature of M, and define the $(0,2)$-tensor η on M by

$$\eta = Hg_{ij} - h_{ij},$$

where g_{ij} and h_{ij} are the first and second fundamental forms of M, respectively. Denote the eigenvalues of $g^{-1}\eta$ by $\lambda(\eta) = (H-\kappa_1, H-\kappa_2, \ldots, H-\kappa_n)$. Substituting $(\kappa_1, \kappa_2, \ldots, \kappa_n)$ by $\lambda(\eta)$ in Eq. (1.2) gives

$$\sigma_k(\lambda(\eta))(X) = \psi(X, \nu(X)),\ \forall X \in M. \tag{1.3}$$

In the complex setting, when $k = n$, Eq. (1.3) is called Monge-Ampère equation for $(n-1)$-plurisubharmonic functions, which is related to the Gauduchon conjecture [12, 33] in complex geometry. Moreover, this type of Eq. (1.3) arises from conformal geometry. The $(0,2)$-tensor η is similar to Schouten tensor A_g, where

$$A_g = \frac{1}{n-2}\big(Ric_g - \frac{R_g}{2(n-1)}g\big),$$

and Ric_g is Ricci tensor, R_g is scalar curvature. Let $[g_0]$ denote the conformal class of g_0 on a smooth closed Riemannian manifold of dimension $n \geq 3$. An interesting problem is to find a metric $g \in [g_0]$ such that

$$\sigma_k(A_g) = \psi(x),$$

that is the well-known σ_k-Yamabe problem. In [9], Chu-Jiao established the curvature estimates and obtained an existence result on the closed star-shaped (η, k)-convex hypersurface satisfying Eq. (1.3). The author in [37] generalized Chu-Jiao's result in Euclidean space to space form. Inspired by Chu-Jiao's result, Chen-Tu-Xiang [6, 7] considered the corresponding quotient Hessian equations

$$\frac{\sigma_k}{\sigma_l}(\lambda(\eta))(X) = \psi(X, \nu(X)),\ \forall X \in M, \tag{1.4}$$

in Euclidean space and the warped product manifold and obtained the similar results. For other relevant results, refer to [8, 10].

Inspired by the above works, we consider a more wider class of prescribed Weingarten curvature equations as below

$$f(\lambda(\eta)) = \psi\big(X, \nu(X)\big),\ \forall X \in M, \tag{1.5}$$

where $0 < \psi \in C^2\big(\mathbb{R}^{n+1} \times \mathbb{S}^n\big)$, f is a symmetric function of $\lambda(\eta)$ and satisfies the following conditions (1.6)-(1.14). Note that except (1.14), all conditions are proposed by Caffarelli-Nireberg-Spruck [3, 4]. The function f is assumed to be defined in a symmetric open and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin,

$$\Gamma_n = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0,\ 1 \leq i \leq n\} \subset \Gamma \subset \Gamma_1 = \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0\}, \tag{1.6}$$

and satisfies the conditions

$$f_i = \frac{\partial f}{\partial \lambda_i} > 0 \ \text{ in } \Gamma, \ 1 \le i \le n, \tag{1.7}$$

$$f \ \text{ is a concave positive function in } \Gamma. \tag{1.8}$$

To get C^0 and C^1 estimate of Eq. (1.5), we assume that there exist two positive constants $r_1 < 1 < r_2$ such that

$$\psi(X, \frac{X}{|X|}) \ge f\big(\frac{n-1}{r_1}\mathbf{1}\big) \ \text{ for } |X| = r_1, \tag{1.9}$$

$$\psi(X, \frac{X}{|X|}) \le f\big(\frac{n-1}{r_2}\mathbf{1}\big) \ \text{ for } |X| = r_2, \tag{1.10}$$

$$\frac{\partial}{\partial \rho}\big(\rho\psi(X, \nu)\big) \le 0 \ \text{ for } r_1 \le \rho \le r_2, \tag{1.11}$$

where $\mathbf{1} = (1, 1, \dots, 1)$ and $\rho = |X|$. Let $\psi_0 = \inf_{\mathbb{R}^{n+1} \times \mathbb{S}^n} \psi$. We assume that

$$\overline{\lim_{\lambda \to \partial\Gamma}} f(\lambda) \le \overline{\psi}_0 \tag{1.12}$$

for some constant $\overline{\psi}_0 < \psi_0$ and

$$\overline{\psi}_0 < \frac{1}{r_2} f\big((n-1)\mathbf{1}\big).$$

In addition we assume that for every $C > 0$ and every compact set K in Γ there is a number $R = R(C, K)$ such that

$$f(R\lambda) \ge C \text{ for all } \lambda \in \Gamma. \tag{1.13}$$

If $\lambda(\eta) \in \Gamma$ for all $X \in M$, hypersurface $M \subset \mathbb{R}^{n+1}$ is called Γ-convex. Let the principal curvature of M satisfy $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_n$. Finally we assume that for some sufficiently small constant $\theta > 0$, if $|\kappa_i| \le \theta\kappa_1$ for all $i \ge 2$, then there exits a constant $c_0 > 0$ such that

$$\sum_{i=1}^n f_i\big(\lambda(\eta)\big) \ge c_0\kappa_1. \tag{1.14}$$

Remark 1.1. Specially, assumption (1.14) holds for Eqs. (1.3) and (1.4). In addition,

$$\sum_{i=1}^n \frac{\partial \sigma_2(\kappa)}{\partial \kappa_i} = (n-1)\sigma_1(\kappa) \ge (n-1)\kappa_1,$$

assumption (1.14) holds for Eq. (1.2) for $k = 2$. It is worth emphasizing that assumption (1.14) is critical for curvature estimates.

Theorem 1.2. *Let M be a closed star-shaped Γ-convex hypersurface satisfying the curvature Eq.* (1.5)*. Suppose that f and ψ satisfy* (1.6)-(1.8)*,* (1.12)-(1.14)*. Then we have*

$$\max_{X\in M}\left|\kappa_i(X)\right| \le C, \quad 1\le i\le n, \tag{1.15}$$

where C is a constant depending on n, $\left|X\right|_{C^1}$ and $\left|\psi\right|_{C^2}$.

From Theorem 1.2, we obtain the following result by applying the continuity method.

Theorem 1.3. *Suppose f and ψ satisfy* (1.6)-(1.14)*. Then there exists a unique $C^{3,\delta}$ closed star-shaped Γ-convex hypersurface M satisfying Eq.* (1.5) *for any $\delta\in(0,1)$.*

Another interesting question in geometric analysis is whether interior curvature estimates hold. There are no interior curvature bounds for graphs of prescribed k-th ($k\ge 3$) mean curvature unless we make some additional assumptions (see [35, 36]). Purely interior curvature bounds had been obtained under a weakened condition in [20] for prescribed scalar curvature equations with general right hand sides. Purely interior curvature estimates for 3d scalar curvature equations were proved completely by Qiu [27]. Sheng, Urbas and Wang [31] proved interior curvature estimates for a class of fully nonlinear elliptic equations subject to affine Dirichlet data, generalizing the well-known Pogorelov estimates [26]. Recently, Chen, Dong and Han [5] proved Pogorelov type estimates for the equation

$$\frac{\sigma_k(\eta)}{\sigma_l(\eta)} = \psi(x),$$

which imply Liouville theorem for such equation. We also consider Pogorelov type curvature estimation in this paper.

Let Ω be a bounded domain in $\mathbb{R}^n$, $u\in C^2(\Omega)$. $\kappa=(\kappa_1,\dots,\kappa_n)$ and $\nu(x)$ denote the principle curvatures and the upward unit normal vector field of the graph $X=(x,u(x))$ at x in Ω,

$$\nu(x) = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}.$$

Setting $\lambda(\eta)=(H-\kappa_1,\dots,H-\kappa_n)$, we consider the following Dirichlet problem

$$\begin{cases} f\big(\lambda(\eta)\big) = \psi\big(X,\nu(x)\big), & x\in\Omega, \\ u(x)=\phi(x), & x\in\partial\Omega, \end{cases} \tag{1.16}$$

where $0<\psi\in C^2(\overline{\Omega}\times\mathbb{R}\times\mathbb{S}^n)$, ϕ is affine and $\lambda(\eta)\in\Gamma$. Using the method of proof of Theorem 1.2, we establish the following Pogorelov type curvature estimates.

Theorem 1.4. *Suppose that f satisfy* (1.6)-(1.8)*,* (1.12)-(1.14) *and let $u\in C^4(\Omega)\cap C^{0,1}(\overline{\Omega})$ be a solution of the Dirichlet problem* (1.16)*. Then there* *exists a constant $\beta > 0$ such that the second fundamental form A of graph $\{(x, u(x))\}$ satisfies*

$$|A(x)| \le \frac{C}{(\phi - u)^{\beta}}, \quad \forall x \in \Omega, \tag{1.17}$$

where C is a constant depending on c_0, n, $|u|_{C^1(\overline{\Omega})}$, $|\phi|_{C^1(\overline{\Omega})}$ and $|\psi|_{C^2(\overline{\Omega}\times\mathbb{R}\times\mathbb{S}^n)}$.

The rest of this paper is organized as follows. In Section 2, we give some definitions and important formulas. In Section 3, we give the curvature estimates, that is Theorem 1.2. In Section 4, we give the proof for the existence, that is Theorem 1.3. In Section 5, we derive the interior curvature bounds for solutions of Eq. (1.16), that is Theorem 1.4.

2. Preliminaries

In this section, we recall some geometric quantities and related formulas on hypersurfaces in $\mathbb{R}^{n+1}$. We choose an orthonormal frame in $\mathbb{R}^{n+1}$ such that $\{e_1, \ldots, e_n\}$ are tangent to M and e_{n+1} is the unit outer normal of M. Let $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$ be the corresponding coframe. On M,

$$\omega_{n+1} = 0.$$

The second fundamental form of M can be written as

$$\omega_{i,n+1} = h_{ij}\omega_j.$$

The following formulas are well known for hypersurfaces in $\mathbb{R}^{n+1}$.

$$X_{ij} = -h_{ij}e_{n+1} \qquad \text{Gauss formula,} \tag{2.1}$$

$$\big(e_{n+1}\big)_i = h_{ij}e_j \qquad \text{Weingarten equation,} \tag{2.2}$$

$$h_{ijk} = h_{ikj} \qquad \text{Codazzi formula,} \tag{2.3}$$

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \qquad \text{Gauss equation,} \tag{2.4}$$

where R_{ijkl} is the Riemannian curvature tensor. From (2.1)-(2.4), we have

$$h_{ijkl} = h_{klij} + h_{mk}(h_{mj}h_{il} - h_{ml}h_{ij}) + h_{mi}(h_{mj}h_{kl} - h_{ml}h_{kj}). \tag{2.5}$$

We assume the origin is inside the body enclosed by M. Since M is a star-shaped hypersurface in $\mathbb{R}^{n+1}$, the position vector X of M is expressed as

$$X(x) = \rho(x)x.$$

Following the notations in [19], let ∇ be the gradient on $\mathbb{S}^n$. Then the induced metric, unit normal vector and second fundamental form on M are given, respectively, by

$$g_{ij} = \rho^2\delta_{ij} + \nabla_i\rho\nabla_j\rho, \quad g^{ij} = \frac{1}{\rho^2}\big(\delta_{ij} - \frac{\nabla_i\rho\nabla_j\rho}{\rho^2 + |\nabla\rho|^2}\big),$$

$$\nu = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}, \tag{2.6}$$

$$h_{ij} = \frac{1}{\sqrt{\rho^2 + |\nabla\rho|^2}}\big(\rho^2\delta_{ij} + 2\nabla_i\rho\nabla_j\rho - \rho\nabla_{ij}\rho\big). \tag{2.7}$$

The support function of M can be expressed as $u = \langle X, \nu\rangle$, so we have

$$u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{2.8}$$

For simplicity, we introduce the following notations:

$$G(h_{ij}) = f\big(\lambda(\eta)\big), \quad G^{ij} = \frac{\partial G}{\partial h_{ij}}, \quad G^{ij,rs} = \frac{\partial^2 G}{\partial h_{ij} h_{rs}},$$

$$F\big(\eta_{ij}\big) = f\big(\lambda(\eta)\big), \quad F^{ij} = \frac{\partial F}{\partial \eta_{ij}}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial \eta_{ij}\eta_{rs}}.$$

Equation (1.5) can be written as

$$G(h_{ij}) = F\big(\eta_{ij}\big) = \psi\big(X, \nu(X)\big). \tag{2.9}$$

If (h_{ij}) is diagonal and $h_{11} \geq h_{22} \geq \cdots \geq h_{nn}$, then

$$G^{11} \leq G^{22} \leq \cdots \leq G^{nn}.$$

Since $\eta_{ii} = \sum_{k\neq i} h_{kk}$, $\eta_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn}$. Then

$$F^{11} \geq F^{22} \geq \cdots \geq F^{nn}.$$

Lemma 2.1. *Let M be a closed star-shaped Γ-convex hypersurface satisfying Eq.* (1.5)*. Then*

(1) $G(h_{ij}) = f\big(\lambda(\eta)\big)$ *is elliptic on M,*
(2) $G(h_{ij})$ *is concave with respect to (h_{ij}),*
(3) $\sum_{i=1}^{n} G^{ii} = (n-1)\sum_{i=1}^{n} F^{ii}$,
(4) $G^{ij}h_{ij} = F^{ij}\eta_{ij}$,
(5) $G^{ij}h_{ijk} = F^{ij}\eta_{ijk}$,
(6) $G^{ij,kl}h_{ijr}h_{kls} = F^{ij,kl}\eta_{ijr}\eta_{kls}$.

Proof. (1) By the chain rule, we have

$$\big(G^{ij}\big) = \begin{pmatrix} \sum\limits_{i\neq 1} F^{ii} & -F^{12} & \cdots & -F^{1n} \\ -F^{21} & \sum\limits_{i\neq 2} F^{ii} & \cdots & -F^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -F^{n1} & -F^{n2} & \cdots & \sum\limits_{i\neq n} F^{ii} \end{pmatrix}.$$

Obviously, $(f_1, f_2, \ldots, f_n)$ are the eigenvalues of $\big(F^{ij}\big)$. By an orthogonal transformation, we know the eigenvalues of $\big(G^{ij}\big)$ are $\big(\sum_{i\neq 1} f_i, \sum_{i\neq 2} f_i, \ldots, \sum_{i\neq n} f_i\big)$. From (1.7), we have $\big(G^{ij}\big) > 0$, which means that $\big(G^{ij}\big)$ is elliptic.

(2) By (1.8), we know $F(\eta_{ij})$ is concave with respect to (η_{ij}). Since η_{ij} is linear dependent on h_{ij}, $G(h_{ij})$ is also concave respect to $\big(h_{ij}\big)$.

(3) From $G^{ii} = \frac{\partial F}{\partial \eta_{kl}} \frac{\partial \eta_{kl}}{\partial h_{ii}} = \sum_{j \neq i} F^{jj}$, we have

$$\sum_{i=1}^{n} G^{ii} = (n-1) \sum_{i=1}^{n} F^{ii}.$$

(4) A straightforward calculation yields

$$G^{ij} h_{ij} = \frac{\partial F}{\partial \eta_{kl}} \frac{\partial \eta_{kl}}{\partial h_{ij}} h_{ij} = F^{ij} \eta_{ij}.$$

(5) Similar to (4), we have

$$G^{ij} h_{ijk} = \frac{\partial F}{\partial \eta_{kl}} \frac{\partial \eta_{kl}}{\partial h_{ij}} h_{ijk} = F^{ij} \eta_{ijk}.$$

(6) Applying the chain rule, we have

$$\begin{aligned} G^{ij,kl} h_{ijr} h_{kls} &= \frac{\partial^2 F}{\partial \eta_{pq} \partial \eta_{mn}} \frac{\partial \eta_{pq}}{\partial h_{ij}} \frac{\partial \eta_{mn}}{\partial h_{kl}} h_{ijr} h_{kls} + \frac{\partial F}{\partial \eta_{pq}} \frac{\partial^2 \eta_{pq}}{\partial h_{ij} \partial h_{kl}} h_{ijr} h_{kls} \\ &= \frac{\partial^2 F}{\partial \eta_{pq} \partial \eta_{mn}} \frac{\partial \eta_{pq}}{\partial h_{ij}} \frac{\partial \eta_{mn}}{\partial h_{kl}} h_{ijr} h_{kls} \\ &= F^{ij,kl} \eta_{ijr} \eta_{kls}. \end{aligned}$$

□

The following lemma can be found in [13].

Lemma 2.2. *For any symmetric matrix (η_{ij}), we have*

$$G^{ij,kl} \eta_{ij} \eta_{kl} = G^{ii,jj} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{G^{ii} - G^{jj}}{h_{ii} - h_{jj}} \eta_{ij}^2.$$

The second term on the right-hand is nonpositive if G is concave, and it is interpreted as a limit if $h_{ii} = h_{jj}$.

3. Curvature estimates

In this section, we give the proof of Theorem 1.2, that is, we prove that the principal curvatures have uniform bounds.

From (2.8), we see that there exists a positive constant C depending on $\inf_M \rho$ and $\|\rho\|_{C^1}$ such that

$$\frac{1}{C} \leq \inf_M u \leq u \leq \sup_M u \leq C.$$

Let $\kappa_{\max}$ be the largest principal curvature. Since $\lambda(\eta) \in \Gamma \subset \Gamma_1$, we have

$$\sum_{i=1}^{N} \eta_{ii} = (n-1)H > 0.$$

It suffices to prove $\kappa_{\max}$ is uniformly bounded from above. We consider the following auxiliary function

$$Q = \log \kappa_{\max} - \log(u - a) + \frac{A}{2} |X|^2,$$

where $a = \frac{1}{2}\inf_M u$. Suppose that Q achieves its maximum value at a point X_0. We choose a local orthonormal frame $\{e_1, \dots, e_n\}$ near X_0 such that

$$h_{ij} = h_{ii}\delta_{ij}, \quad h_{11} \geq \cdots \geq h_{nn}, \quad \text{at } X_0.$$

Recalling $\eta_{ij} = H\delta_{ij} - h_{ij}$, we have $\eta_{ij} = \big(\sum_{k\neq i} h_{kk}\big)\delta_{ij}$. Define a new function $\widetilde{Q}$ near X_0 by

$$\widetilde{Q} = \log h_{11} - \log(u-a) + \frac{A}{2}|X|^2.$$

Obviously, $\widetilde{Q}$ achieves its maximum value at X_0. From now on, all calculations will be done at X_0. Hence

$$\widetilde{Q}_i = \frac{h_{11i}}{h_{11}} - \frac{u_i}{u-a} + A\langle X, e_i\rangle = 0, \tag{3.1}$$

and

$$\begin{aligned} 0 \geq G^{ij}\widetilde{Q}_{ij} = {} & G^{ij}\left(\frac{h_{11ij}}{h_{11}} - \frac{h_{11i}h_{11j}}{h_{11}^2}\right) - G^{ij}\left(\frac{u_{ij}}{u-a} - \frac{u_iu_j}{(u-a)^2}\right) \\ & + A\sum_{i=1}^n G^{ii} - AuG^{ij}h_{ij}. \end{aligned} \tag{3.2}$$

Using (2.5) gives that

$$h_{11ij} = h_{ij11} + h_{im}\big(h_{j1}h_{1m} - h_{jm}h_{11}\big) + h_{1m}\big(h_{ij}h_{1m} - h_{jm}h_{1i}\big),$$

which means that

$$G^{ij}h_{11ij} = G^{ij}h_{ij11} + G^{ij}h_{ij}h_{11}^2 - G^{ii}h_{ii}^2h_{11}. \tag{3.3}$$

Differentiating Eq. (2.9) with respect to e_1 twice, we obtain

$$G^{ij}h_{ij1} = \psi_1, \tag{3.4}$$

$$G^{ij}h_{ij11} = -G^{ij,ml}h_{ij1}h_{ml1} + \psi_{11}. \tag{3.5}$$

Direct calculation yields that

$$\psi_1 = \big(d_X\psi\big)(e_1) + h_{11}\big(d_\nu\psi\big)(e_1) \geq -C - Ch_{11}, \tag{3.6}$$

$$\begin{aligned} \psi_{11} = {} & \big(d_{XX}\psi\big)(e_1,e_1) + 2h_{11}\big(d_{X\nu}\psi\big)(e_1,e_1) \\ & - h_{11}\big(d_X\psi\big)(\nu) + h_{11}^2\big(d_{\nu\nu}\psi\big)(e_1,e_1) \\ & + \sum_{i=1}^n h_{11i}\big(d_\nu\psi\big)(e_i) + h_{11}^2\big(d_\nu\psi\big)(\nu) \\ \geq {} & \sum_{i=1}^n h_{11i}\big(d_\nu\psi\big)(e_i) - h_{11}^2C. \end{aligned} \tag{3.7}$$

Combined (3.3), (3.5) and (3.7), we have

$$\frac{G^{ij}h_{11ij}}{h_{11}}$$

$$= -G^{ij,ml}\frac{h_{ij1}h_{ml1}}{h_{11}}+\frac{\psi_{11}}{h_{11}}+G^{ij}h_{ij}h_{11}-G^{ii}h_{ii}^2$$

$$\geq -G^{ij,ml}\frac{h_{ij1}h_{ml1}}{h_{11}}+G^{ij}h_{ij}h_{11}-G^{ii}h_{ii}^2+\sum_{i=1}^{n}\frac{h_{11i}}{h_{11}}\big(d_\nu\psi\big)(e_i)-h_{11}C. \tag{3.8}$$

Thus (2.1), (2.2) and Codazzi formula gives that

$$-G^{ij}\frac{u_{ij}}{u-a}=-\frac{1}{u-a}\sum_{k=1}^{n}G^{ij}h_{ijk}\langle X,e_k\rangle-\frac{1}{u-a}G^{ij}h_{ij}+\frac{u}{u-a}G^{ii}h_{ii}^2. \tag{3.9}$$

Taking (3.4) and (3.6) into (3.9), we have

$$\begin{aligned}-G^{ij}\frac{u_{ij}}{u-a}\geq\ &-\frac{1}{u-a}\sum_{i=1}^{n}h_{ii}\big(d_\nu\psi\big)(e_i)\langle X,e_i\rangle+\frac{u}{u-a}G^{ii}h_{ii}^2\\ &-\frac{1}{u-a}G^{ij}h_{ij}-C.\end{aligned} \tag{3.10}$$

From (3.1) it follows that

$$\frac{h_{11i}}{h_{11}}-\frac{h_{ii}\langle X,e_i\rangle}{u-a}=-A\langle X,e_i\rangle,$$

which implies that

$$\frac{h_{11i}}{h_{11}}\big(d_\nu\psi\big)(e_i)-\frac{h_{ii}\langle X,e_i\rangle\big(d_\nu\psi\big)(e_i)}{u-a}\geq -AC. \tag{3.11}$$

Substituting (3.8), (3.10) and (3.11) into (3.2), we find

$$\begin{aligned}0\geq\ &\frac{a}{u-a}G^{ii}h_{ii}^2+G^{ij}\frac{u_iu_j}{(u-a)^2}+A\sum_{i=1}^{n}G^{ii}+\big(h_{11}-\frac{1}{u-a}-Au\big)G^{ij}h_{ij}\\ &-h_{11}C-CA-\frac{1}{h_{11}}G^{ij,kl}h_{ij1}h_{kl1}-G^{ij}\frac{h_{11i}h_{11j}}{h_{11}^2}.\end{aligned} \tag{3.12}$$

We now consider two cases.

Case 1. There is a positive constant $\theta>0$ to be chosen later such that $h_{22}>\theta h_{11}$ or $h_{nn}<-\theta h_{11}$. From (3.1) it follows that

$$\begin{aligned}G^{ij}\frac{h_{11i}h_{11j}}{h_{11}^2}&=G^{ij}\big(\frac{u_i}{u-a}-A\langle X,e_i\rangle\big)\big(\frac{u_j}{u-a}-A\langle X,e_j\rangle\big)\\ &\leq(1+\epsilon)G^{ij}\frac{u_iu_j}{(u-a)^2}+(1+\frac{1}{\epsilon})A^2G^{ij}\langle X,e_i\rangle\langle X,e_j\rangle\\ &\leq(1+\epsilon)G^{ij}\frac{u_iu_j}{(u-a)^2}+\frac{CA^2}{\epsilon}G^{ij}\langle X,e_i\rangle\langle X,e_j\rangle.\end{aligned} \tag{3.13}$$

Choosing $\epsilon=\frac{a^2}{4\rho_{\max}^2}$, we see that

$$\epsilon G^{ij}\frac{u_iu_j}{(u-a)^2}=\varepsilon G^{ij}\frac{h_{ii}h_{jj}\langle X,e_i\rangle\langle X,e_j\rangle}{(u-a)^2}$$

$$\leq \frac{a}{4(u-a)} G^{ii} h_{ii}^2. \tag{3.14}$$

The concavity of G tells us

$$-\frac{1}{h_{11}} G^{ij,kl} h_{ij1} h_{kl1} \geq 0. \tag{3.15}$$

Putting (3.13), (3.14) and (3.15) into (3.12), we obtain

$$\begin{aligned} 0 \geq\ & \frac{a}{2(u-a)} G^{ii} h_{ii}^2 + \big(h_{11} - \frac{1}{u-a} - Au\big) G^{ij} h_{ij} \\ & - h_{11} C - CA + (A - CA^2) \sum_{i=1}^n G^{ii} \\ \geq\ & \frac{a}{2(u-a)} \big(G^{22} h_{22}^2 + G^{nn} h_{nn}^2\big) + \big(h_{11} - \frac{1}{u-a} - Au\big) G^{ij} h_{ij} \\ & - h_{11} C - CA + (A - CA^2) \sum_{i=1}^n G^{ii}. \end{aligned} \tag{3.16}$$

From (1.8) and (1.14) it follows that

$$\sum_{i=1}^n f_i \lambda_i \geq 0.$$

Combining this with (4) in Lemma 2.1, we have

$$G^{ij} h_{ij} = F^{ij} \eta_{ij} = \sum_{i=1}^n f_i \eta_{ii} \geq 0. \tag{3.17}$$

From (3) in Lemma 2.1, we get

$$\begin{aligned} G^{22} &= F^{11} + F^{33} + \cdots + F^{nn} \\ &\geq \frac{1}{2} F^{11} + \frac{1}{2} F^{22} + F^{33} + \cdots + F^{nn} \\ &\geq \frac{1}{2} \sum_{i=1}^n F^{ii} \geq \frac{1}{2(n-1)} \sum_{i=1}^n G^{ii}, \end{aligned} \tag{3.18}$$

$$G^{nn} \geq \frac{1}{n} \sum_{i=1}^n G^{ii}. \tag{3.19}$$

From this we see that in **Case 1**

$$\frac{a}{2(u-a)} \big(G^{22} h_{22}^2 + G^{nn} h_{nn}^2\big) \geq \frac{\theta^2 a}{4n(u-a)} h_{11}^2 \sum_{i=1}^n G^{ii}. \tag{3.20}$$

Without loss of generality, we have

$$h_{11} \geq \max\left\{ \frac{1}{u-a} + Au, \sqrt{\frac{8n(u-a)}{a} C \frac{A}{\theta}} \right\},$$

then

$$\big(h_{11} - \frac{1}{u-a} - Au\big)G^{ij}h_{ij} \geq 0, \tag{3.21}$$

$$\frac{\theta^2 a}{8n(u-a)} h_{11}^2 \sum_{i=1}^{n} G^{ii} + (A - CA^2) \sum_{i=1}^{n} G^{ii} \geq 0. \tag{3.22}$$

Using (3.16), (3.21) and (3.22), we obtain

$$0 \geq \frac{\theta^2 a}{8n(u-a)} h_{11}^2 \sum_{i=1}^{n} G^{ii} - Ch_{11} - CA. \tag{3.23}$$

Similar to the conditions (11) and (12) in [4], from (1.13) it follows that

$$\sum_{i=1}^{n} G^{ii} = (n-1) \sum_{i=1}^{n} F^{ii} = (n-1) \sum_{i=1}^{n} f_i \geq C_0 \tag{3.24}$$

for some constant $C_0 > 0$. From (3.23) and (3.24), we find

$$0 \geq \frac{\theta^2 a C_0}{8n(u-a)} h_{11}^2 - Ch_{11} - CA.$$

Therefore we have

$$\kappa_{\max} \leq C.$$

Case 2. We now assume that $|h_{ii}| \leq \theta h_{11}$ for all $i = 2, \ldots, n$. Divide the indices $\{1, 2, \ldots, n\}$ into two cases

$$I = \{i \,|\, G^{ii} \leq 4G^{11}\}, \quad J = \{i \,|\, G^{ii} > 4G^{11}\},$$

where G^{ii} is evaluated at X_0. Similar to (3.13), we have

$$\begin{aligned} \sum_{i\in I} G^{ii} \frac{|h_{11i}|^2}{h_{11}^2} &= \sum_{i\in I} G^{ii} \big(\frac{u_i}{u-a} - A\langle X, e_i\rangle\big)^2 \\ &\leq (1+\epsilon) \sum_{i\in I} G^{ii} \frac{|u_i|^2}{(u-a)^2} + \frac{CA^2}{\epsilon} \sum_{i\in I} G^{ii} \langle X, e_i\rangle^2 \\ &\leq (1+\epsilon) \sum_{i\in I} G^{ii} \frac{|u_i|^2}{(u-a)^2} + \frac{CA^2}{\epsilon} G^{11}. \end{aligned} \tag{3.25}$$

From $\epsilon = \frac{a^2}{4\rho_{\max}^2}$, we get

$$\epsilon \sum_{i\in I} G^{ii} \frac{|u_i|^2}{(u-a)^2} = \epsilon \sum_{i\in I} G^{ii} \frac{h_{ii}^2 |\langle X, e_i\rangle|^2}{(u-a)^2} \leq \frac{a}{4(u-a)} G^{ii} h_{ii}^2. \tag{3.26}$$

Without loss of generality, we assume that

$$h_{11}^2 \geq \frac{4(u-a)}{a} CA^2,$$

then

$$CA^2G^{11} \le \frac{a}{4(u-a)}G^{ii}h_{ii}^2. \tag{3.27}$$

Substitute (3.26) and (3.27) into (3.25)

$$\sum_{i\in I} G^{ii}\frac{|h_{11i}|^2}{h_{11}^2} \le G^{ii}\frac{|u_i|^2}{(u-a)^2} + \frac{a}{2(u-a)}G^{ii}h_{ii}^2. \tag{3.28}$$

It follows from the concave of G and Lemma 2.2 that

$$-\frac{1}{h_{11}}G^{ij,kl}h_{ij1}h_{kl1} \ge \frac{2}{h_{11}}\frac{G^{ii}-G^{11}}{h_{11}-h_{ii}}h_{11i}^2. \tag{3.29}$$

We need to show

$$\frac{2}{h_{11}}\frac{G^{ii}-G^{11}}{h_{11}-h_{ii}} \ge \frac{G^{ii}}{h_{11}^2}, \quad i \in J,$$

which is equivalent to

$$G^{ii}h_{11} + G^{ii}h_{ii} \ge 2G^{11}h_{11}, \quad i \in J. \tag{3.30}$$

For $i \in J$, $G^{ii} > 4G^{11}$. (3.30) can be obtained if we can show

$$4G^{11}h_{11} + 4G^{11}h_{ii} \ge 2G^{11}h_{11}, \quad i \in J. \tag{3.31}$$

If $h_{ii} \ge 0$, (3.31) holds obviously. If $h_{ii} < 0$, then $h_{ii} \ge -\theta h_{11}$ by our assumption of **Case 2**. We find that (3.31) also holds if $\theta \le \frac{1}{2}$. Therefore

$$-\frac{1}{h_{11}}G^{ij,kl}h_{ij1}h_{kl1} \ge \sum_{i\in J} G^{ii}\frac{h_{11i}^2}{h_{11}^2}. \tag{3.32}$$

Inserting (3.28) and (3.32) into (3.12), we have

$$0 \ge \frac{a}{2(u-a)}G^{ii}h_{ii}^2 + A\sum_{i=1}^n G^{ii} - h_{11}C - CA. \tag{3.33}$$

Using (1.14) yields that

$$\sum_{i=1}^n G^{ii} = (n-1)\sum_{i=1}^n F^{ii} \ge (n-1)c_0h_{11}.$$

Choosing A sufficiently large, we obtain

$$\frac{A}{2}\sum_{i=1}^n G^{ii} - h_{11}C \ge 0.$$

Combining this and (3.33), we have

$$\kappa_{\max} \le C.$$

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. To obtain a solution to Eq. (1.5) by the continuity method, we need to derive C^0 and C^1 estimates.

Let us consider a family of functions for $t \in [0,1]$

$$\psi^t = t\psi(X,\nu)+(1-t)\left(\frac{f\big((n-1)\mathbf{1}\big)}{|X|}+\varepsilon\left(\frac{f\big((n-1)\mathbf{1}\big)}{|X|}-f\big((n-1)\mathbf{1}\big)\right)\right), \tag{4.1}$$

where the constant ε is small sufficiently such that

$$\overline{\psi}_0 < \min_{r_1\le\rho\le r_2}\left(\frac{f\big((n-1)\mathbf{1}\big)}{\rho}+\varepsilon\left(\frac{f\big((n-1)\mathbf{1}\big)}{\rho}-f\big((n-1)\mathbf{1}\big)\right)\right).$$

Then we have by (1.12)

$$\overline{\psi}_0 \le \min_{0\le t\le 1}\psi^t(X,\nu).$$

From concavity of f we see that for $0\le p\le 1$,

$$f\big(p\lambda+(1-p)\mu\big)\ge pf\big(\lambda\big)+(1-p)f\big(\mu\big),\quad \lambda,\mu\in\Gamma.$$

Let $\mu\to\mathbf{0}$, then we have

$$f\big(p\lambda\big)\ge pf\big(\lambda\big). \tag{4.2}$$

When $p=r_1$ and $\lambda=\frac{(n-1)\mathbf{1}}{r_1}$ in (4.2), we obtain

$$\frac{f\big((n-1)\mathbf{1}\big)}{r_1}\ge f\big(\frac{(n-1)\mathbf{1}}{r_1}\big). \tag{4.3}$$

When $p=\frac{1}{r_2}$ and $\lambda=(n-1)\mathbf{1}$ in (4.2), we get

$$f\big(\frac{(n-1)\mathbf{1}}{r_2}\big)\ge\frac{f\big((n-1)\mathbf{1}\big)}{r_2}. \tag{4.4}$$

From (4.3) and (4.4) it follows that the function $\psi^t(X,\nu)$ satisfies (1.9) and (1.10) with strict inequalities.

To prove Theorem 1.3, we consider the following family of equations

$$f\big(\lambda(\eta)\big)=\psi^t(X,\nu),\quad 0\le t\le 1. \tag{4.5}$$

Now, we prove that the solutions of Eq. (4.5) have uniform C^0 bounds.

Theorem 4.1. *Let M be a closed star-shaped Γ-convex hypersurface satisfying the curvature Eq.* (4.5). *Suppose that f and ψ satisfy* (1.6)-(1.10), (1.12)-(1.13). *Then*

$$r_1<\rho(x)<r_2\quad\forall x\in\mathbb{S}^n.$$

Proof. Suppose that $\rho(x)$ attains its maximum at $x_0\in\mathbb{S}^n$ and $\rho(x_0)=r_2$. Then at x_0 we have $\nabla\rho=0$ and $\kappa_i\ge\frac{1}{r_2}$ for each principal curvature. From (2.7) we get

$$h^i_j(x_0)=g^{ik}h_{kj}=\frac{1}{r_2}\delta_{ij}-\frac{\rho_{ij}(x_0)}{\rho^2(x_0)}\ge\frac{1}{r_2}\delta_{ij}.$$

It follows that

$$\lambda(\eta) \geq \frac{n-1}{r_2}\mathbf{1},$$

which means that

$$f\big(\lambda(\eta)\big) \geq f\big(\frac{n-1}{r_2}\mathbf{1}\big).$$

On the other hand, the unit outer normal vector $\nu = \frac{X}{|X|}$ at x_0 and ψ^t satisfies (1.10) with strict inequality for $0 \leq t < 1$. Then

$$f\big(\frac{n-1}{r_2}\mathbf{1}\big) > \psi^t(X, \frac{X}{|X|}) = \psi^t(X, \nu) = f\big(\lambda(\eta)\big),$$

which is a contradiction. Hence $\rho(x) < r_2$. Similarly, we can prove $\rho(x) > r_1$. □

Now, we prove the following uniqueness result.

Proposition 4.2. *For $t = 0$, there exists the unique solution $\rho \equiv 1$ to Eq.* (4.5)*, namely, $M = \mathbb{S}^n$.*

Proof. Let $X = \rho x$ be a solution of Eq. (4.5) for $t = 0$. Assume the function ρ achieves its maximum $\rho_{\max}$ at $x_0 \in \mathbb{S}^n$, then at $x_0 \in \mathbb{S}^n$

$$\lambda(\eta) \geq \frac{n-1}{\rho_{\max}}\mathbf{1},$$

which derives that

$$f\big(\lambda(\eta)\big) \geq f\big(\frac{n-1}{\rho_{\max}}\mathbf{1}\big) \geq \frac{f\big((n-1)\mathbf{1}\big)}{\rho_{\max}}. \tag{4.6}$$

From (4.5), we obtain

$$f\big(\lambda(\eta)\big) = \left(\frac{f\big((n-1)\mathbf{1}\big)}{\rho_{\max}} + \varepsilon\left(\frac{f\big((n-1)\mathbf{1}\big)}{\rho_{\max}} - f\big((n-1)\mathbf{1}\big)\right)\right). \tag{4.7}$$

Combining (4.7) with (4.6),

$$\frac{f\big((n-1)\mathbf{1}\big)}{\rho_{\max}} - f\big((n-1)\mathbf{1}\big) \geq 0,$$

which gives

$$\rho_{\max} \leq 1.$$

Similarly,

$$\rho_{\min} \geq 1.$$

Thus, $\rho = 1$ is the unique solution of Eq. (4.5) for $t = 0$. □

Now, we establish gradient estimates for the solutions of Eq. (4.5).

Theorem 4.3. *Let M be a closed star-shaped Γ-convex hypersurface satisfying Eq.* (1.5). *Suppose that f and ψ satisfy* (1.6)-(1.8), (1.11)-(1.13). *Then*

$$\max_{\mathbb{S}^n} \big|\nabla\rho\big| \leq C, \tag{4.8}$$

where C is a constant depending on n, $\big|\rho\big|_{C^0}$ and $\big|\psi\big|_{C^2}$.

Proof. According to (2.8), it is sufficient to get a positive lower bound of u. We consider the following auxiliary function

$$\varphi = -\log u + \gamma(|X|^2),$$

where $\gamma(\cdot)$ is a positive function will be determined later. Assume X_0 is the maximum value point of φ. If X is parallel to ν at X_0, then

$$u(X_0) = \langle X, \nu\rangle = \rho(X_0) \geq \inf_M \rho > r_1.$$

Therefore,

$$-\log u + \gamma(|X|^2) \leq -\log u(X_0) + \gamma(|X_0|^2),$$

which means that u has a positive lower bound. If X is not parallel to ν at X_0, we can choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ near X_0 such that

$$\langle X, e_1\rangle \neq 0, \quad \langle X, e_i\rangle = 0, \quad i \geq 2,$$

which yields that

$$X = \langle X, e_1\rangle e_1 + \langle X, \nu\rangle\nu.$$

From now on, all calculations will be done at X_0. Then we have

$$0 = \varphi_i = -\frac{u_i}{u} + 2\gamma'\langle X, e_i\rangle = -\frac{h_{i1}\langle X, e_1\rangle}{u} + 2\gamma'\langle X, e_i\rangle, \tag{4.9}$$

which yields that

$$h_{11} = 2u\gamma', \quad h_{1i} = 0, \quad i \geq 2. \tag{4.10}$$

Therefore, after rotating $\{e_2, \ldots, e_n\}$, we can assume that h_{ij} is diagonal. Applying the maximum principle,

$$\begin{aligned} 0 &\geq G^{ij}\varphi_{ij} \\ &= -\frac{G^{ij}u_{ij}}{u} + G^{ij}\frac{u_i u_j}{u^2} + 4\gamma'' G^{ij}\langle X, e_i\rangle\langle X, e_j\rangle + 2\gamma'\sum_{i=1}^n G^{ii} - 2u\gamma' G^{ij}h_{ij} \\ &= -\frac{G^{ij}u_{ij}}{u} + 4\big((\gamma')^2 + \gamma''\big)G^{11}\langle X, e_1\rangle^2 + 2\gamma'\sum_{i=1}^n G^{ii} - 2u\gamma' G^{ij}h_{ij}. \end{aligned} \tag{4.11}$$

Using (2.1), (2.2) and (2.3) yields that

$$u_{ij} = h_{ij1}\langle X, e_1\rangle + h_{ij} - u\sum_{k=1}^n h_{ik}h_{kj},$$

which gives that

$$-G^{ij}\frac{u_{ij}}{u} = -\frac{G^{ij}h_{ij}}{u} - \frac{G^{ij}h_{ij1}\langle X, e_1\rangle}{u} + G^{ij}h_{ik}h_{kj}. \tag{4.12}$$

From (3.4) and (3.6) it follows that

$$-\frac{G^{ij}h_{ij1}\langle X,e_1\rangle}{u}=-\frac{1}{u}\Big(\big(d_X\psi\big)(e_1)\langle X,e_1\rangle+h_{11}\big(d_\nu\psi\big)(e_1)\langle X,e_1\rangle\Big).\tag{4.13}$$

From concavity of f,

$$f\big(\lambda\big)\geq f\big(p\lambda\big)+(1-p)f_i\big(\lambda(\eta)\big)\lambda_i,$$

letting $p\to 0$, we have

$$\psi=f\big(\lambda\big)\geq f_i\lambda_i=F^{ii}\eta_{ii}=G^{ij}h_{ij}.\tag{4.14}$$

Inserting (4.12) and (4.13) into (4.11) and using (4.14), we obtain

$$\begin{aligned}0\geq{}&4\big((\gamma')^2+\gamma''\big)G^{11}\langle X,e_1\rangle^2-\frac{1}{u}\Big(\psi+\big(d_X\psi\big)(e_1)\langle X,e_1\rangle\Big)\\&-2\gamma'\big(d_\nu\psi\big)(e_1)\langle X,e_1\rangle+G^{ii}h_{ii}^2+2\gamma'\sum_{i=1}^n G^{ii}-2u\gamma'G^{ij}h_{ij}.\end{aligned}\tag{4.15}$$

From (1.11), we find that

$$\begin{aligned}0&\geq\frac{\partial}{\partial\rho}\big(\rho\psi(X,\nu)\big)\\&=\psi+(d_X\psi)(\langle X,e_1\rangle e_1+u\nu)\\&=\psi+\langle X,e_1\rangle(d_X\psi)(e_1)+u(d_X\psi)(\nu).\end{aligned}\tag{4.16}$$

Putting (4.16) into (4.15)

$$\begin{aligned}0\geq{}&4\big((\gamma')^2+\gamma''\big)G^{11}\langle X,e_1\rangle^2+(d_X\psi)(\nu)\\&-2\gamma'\big(d_\nu\psi\big)(e_1)\langle X,e_1\rangle+2\gamma'\sum_{i=1}^n G^{ii}-2u\gamma'G^{ij}h_{ij}.\end{aligned}\tag{4.17}$$

Without loss of generality, we assume that

$$\langle X,e_1\rangle^2\geq\frac{1}{2}\inf_M\rho^2.\tag{4.18}$$

Otherwise,

$$u^2=\langle X,\nu\rangle^2=|X|^2-\langle X,e_1\rangle^2\geq\frac{1}{2}\inf_M\rho^2.$$

Now we choose

$$\gamma(t)=\frac{\alpha}{t},$$

where α is a constant to be determined later. Since $h_{11}=2\gamma' u<0$ and $\sum_{i=1}^n\eta_{ii}=(n-1)H>0$, there exists some $i=2,\dots,n$ such that $h_{ii}>0$. Then $\eta_{11}=H-h_{11}>\eta_{ii}=H-h_{ii}$ and $F^{11}\leq F^{ii}$. It follows from this and (3.24) that

$$\begin{aligned}G^{11}&=F^{22}+\cdots+F^{ii}+\cdots+F^{nn}\\&\geq\frac{1}{2}F^{11}+\cdots+\frac{1}{2}F^{ii}+\cdots+F^{nn}\end{aligned}$$

$$\geq \frac{1}{2}\sum_{i=1}^{n} F^{ii} = \frac{1}{2(n-1)}\sum_{i=1}^{n} G^{ii}$$

$$\geq \frac{1}{2(n-1)}C_0. \tag{4.19}$$

Using (3.17), we have

$$-2u\gamma' G^{ij}h_{ij} \geq 0. \tag{4.20}$$

Substitute (4.19) and (4.20) into (4.17)

$$0 \geq \frac{2}{n-1}\Big(\frac{\alpha^2}{\rho^8} + \frac{2\alpha}{\rho^6}\Big)|\langle X, e_1\rangle|^2 \sum_{i=1}^{n} G^{ii} - C\sum_{i=1}^{n} G^{ii}$$

$$- \frac{C\alpha}{\rho^4}|\langle X, e_1\rangle| \sum_{i=1}^{n} G^{ii} - \frac{C\alpha}{\rho^4}\sum_{i=1}^{n} G^{ii}. \tag{4.21}$$

Choose α sufficiently large, then we obtain a contradiction from (4.21). Therefore,

$$u^2 \geq \frac{1}{2}\inf_M \rho^2.$$

□

Using Theorem 4.1, Proposition 4.2, Theorem 4.3 and Theorem 1.2, we get C^2 estimates. Higher order estimates follows from Evans-Krylov theory [11]. Applying the similar argument of [4], we obtain the existence and uniqueness of solution to Eq. (1.5).

5. Proof of Theorem 1.4

In this section, we establish the interior curvature bounds for solutions of Eq. (1.16). Let

$$G(h_{ij}) = F(\eta_{ij}) = f\big(\lambda(\eta)\big) = \psi(X, \nu). \tag{5.1}$$

From Remark 1.2 in [31], we know that $\phi - u \geq c(\Omega') > 0$ for any $\Omega' \subset\subset \Omega$. Without loss of generality, we may assume that $u \in C^4(\overline{\Omega})$ in view of replacing u by $u + \epsilon$ and Ω by $\{x \in \Omega : u(x) + \epsilon < \phi(x)\}$ for small enough $\epsilon > 0$. We consider the auxiliary function

$$\widetilde{H}(X, \xi) = \frac{h_{\xi\xi}(\phi - u)^\beta}{\langle \nu, E_{n+1}\rangle - a} e^{\frac{A}{2}|X|^2},$$

where A and β are constants to be determined later, $E_{n+1} = (0, \dots, 0, 1)$ and $a = \frac{1}{2}\inf_{\overline{\Omega}}\langle \nu, E_{n+1}\rangle$. Suppose $\widetilde{H}(X, \xi)$ attains its maximum at an interior point $x_0 \in \Omega$, in a direction $\xi_0 \in T_{X(x_0)}M$ which we may take to be e_1. Choose the coordinates such that (h_{ij}) is diagonal at $(x_0, u(x_0))$ and

$$h_{11} \geq \cdots \geq h_{nn}.$$

Therefore the function

$$H = \log h_{11} + \beta \log(\phi - u) - \log\big(\langle \nu, E_{n+1}\rangle - a\big) + \frac{A}{2}|X|^2$$

has a maximum at x_0. From now on, all the calculations will be done at x_0. Differentiating H once gives

$$0 = H_i = \frac{h_{11i}}{h_{11}} + \beta\frac{(\phi - u)_i}{\phi - u} - \frac{\langle \nu, E_{n+1}\rangle_i}{\langle \nu, E_{n+1}\rangle - a} + A\langle X, e_i\rangle. \tag{5.2}$$

Differentiating H twice, we have

$$\begin{aligned} 0 \geq{}& G^{ij}H_{ij} \\ ={}& G^{ij}\big(\frac{h_{11ij}}{h_{11}} - \frac{h_{11i}h_{11j}}{h_{11}^2}\big) + \beta G^{ij}\big(\frac{(\phi-u)_{ij}}{\phi - u} - \frac{(\phi-u)_i(\phi-u)_j}{(\phi-u)^2}\big) \\ &- G^{ij}\big(\frac{\langle \nu, E_{n+1}\rangle_{ij}}{\langle \nu, E_{n+1}\rangle - a} - \frac{\langle \nu, E_{n+1}\rangle_i\langle \nu, E_{n+1}\rangle_j}{(\langle \nu, E_{n+1}\rangle - a)^2}\big) \\ &+ A\sum_{i=1}^n G^{ii} - AG^{ij}h_{ij}\langle X, \nu\rangle. \end{aligned} \tag{5.3}$$

By (3.7) and (3.8), we have

$$\begin{aligned} \frac{G^{ij}h_{11ij}}{h_{11}} \geq{}& - G^{ij,ml}\frac{h_{ij1}h_{ml1}}{h_{11}} + G^{ij}h_{ij}h_{11} - G^{ii}h_{ii}^2 \\ &+ \sum_{i=1}^n \frac{h_{11i}}{h_{11}}\big(d_\nu\psi\big)(e_i) - h_{11}C - C. \end{aligned} \tag{5.4}$$

Direct calculation yields that

$$\langle \nu, E_{n+1}\rangle_{ij} = h_{ijk}\langle e_k, E_{n+1}\rangle - h_{ik}h_{kj}\langle \nu, E_{n+1}\rangle.$$

Combining with (3.4) and (3.6), we obtain

$$\begin{aligned} -G^{ij}\frac{\langle \nu, E_{n+1}\rangle_{ij}}{\langle \nu, E_{n+1}\rangle - a} \geq{}& \frac{\langle \nu, E_{n+1}\rangle}{\langle \nu, E_{n+1}\rangle - a}G^{ij}h_{ik}h_{kj} \\ &- \frac{\sum_{i=1}^n h_{ii}(d_\nu\psi)(e_i)\langle e_i, E_{n+1}\rangle}{\langle \nu, E_{n+1}\rangle - a} - C. \end{aligned} \tag{5.5}$$

Using Gauss's formula and the assumption that ϕ is affine, we get

$$G^{ij}\frac{(\phi-u)_{ij}}{\phi - u} = -d_X\phi(\nu)G^{ij}h_{ij} + G^{ij}h_{ij}\langle \nu, E_{n+1}\rangle. \tag{5.6}$$

Applying (5.2) gives that

$$- \frac{\sum_{i=1}^n h_{ii}(d_\nu\psi)(e_i)\langle e_i, E_{n+1}\rangle}{\langle \nu, E_{n+1}\rangle - a} + \sum_{i=1}^n \frac{h_{11i}}{h_{11}}\big(d_\nu\psi\big)(e_i)$$

$$= -\sum_{i=1}^{n}\left(\beta\frac{(\phi-u)_i}{\phi-u}+A\langle X,e_i\rangle\right)(d_\nu\psi)(e_i)$$

$$\geq -\frac{\beta C}{\phi-u}-AC. \tag{5.7}$$

Inserting (5.4)-(5.7) into (5.3) yields

$$\begin{aligned}
0\geq\ & -G^{ij,ml}\frac{h_{ij1}h_{ml1}}{h_{11}}-G^{ij}\frac{h_{11i}h_{11j}}{h_{11}^2}-\beta G^{ij}\frac{(\phi-u)_i(\phi-u)_j}{(\phi-u)^2}\\
&+G^{ij}h_{ij}\left(h_{11}-\beta\frac{d_X\phi(\nu)}{\phi-u}+\beta\frac{\langle\nu,E_{n+1}\rangle}{\phi-u}-A\langle X,\nu\rangle\right)\\
&+G^{ij}\frac{\langle\nu,E_{n+1}\rangle_i\langle\nu,E_{n+1}\rangle_j}{(\langle\nu,E_{n+1}\rangle-a)^2}+\frac{a}{\langle\nu,E_{n+1}\rangle-a}G^{ii}h_{ii}^2+A\sum_{i=1}^{n}G^{ii}\\
&-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C.
\end{aligned}\tag{5.8}$$

We now consider two cases.

Case 1. There is a positive constant $\theta>0$ to be chosen later such that $h_{22}>\theta h_{11}$ or $h_{nn}<-\theta h_{11}$. From (5.2) it follows that

$$\begin{aligned}
&G^{ij}\frac{h_{11i}h_{11j}}{h_{11}^2}\\
&=G^{ij}\left(\beta\frac{(\phi-u)_i}{\phi-u}-\frac{\langle\nu,E_{n+1}\rangle_i}{\langle\nu,E_{n+1}\rangle-a}+A\langle X,e_i\rangle\right)\\
&\quad\left(\beta\frac{(\phi-u)_j}{\phi-u}-\frac{\langle\nu,E_{n+1}\rangle_j}{\langle\nu,E_{n+1}\rangle-a}+A\langle X,e_j\rangle\right)\\
&\leq(1+\gamma)G^{ij}\frac{\langle\nu,E_{n+1}\rangle_i\langle\nu,E_{n+1}\rangle_j}{\big(\langle\nu,E_{n+1}\rangle-a\big)^2}+\beta^2(2+\frac{2}{\gamma})G^{ij}\frac{(\phi-u)_i(\phi-u)_j}{(\phi-u)^2}\\
&\quad+A^2(2+\frac{2}{\gamma})G^{ij}\langle X,e_i\rangle\langle X,e_j\rangle.
\end{aligned}\tag{5.9}$$

Choosing $\gamma=\frac{a^2}{2}$, we see that

$$\begin{aligned}
\gamma G^{ij}\frac{\langle\nu,E_{n+1}\rangle_i\langle\nu,E_{n+1}\rangle_j}{\big(\langle\nu,E_{n+1}\rangle-a\big)^2}&=\gamma G^{ij}\frac{h_{ii}\langle e_i,E_{n+1}\rangle h_{jj}\langle e_j,E_{n+1}\rangle}{\big(\langle\nu,E_{n+1}\rangle-a\big)^2}\\
&\leq\frac{a}{2(\langle\nu,E_{n+1}\rangle-a)}G^{ii}h_{ii}^2
\end{aligned}\tag{5.10}$$

and

$$\begin{aligned}
&\beta^2(2+\frac{2}{\gamma})G^{ij}\frac{(\phi-u)_i(\phi-u)_j}{(\phi-u)^2}+A^2(2+\frac{2}{\gamma})G^{ij}\langle X,e_i\rangle\langle X,e_j\rangle\\
&\leq\frac{\beta^2C}{(\phi-u)^2}\sum_{i=1}^{n}G^{ii}+A^2C\sum_{i=1}^{n}G^{ii}.
\end{aligned}\tag{5.11}$$

It follows from the concavity of G

$$-\frac{1}{h_{11}}G^{ij,ml}h_{ij1}h_{ml1}\geq 0. \tag{5.12}$$

Putting (5.9)-(5.12) into (5.8), we obtain

$$\begin{aligned}0\geq&\Big(-\frac{\beta^2C}{(\phi-u)^2}-\frac{\beta C}{(\phi-u)^2}-A^2C+A\Big)\sum_{i=1}^n G^{ii}+\frac{a}{2(\langle\nu,E_{n+1}\rangle-a)}G^{ii}h_{ii}^2\\&+G^{ij}h_{ij}\Big(h_{11}-\beta\frac{d_X\phi(\nu)}{\phi-u}+\beta\frac{\langle\nu,E_{n+1}\rangle}{\phi-u}-A\langle X,\nu\rangle\Big)\\&-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C.\end{aligned} \tag{5.13}$$

From (3.18) and (3.19), we get

$$G^{22}\geq\frac{1}{2(n-1)}\sum_{i=1}^n G^{ii},\quad G^{nn}\geq\frac{1}{n}\sum_{i=1}^n G^{ii}. \tag{5.14}$$

By the assumption of **Case 1** we find that

$$\frac{a}{2(\langle\nu,E_{n+1}\rangle-a)}\big(G^{22}h_{22}^2+G^{nn}h_{nn}^2\big)\geq\frac{\theta^2 a}{4n(\langle\nu,E_{n+1}\rangle-a)}h_{11}^2\sum_{i=1}^n G^{ii}. \tag{5.15}$$

Using the fact $G^{ij}h_{ij}\geq 0$ yields that

$$G^{ij}h_{ij}\Big(h_{11}-\beta\frac{d_X\phi(\nu)}{\phi-u}+\beta\frac{\langle\nu,E_{n+1}\rangle}{\phi-u}-A\langle X,\nu\rangle\Big)\geq 0, \tag{5.16}$$

provided that $(\phi-u)h_{11}$ is sufficiently large. Insert (5.15) and (5.16) into (5.13),

$$\begin{aligned}0\geq&\frac{\theta^2 a(\phi-u)^2h_{11}^2}{8n(\langle\nu,E_{n+1}\rangle-a)}\sum_{i=1}^n G^{ii}-\Big(\beta^2C+A^2C(\phi-u)^2\Big)\sum_{i=1}^n G^{ii}\\&+\frac{\theta^2 a(\phi-u)^2h_{11}^2}{8n(\langle\nu,E_{n+1}\rangle-a)}\sum_{i=1}^n G^{ii}-C(\phi-u)h_{11}-\beta C-AC\\\geq&\frac{\theta^2 a(\phi-u)^2h_{11}^2}{8n(\langle\nu,E_{n+1}\rangle-a)}\sum_{i=1}^n G^{ii}-C(\phi-u)h_{11}-\beta C-AC,\end{aligned} \tag{5.17}$$

provided that $(\phi-u)^2h_{11}^2$ is sufficiently large. From (3.24) and (5.17) we obtain

$$0\geq\frac{\theta^2 aC_0(\phi-u)^2h_{11}^2}{8n(\langle\nu,E_{n+1}\rangle-a)}-C(\phi-u)h_{11}-\beta C-AC.$$

Therefore,

$$h_{11}\leq\frac{C}{\phi-u}.$$

Case 2. We now assume that $|h_{ii}| \le \theta h_{11}$ for all $i = 2, \dots, n$. Divide the indices $\{1, 2, \dots, n\}$ into two cases

$$I = \{i \,|\, G^{ii} \le 4G^{11}\}, \quad J = \{i \,|\, G^{ii} > 4G^{11}\},$$

where G^{ii} is evaluated at X_0. Similar to (5.9), we have

$$\begin{aligned}
&\sum_{i\in I} G^{ij} \frac{h_{11i}h_{11j}}{h_{11}^2} \\
&\le (1+\varepsilon) \sum_{i\in I} G^{ij} \frac{\langle \nu, E_{n+1}\rangle_i \langle \nu, E_{n+1}\rangle_j}{\big(\langle \nu, E_{n+1}\rangle - a\big)^2} + \beta^2 (2+\frac{2}{\varepsilon}) \sum_{i\in I} G^{ij} \frac{(\phi-u)_i \phi - u)_j}{(\phi-u)^2} \\
&\quad + A^2 (2+\frac{2}{\varepsilon}) \sum_{i\in I} G^{ij} \langle X, e_i\rangle \langle X, e_j\rangle \\
&\le (1+\varepsilon) \sum_{i\in I} G^{ij} \frac{\langle \nu, E_{n+1}\rangle_i \langle \nu, E_{n+1}\rangle_j}{\big(\langle \nu, E_{n+1}\rangle - a\big)^2} + \frac{C\beta^2}{\varepsilon(\phi-u)^2} G^{11} + \frac{A^2 C}{\varepsilon} G^{11}. \qquad (5.18)
\end{aligned}$$

By (5.2) we have

$$\begin{aligned}
\beta G^{ii} \frac{(\phi-u)_i^2}{(\phi-u)^2} =& \sum_{i\in J} \frac{G^{ii}}{\beta} \left(\frac{h_{11i}}{h_{11}} - \frac{\langle \nu, E_{n+1}\rangle_i}{\langle \nu, E_{n+1}\rangle - a} + A\langle X, e_i\rangle \right)^2 \\
&+ \beta \sum_{i\in I} G^{ii} \frac{(\phi-u)_i^2}{(\phi-u)^2} \\
\le& \sum_{i\in J} \frac{2+\frac{2}{\varepsilon}}{\beta} G^{ii} \frac{h_{11i}^2}{h_{11}^2} + \frac{1+\varepsilon}{\beta} \frac{G^{ii} h_{ii}^2}{(\langle \nu, E_{n+1}\rangle - a)^2} \\
&+ \frac{2+\frac{2}{\varepsilon}}{\beta} A^2 C \sum_{i=1}^n G^{ii} + \frac{\beta C}{(\phi-u)^2} G^{11}. \qquad (5.19)
\end{aligned}$$

Note that $2a = \inf_{\overline{\Omega}} \langle \nu, E_{n+1}\rangle \le 1$. Fixing $\varepsilon = \frac{a^2}{4}$ and assuming that

$$\beta \ge \beta_1 := \frac{8}{a^2},$$

we obtain

$$\frac{a}{\langle \nu, E_{n+1}\rangle - a} - \frac{\varepsilon + \frac{1}{\beta} + \frac{\varepsilon}{\beta}}{(\langle \nu, E_{n+1}\rangle - a)^2} \ge \frac{a}{2}. \qquad (5.20)$$

Take (5.18)-(5.20) into (5.8),

$$\begin{aligned}
0 \ge& -G^{ij,ml} \frac{h_{ij1}h_{ml1}}{h_{11}} - (1+\frac{C}{\beta}) \sum_{i\in J} G^{ii} \frac{h_{11i}^2}{h_{11}^2} \\
&+ G^{ij} h_{ij} \left(h_{11} - \beta \frac{d_X \phi(\nu)}{\phi-u} + \beta \frac{\langle \nu, E_{n+1}\rangle}{\phi-u} - A\langle X, \nu\rangle \right)
\end{aligned}$$

$$+\frac{a}{2}G^{ii}h_{ii}^2-\Big(\frac{C\beta^2}{(\phi-u)^2}+\frac{\beta C}{(\phi-u)^2}+A^2C\Big)G^{11}$$

$$+A\sum_{i=1}^{n}G^{ii}-\frac{A^2C}{\beta}\sum_{i=1}^{n}G^{ii}-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C. \tag{5.21}$$

Without loss of generality, we assume that

$$(\phi-u)h_{11}\ge \max\Big\{\beta d_X\phi(\nu)+A\langle X,\nu\rangle(\phi-u),\ \sqrt{\frac{2}{a}\big(C\beta^2+C\beta+A^2C(\phi-u)^2\big)}\Big\},$$

then

$$0\ge\ -G^{ij,ml}\frac{h_{ij1}h_{ml1}}{h_{11}}-(1+\frac{C}{\beta})\sum_{i\in J}G^{ii}\frac{h_{11i}^2}{h_{11}^2}$$

$$+A\sum_{i=1}^{n}G^{ii}-\frac{A^2C}{\beta}\sum_{i=1}^{n}G^{ii}-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C. \tag{5.22}$$

It follows from the concave of G and Lemma 2.2 that

$$-\frac{1}{h_{11}}G^{ij,ml}h_{ij1}h_{ml1}\ge\frac{2}{h_{11}}\frac{G^{ii}-G^{11}}{h_{11}-h_{ii}}h_{11i}^2. \tag{5.23}$$

We need to show

$$\frac{2}{h_{11}}\frac{G^{ii}-G^{11}}{h_{11}-h_{ii}}\ge(1+\frac{C}{\beta})\frac{G^{ii}}{h_{11}^2},\quad i\in J,$$

which is equivalent to

$$(1-\frac{C}{\beta})G^{ii}h_{11}+(1+\frac{C}{\beta})G^{ii}h_{ii}\ge 2G^{11}h_{11},\quad i\in J, \tag{5.24}$$

provided β is sufficiently large. For $i\in J$, $G^{ii}>4G^{11}$. (5.24) can be obtained if we can show

$$(1-\frac{2C}{\beta})G^{11}h_{11}+2(1+\frac{C}{\beta})G^{11}h_{ii}\ge 0,\quad i\in J. \tag{5.25}$$

Assuming that $\beta\ge\beta_2:=5C$. If $h_{ii}\ge 0$, (5.25) holds obviously; if $h_{ii}<0$, then $h_{ii}\ge-\theta h_{11}$ by the assumption of **Case 2**. We find that (5.25) also holds if $\theta\le\frac{1}{4}$. Therefore,

$$0\ge A\sum_{i=1}^{n}G^{ii}-\frac{A^2C}{\beta}\sum_{i=1}^{n}G^{ii}-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C. \tag{5.26}$$

Without loss of generality, we assume that $\beta\ge\max\{\beta_1,\beta_2,2AC\}$. It follows that

$$0\ge\frac{A}{2}\sum_{i=1}^{n}G^{ii}-Ch_{11}-\frac{\beta C}{\phi-u}-AC-C. \tag{5.27}$$

Use (1.14), then we have

$$\sum_{i=1}^{n} G^{ii} = (n-1)\sum_{i=1}^{n} F^{ii} \geq (n-1)c_0 h_{11}.$$

Choosing A sufficiently large, we obtain

$$\frac{A}{4}\sum_{i=1}^{n} G^{ii} - h_{11}C \geq 0.$$

From this and (5.27), we obtain

$$h_{11} \leq \frac{C}{(\phi - u)}.$$

Combining **Case 1** and **Case 2**, $\widetilde{H}(X, \xi)$ satisfies a similar bound. The curvature bound of Theorem 1.4 follows.

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