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# TIMELIKE TUBULAR SURFACES OF WEINGARTEN TYPES AND LINEAR WEINGARTEN TYPES IN MINKOWSKI 3-SPACE

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Abstract. Let $K$, $H$, $K_{II}$ and $H_{II}$ be the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature of a timelike tubular surface $T_\gamma(\alpha)$ with the radius $\gamma$ along a timelike curve $\alpha(s)$ in Minkowski 3-space $E_1^3$. We prove that $T_\gamma(\alpha)$ must be a $(K, H)$-Weingarten surface and a $(K, H)$-linear Weingarten surface. We also show that $T_\gamma(\alpha)$ is $(X, Y)$-Weingarten type if and only if its central curve is a circle or a helix, where $(X, Y) \in \{(K, K_{II}), (K, H_{II}), (H, K_{II}), (H, H_{II}), (K_{II}, H_{II})\}$. Furthermore, we prove that there exist no timelike tubular surfaces of $(X, Y)$-linear Weingarten type, $(X, Y, Z)$-linear Weingarten type and $(K, H, K_{II}, H_{II})$-linear Weingarten type along a timelike curve in $E_1^3$, where $(X, Y, Z) \in \{(K, H, K_{II}), (K, H, H_{II}), (K, K_{II}, H_{II}), (H, K_{II}, H_{II})\}$.

## 1. Introduction

A canal surface is the envelop of a one-parameter set of moving spheres. The research history of canal surfaces can be traced back to the work of Monge. In 1850, the class of surfaces formed by sweeping a sphere with a radius function $r(s)$ was first investigated by Monge. In special, choosing the radius function $r(s)$ as a constant, we obtain a tubular surface. Tubular surfaces are actively applied in surface modeling for CAD/CAM, shape control and robots path planning (cf. [1, 14]).

The properties of a surface are largely determined by its curvatures. Weingarten surfaces are exactly such a kind of surfaces whose curvatures satisfy some nontrivial functional relationships. More precisely, a surface in a 3-dimensional Euclidean space $\mathbb{R}^3$ is called a Weingarten surface if its two principal curvatures $k_1$ and $k_2$ satisfy a nontrivial functional relation $\Phi(k_1, k_2) = 0$. The relation

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$\Phi(k_1, k_2) = 0$ implies that its mean curvature $H$ and Gauss curvature $K$ satisfy $\Phi(K, H) = 0$. Weingarten surfaces were introduced by Weingarten [21, 22] in 1861. The research of Weingarten surfaces has been an important topic in differential geometry. After the works of Chern [2], Hopf [7] and Voss [20], there has been increasing attention on this field. Ro and Yoon [16] established a classification of Weingarten tubes in $\mathbb{R}^3$. Tunçer, Yoon and Karacan [18] gave some results for characteristics and existences of tubular surfaces of Weingarten type and linear Weingarten type in $\mathbb{R}^3$. For more results about Weingarten surfaces in $\mathbb{R}^3$, we refer to [6, 10, 11, 13, 17] and the references therein.

As we known, a Euclidean space is a special Riemannian manifold with a flat Riemannian metric. As a generalization of a Riemannian manifold, a pseudo-Riemannian manifold, also called a semi-Riemannian manifold, is equipped with a pseudo-Riemannian metric in which the requirement of positive-definiteness is relaxed. Unlike Riemannian manifolds, an indefinite signature of pseudo-Riemannian manifolds allows tangent vectors to be classified into spacelike, timelike or lightlike. Just as Euclidean space $\mathbb{R}^n$ can be thought of as the Riemannian manifold model, Minkowski space $E_1^n$ is the pseudo-Riemannian manifold model. Although initially developed by mathematician Hermann Minkowski for Maxwell's equations of electromagnetism, Minkowski space is closely associated with Einstein's theories of general relativity.

With the development of geometry and physics, geometers and physicians extended some topics of Riemannian geometry to that of pseudo-Riemannian manifolds, especially Minkowski 3-space $E_1^3$. Minkowski 3-space is a 3-dimensional pseudo-Riemannian manifold with a flat metric

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ denotes the canonical coordinates in $E_1^3$. It is also called a 3-dimensional Lorentzian space or a 3-dimensional semi-Euclidean space. Similar to that in $\mathbb{R}^3$, we can give the definitions of Weingarten type and linear Weingarten type surfaces in $E_1^3$.

In recent years, surfaces of Weingarten type in $E_1^3$ have attracted great attentions. Dillen and Kühnel [3], Dillen and Sodsiri [4], and Kim and Yoon [9] derived some interesting results for ruled surfaces of Weingarten type in $E_1^3$. Moreover, several interesting results for tubular surfaces of Weingarten type in $E_1^3$ have been obtained. In 2014, Karacan, Yoon and Tunçer [8] gave some results for spacelike tubular surfaces of Weingarten type and linear Weingarten type in $E_1^3$. In 2016, Uçum and İlarslan [19] studied spacelike tubular surfaces of Weingarten type and linear Weingarten type obtained from pseudo-hyperbolic spheres $\mathbb{H}_0^2$ in $E_1^3$. In 2019, by classifying the linear Weingarten surfaces, Fu, Jung, Qian and Su [5, 15] showed some geometric properties of canal surfaces foliated by pseudo spheres $\mathbb{S}_1^2$ and pseudo-hyperbolic spheres $\mathbb{H}_0^2$ along space curves in $E_1^3$.

To the authors' knowledge, there has been no detailed discussion about timelike tubular surfaces of Weingarten type and linear Weingarten type along a timelike curve in Minkowski 3-space. Hence the aim of this paper is to investigate their characteristics and existence. In fact, if the second fundamental form $II$ of a surface $M$ in $E_1^3$ is non-degenerate, then it can be regarded as a pseudo-Riemannian metric. We may extend the classical concept of Weingarten and linear Weingarten surfaces by taking the second Gaussian curvature $K_{II}$ and the second mean curvature $H_{II}$ into consideration. Hereby, we provide the following definition:

**Definition 1.1.** Let $\Omega_1$ and $\Omega_2$ be sets of pairs and triples of the Gaussian curvature $K$, the mean curvature $H$, the second Gaussian curvature $K_{II}$ and the second mean curvature $H_{II}$ of a surface $M$ in Minkowski 3-space $E_1^3$, namely

$$\Omega_1 = \Big\{(K,H),(K,K_{II}),(K,H_{II}),(H,K_{II}),(H,H_{II}),(K_{II},H_{II})\Big\}$$

and

$$\Omega_2 = \Big\{(K,H,K_{II}),(K,H,H_{II}),(K,K_{II},H_{II}),(H,K_{II},H_{II})\Big\}.$$

We define the following terms:

(I) $M$ is said to be an $(X,Y)$-Weingarten surface if for $(X,Y)\in\Omega_1$,

$$\Phi(X,Y) = \begin{vmatrix} X_s & X_\theta \\ Y_s & Y_\theta \end{vmatrix} = 0. \tag{1.1}$$

(II) $M$ is said to be an $(X,Y)$-linear Weingarten surface if for $(X,Y)\in\Omega_1$, there exist a constant $m$ and two nonzero constants $a$ and $b$ such that

$$aX + bY = m. \tag{1.2}$$

(III) $M$ is said to be an $(X,Y,Z)$-linear Weingarten surface if for $(X,Y,Z)\in\Omega_2$, there exist a constant $m$ and three nonzero constants $a$, $b$ and $c$ such that

$$aX + bY + cZ = m. \tag{1.3}$$

(IV) $M$ is said to be a $(K,H,K_{II},H_{II})$-linear Weingarten surface if there exist a constant $m$ and four nonzero constants $a$, $b$, $c$ and $d$ such that

$$aK + bH + cK_{II} + dH_{II} = m. \tag{1.4}$$

The main results of this paper are as follows. In Section 3, we investigate timelike tubular surfaces of Weingarten types along a timelike curve in Minkowski 3-space $E_1^3$. We first prove that a timelike tubular surface along a timelike curve in $E_1^3$ must be a $(K,H)$-Weingarten surface in Theorem 3.1. Moreover, in Theorem 3.2, for $(X,Y)\in\Omega_1\backslash\{(K,H)\}$, we give the characteristic of a timelike tubular surface of $(X,Y)$-Weingarten type along a timelike curve in $E_1^3$. We show that its central curve is a circle or a helix. Furthermore, we give three examples of timelike tubular surfaces of Weingarten types. In addition, for the sake of visualization, their graphs are drawn by using MATLAB.

In Section 4, we consider timelike tubular surfaces of linear Weingarten types along a timelike curve in $E_1^3$. In Theorem 4.1, we demonstrate that a timelike tubular surface along a timelike curve in $E_1^3$ must be a $(K, H)$-linear Weingarten surface. Moreover, we confirm that there exist no timelike tubular surfaces of $(X, Y)$-linear Weingarten type, $(X, Y, Z)$-linear Weingarten type and $(K, H, K_{II}, H_{II})$-linear Weingarten type along a timelike curve in $E_1^3$, where $(X, Y) \in \Omega_1 \setminus \{(K, H)\}$ and $(X, Y, Z) \in \Omega_2$.

## 2. Preliminaries

The Minkowski 3-space $E_1^3$ is the Euclidean 3-space $\mathbb{R}^3$ equipped with the Lorentzian product

$$\langle u, v\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

where $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3)$. A vector $v$ in $E_1^3$ is said to be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and lightlike (null) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $E_1^3$ is locally spacelike, timelike or lightlike, if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or lightlike, respectively. A surface $M$ in $E_1^3$ is called a timelike surface, a spacelike surface or a lightlike surface if its normal vector $U$ is spacelike, timelike or lightlike.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the moving Frenet frame along a timelike curve $\alpha(s)$ in $E_1^3$. Then the Frenet equations are given by (cf. [12])

$$\begin{bmatrix}\mathbf{T}'(s)\\ \mathbf{N}'(s)\\ \mathbf{B}'(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ \kappa(s) & 0 & -\tau(s)\\ 0 & \tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{bmatrix}, \tag{2.2}$$

where functions $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion of $\alpha(s)$. Moreover, the following conditions hold:

$$\langle \mathbf{T}(s), \mathbf{T}(s)\rangle = -1, \tag{2.3}$$

$$\langle \mathbf{N}(s), \mathbf{N}(s)\rangle = \langle \mathbf{B}(s), \mathbf{B}(s)\rangle = 1 \tag{2.4}$$

and

$$\langle \mathbf{T}(s), \mathbf{N}(s)\rangle = \langle \mathbf{T}(s), \mathbf{B}(s)\rangle = \langle \mathbf{N}(s), \mathbf{B}(s)\rangle = 0. \tag{2.5}$$

A canal surface $M$ in $E_1^3$ is the envelope formed by sweeping a family of pseudo-spheres $\mathbb{S}_1^2$ whose centers lie on a timelike curve $\alpha(s)$ framed by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$. Thus $M$ can be parametrized by

$$\begin{aligned} x(s,\theta) = \alpha(s) + r(s)\big(&r'(s)\mathbf{T}(s) + \sqrt{1 + r'(s)^2}\cos\theta\mathbf{N}(s)\\ &+ \sqrt{1 + r'(s)^2}\sin\theta\mathbf{B}(s)\big), \end{aligned} \tag{2.6}$$

where the curve $\alpha(s)$ is called the center curve and $r(s)$ is called the radial function of $M$. If $r(s)$ is constant, then $M$ is a tubular surface.

Throughout the paper, all the surfaces we are dealing with are smooth, regular and topologically connected. A surface $M$ in $E_1^3$ can be denoted by

$$x(s,\theta) = (x_1(s,\theta), x_2(s,\theta), x_3(s,\theta)).$$

Let $U$ be the standard unit normal vector field of $M$ defined by

$$U = -\frac{x_s \times x_\theta}{\|x_s \times x_\theta\|}. \tag{2.7}$$

The first fundamental form and the second fundamental form of $M$ are

$$I = Eds^2 + 2Fdsd\theta + Gd\theta^2 \tag{2.8}$$

and

$$II = eds^2 + 2fdsd\theta + gd\theta^2, \tag{2.9}$$

where

$$\begin{array}{lll} E = \langle x_s, x_s\rangle, & F = \langle x_s, x_\theta\rangle, & G = \langle x_\theta, x_\theta\rangle, \\ e = -\langle x_s, U_s\rangle, & f = -\langle x_s, U_\theta\rangle, & g = -\langle x_\theta, U_\theta\rangle. \end{array} \tag{2.10}$$

The Gaussian curvature and the mean curvature are

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2}. \tag{2.11}$$

From Brioschi's formula in a Minkowski 3-space, we are able to compute $K_{II}$ of a surface by replacing the components of the first fundamental form $E, F, G$ with the components of the second fundamental form $e, f, g$, respectively. Thus, the second Gaussian curvature $K_{II}$ of a surface is

$$K_{II} = \frac{1}{(eg - f^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}e_{\theta\theta} + f_{s\theta} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_\theta \\ f_\theta - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_\theta & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_\theta & \frac{1}{2}g_s \\ \frac{1}{2}e_\theta & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}\right\}. \tag{2.12}$$

Denote by $L_{ij}$ the coefficients of second fundamental forms. Then the second mean curvature $H_{II}$ is defined by

$$H_{II} = H - \frac{1}{2\sqrt{|\det II|}}\sum_{i,j=1}^{2}\frac{\partial}{\partial u^i}\left(\sqrt{|\det II|}L^{ij}\frac{\partial}{\partial u^i}(\ln\sqrt{|K|})\right), \tag{2.13}$$

where $(L^{ij}) = (L_{ij})^{-1}$, and $u^1$ and $u^2$ stand for $s$ and $\theta$, respectively.

## 3. Timelike tubular surfaces of Weingarten types in Minkowski 3-space

In this section, we concentrate on timelike tubular surfaces of Weingarten types along timelike curves in Minkowski 3-space $E_1^3$. We first prove that it must be a $(K, H)$-Weingarten surface. Furthermore, we give the necessary and sufficient condition that a timelike tubular surface along a timelike curve in $E_1^3$ is an $(X, Y)$-Weingarten surface, where $(X, Y) \in \Omega_1 \setminus \{(K, H)\}$. More precisely, we obtain the following theorems.

**Theorem 3.1.** *A timelike tubular surface along a timelike curve in Minkowski 3-space must be a $(K, H)$-Weingarten surface.*

**Theorem 3.2.** *Let $T_\gamma(\alpha)$ be a timelike tubular surface along a timelike curve $\alpha(s)$ in Minkowski 3-space. For*

$$(X, Y) \in \{(K, K_{II}), (K, H_{II}), (H, K_{II}), (H, H_{II}), (K_{II}, H_{II})\},$$

*$T_\gamma(\alpha)$ is an $(X, Y)$-Weingarten surface if and only if its central curve is a circle or a helix.*

*Proofs of Theorems 3.1 and 3.2.* On the basis of (2.6), we may get the parametrization of a timelike tubular surface $T_\gamma(\alpha)$ with a constant radius $\gamma$ along a timelike curve $\alpha(s)$:

$$x(s, \theta) = \alpha(s) + \gamma\left(\cos\theta \mathbf{N}(s) + \sin\theta \mathbf{B}(s)\right), \tag{3.1}$$

where $s \in [a, b]$, $\theta \in [0, 2\pi)$ and $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the moving Frenet frame along $\alpha(s)$. Initially, it is from (2.2)-(2.5) and (3.1) that

$$x_s = (1 + \gamma\kappa\cos\theta)\mathbf{T}(s) + \gamma\tau\sin\theta \mathbf{N}(s) - \gamma\tau\cos\theta \mathbf{B}(s) \tag{3.2}$$

and

$$x_\theta = -\gamma\sin\theta \mathbf{N}(s) + \gamma\cos\theta \mathbf{B}(s). \tag{3.3}$$

Then from (2.7) and (3.2)-(3.3), we deduce that the standard unit normal vector field of $T_\gamma(\alpha)$ is

$$U = \cos\theta \mathbf{N}(s) + \sin\theta \mathbf{B}(s).$$

Differentiating $U$ with respect to $s$ and $\theta$, we have

$$U_s = \kappa\cos\theta \mathbf{T}(s) + \tau\sin\theta \mathbf{N}(s) + \tau\cos\theta \mathbf{B}(s) \tag{3.4}$$

and

$$U_\theta = -\sin\theta \mathbf{N}(s) + \cos\theta \mathbf{B}(s). \tag{3.5}$$

Using (2.10) and (3.2)-(3.5), we know that the component functions of the first fundamental form and the second fundamental form are given by

$$E = \gamma^2\tau^2 - (1 + \gamma\kappa\cos\theta)^2, \quad F = -\gamma^2\tau, \quad G = \gamma^2, \tag{3.6}$$

$$e = \kappa\cos\theta(1 + \gamma\kappa\cos\theta) - \gamma\tau^2, \quad f = \gamma\tau, \quad \text{and} \quad g = -\gamma. \tag{3.7}$$

Then we investigate each type of $(X, Y)$-Weingarten surfaces. In order to discuss $(K, H)$-Weingarten surfaces, by using (2.11), (3.6) and (3.7), we obtain

$$K = \frac{\kappa\cos\theta}{\gamma(1 + \gamma\kappa\cos\theta)} \quad \text{and} \quad H = \frac{1 - 2(1 + \gamma\kappa\cos\theta)}{2\gamma(1 + \gamma\kappa\cos\theta)}. \tag{3.8}$$

Differentiating $K$ and $H$ with respect to $s$ and $\theta$, we get

$$K_s = \frac{\kappa'\cos\theta}{\gamma(1 + \gamma\kappa\cos\theta)^2}, \tag{3.9}$$

$$K_\theta = \frac{-\kappa \sin\theta}{\gamma(1+\gamma\kappa\cos\theta)^2}, \tag{3.10}$$

$$H_s = \frac{-\kappa' \cos\theta}{2(1+\gamma\kappa\cos\theta)^2} \tag{3.11}$$

and

$$H_\theta = \frac{\kappa \sin\theta}{2(1+\gamma\kappa\cos\theta)^2}. \tag{3.12}$$

It is from (3.9)-(3.12) that the Jacobi function of $K$ and $H$ is

$$\Phi(K,H) = K_s H_\theta - K_\theta H_s = 0. \tag{3.13}$$

Therefore, we can conclude that $T_\gamma(\alpha)$ is a $(K,H)$-Weingarten surface.

For an $(H, H_{II})$-Weingarten surface, using (2.11), (2.13) and (3.8), we derive

$$H_{II} = \frac{-1}{8\gamma^2\kappa^3\cos^3\theta(1+\gamma\kappa\cos\theta)^3}\sum_{i=0}^{6} u_i \cos^i\theta, \tag{3.14}$$

where

$$\begin{aligned}
u_0 &= -3\gamma^2\kappa^2,\\
u_1 &= 2\gamma^2\kappa\left[(\gamma'\tau-\gamma\tau')\sin\theta - 4\gamma\kappa^2\right],\\
u_2 &= \gamma^2\left[2\gamma\kappa^2(2\kappa\tau'-3\kappa'\tau)\sin\theta - 8\gamma^2\kappa^4+\kappa^2-3\kappa'^2+2\kappa\kappa''\right],\\
u_3 &= 2\gamma\kappa\left[2\gamma^3\kappa^2(2\kappa'\tau-\kappa\tau')\sin\theta+\gamma^2(\kappa^2-4\kappa'^2+3\kappa\kappa'')+2\kappa^2\right],\\
u_4 &= 4\gamma^2\kappa^2\left[\gamma^2(\kappa^2-2\kappa'^2)+4\kappa^2\right],\\
u_5 &= 20\gamma^3\kappa^5
\end{aligned}$$

and

$$u_6 = 8\gamma^4\kappa^6.$$

Differentiating (3.14) with respect to $s$, we get

$$H_{IIs} = \frac{-1}{8\gamma^3\kappa^4\cos^3\theta(1+\gamma\kappa\cos\theta)^4}\sum_{i=0}^{5} v_i \cos^i\theta, \tag{3.15}$$

where

$$\begin{aligned}
v_0 &= 3\gamma^3\kappa^2\kappa',\\
v_1 &= 2\gamma^3\kappa\left[\kappa(\kappa''\tau-\kappa\tau'')+2\kappa'(\kappa\tau'-\kappa'\tau)\sin\theta+6\gamma\kappa^2\kappa'\right],\\
v_2 &= \gamma^3\Big\{2\gamma\kappa^2\left[\kappa(3\kappa''\tau-4\kappa\tau'')+7\kappa'(\kappa\tau'-\kappa'\tau)\right]\sin\theta\\
&\qquad +\kappa(16\gamma\kappa^3\kappa'-\kappa\kappa'+2\kappa\kappa'''-10\kappa'\kappa'')+9\kappa'^3\Big\},\\
v_3 &= 2\gamma^4\kappa\Big\{\gamma\kappa^2\left[\kappa(6\kappa''\tau-5\kappa\tau'')+\kappa'(13\kappa\tau'-8\kappa'\tau)\right]\sin\theta
\end{aligned}$$

$$+\kappa(8\gamma^2\kappa^3\kappa' - 2\kappa\kappa' + 4\kappa\kappa''' - 19\kappa'\kappa'') + 17\kappa'^3\Big\},$$

$$v_4 = 2\gamma^3\kappa^2\Big\{\gamma^3\kappa^2\left[2\kappa(2\kappa''\tau - \kappa\tau'') + \kappa'(13\kappa\tau' - 12\kappa'\tau)\right]\sin\theta$$

$$+\gamma^2\left[\kappa^2(5\kappa''' - \kappa') + 4\kappa'(6\kappa'^2 - 7\kappa\kappa'')\right] + 2\kappa^2\kappa'\Big\}$$

and

$$v_5 = 4\gamma^4\kappa^3\Big\{\gamma^2\left[\kappa^2(\kappa''' - 2\kappa') + \kappa'(8\kappa'^2 - 7\kappa\kappa'')\right] + 2\kappa^2\kappa'\Big\}.$$

Moreover, differentiating (3.14) with respect to $\theta$, we have

$$H_{II\theta} = \frac{-1}{8\gamma^3\kappa^4\cos^4\theta(1+\gamma\kappa\cos\theta)^4}\sum_{i=0}^{6} w_i\cos^i\theta, \tag{3.16}$$

where

$$w_0 = 9\gamma^2\kappa^2\sin\theta,$$

$$w_1 = 34\gamma^3\kappa^3\sin\theta + 4\gamma^2\kappa(\kappa\tau' - \kappa'\tau),$$

$$w_2 = \gamma^2\Big\{\left[\kappa^2(48\gamma^2\kappa^2 - 1) + \kappa'(3\kappa' - 2\kappa'')\right]\sin\theta + 2\gamma\kappa^2(8\kappa\tau' - 7\kappa'\tau)\Big\},$$

$$w_3 = 2\gamma^2\kappa\Big\{\gamma\left[\kappa^2(8\gamma^2\kappa^2 - 1) + \kappa'(3\kappa' - 2\kappa'')\right]\sin\theta$$

$$+4\gamma^2\kappa^2(3\kappa\tau' - 2\kappa'\tau) + (\kappa'\tau - \kappa\tau')\Big\},$$

$$w_4 = 2\gamma^2\kappa^2\Big\{\left[\gamma^2(8\kappa'^2 - \kappa^2 - 7\kappa\kappa'') + 2\kappa^2\right]\sin\theta$$

$$+2\gamma\left[3\gamma^2\kappa^2(\kappa\tau' - 2\kappa'\tau) + (\kappa'\tau - \kappa\tau')\right]\Big\},$$

$$w_5 = \gamma^3\kappa^3\Big\{\left[\gamma^2(2\kappa'^2 - \kappa^2 - \kappa\kappa'') + \kappa^2\right]\sin\theta + 2\gamma(\kappa'\tau - 7\kappa\tau')\Big\}$$

and

$$w_6 = 4\gamma^4\kappa^4\left[\gamma^2\sin\theta + 2\gamma(2\kappa'\tau - \kappa\tau')\right].$$

Then we may obtain

$$\Phi(H, H_{II}) = H_s H_{II\theta} - H_\theta H_{IIs} \tag{3.17}$$

$$= \frac{-1}{16\gamma^4\kappa^3\cos^4\theta(1+\gamma\kappa\cos\theta)^5}\sum_{i=0}^{5} t_i\cos^i\theta,$$

where

$$t_0 = 6\gamma^4\kappa^2\kappa'\sin\theta, \tag{3.18}$$

$$t_1 = 2\gamma^4\kappa^2(\kappa\tau'' - \kappa''\tau), \tag{3.19}$$

$$t_2 = 2\gamma^4\Big\{ \big[\kappa^2(8\gamma^2\kappa^2\kappa' - \kappa''') + \kappa'(4\kappa\kappa'' - 3\kappa'^2)\big]\sin\theta + \gamma\kappa^3(3\kappa\tau'' + \kappa'\tau' - 2\kappa''\tau)\Big\}, \tag{3.20}$$

$$t_3 = 2\gamma^4\kappa\Big[\gamma\kappa(11\kappa'\kappa'' - 3\kappa\kappa''')\sin\theta + \gamma^3\kappa^3(\kappa\tau'' - \kappa'\tau' - 2\kappa''\tau) + \kappa(\kappa''\tau - \kappa\tau'') + \kappa'(\kappa\tau' - \kappa'\tau)\Big], \tag{3.21}$$

$$t_4 = 2\gamma^5\kappa^2\Big[2\gamma(5\kappa\kappa'\kappa'' - 4\kappa'^3 - \kappa^2\kappa''')\sin\theta + \kappa(2\kappa''\tau - 3\kappa\tau'') + 2\kappa'(\kappa\tau' - \kappa'\tau)\Big] \tag{3.22}$$

and

$$t_5 = 4\gamma^6\kappa^3\Big[\kappa(2\kappa''\tau - \kappa\tau'') + 2\kappa'(\kappa\tau' - \kappa'\tau)\Big]. \tag{3.23}$$

Assume that $T_\gamma(\alpha)$ is an $(H, H_{II})$-Weingarten surface in $E_1^3$. Then the Jacobi function (3.17) vanishes. Due to the polynomial in (3.17) is equal to zero for every $\theta$, all its coefficients must be zero. Thus we have

$$t_0 = t_1 = t_2 = t_3 = t_4 = t_5 = 0.$$

Noticing that $T_\gamma(\alpha)$ has non-degenerate second fundamental form, we know that $\kappa \neq 0$. Therefore, the solutions of $t_0 = t_1 = t_2 = t_3 = t_4 = t_5 = 0$ are $\kappa' = \tau = 0$ and $\kappa' = \tau' = 0$. That is to say, the central curve $\alpha(s)$ of $T_\gamma(\alpha)$ is a circle or a helix in $E_1^3$.

Conversely, suppose that the central curve $\alpha(s)$ of $T_\gamma(\alpha)$ is a circle or a helix in $E_1^3$. It is easy to see that $\Phi(H, H_{II}) = 0$ is satisfied for the cases of both $\kappa' = \tau = 0$ and $\kappa' = \tau' = 0$. Hence we can know that $T_r(\alpha)$ is an $(H, H_{II})$-Weingarten surface in $E_1^3$.

For $(K, K_{II})$-Weingarten surfaces, $(K, H_{II})$-Weingarten surfaces, $(H, K_{II})$-Weingarten surfaces and $(K_{II}, H_{II})$-Weingarten surfaces in $E_1^3$, we can make a similar discussion about $(H, H_{II})$-Weingarten surfaces and get the same results. For the sake of briefness, we omit the derivation and only give the corresponding Jacobi functions $\Phi(K, K_{II})$, $\Phi(H, K_{II})$, $\Phi(K, H_{II})$ and $\Phi(K_{II}, H_{II})$ as follows:

$$\Phi(K, K_{II}) = -\frac{\kappa'\sin\theta}{2\gamma^2\cos^2\theta(1+\gamma\kappa\cos\theta)^4}, \tag{3.24}$$

$$\Phi(H, K_{II}) = \frac{\kappa'\sin\theta}{4\gamma\cos^2\theta(1+\gamma\kappa\cos\theta)^4}, \tag{3.25}$$

$$\Phi(K, H_{II}) = -\frac{1}{8\gamma^4\kappa^3\cos^3\theta(1+\gamma\kappa\cos\theta)^5}\sum_{i=0}^{5} h_i\cos^i\theta \tag{3.26}$$

and

$$\Phi(K_{II},H_{II}) = -\frac{1}{32\gamma^4\kappa^4\cos^6\theta(1+\gamma\kappa\cos\theta)^7}\sum_{i=0}^{9} q_i\cos^i\theta, \tag{3.27}$$

where

$$h_0 = 6\gamma^3\kappa^2\kappa'\sin\theta, \tag{3.28}$$

$$h_1 = 16\gamma^4\kappa^3\kappa'\sin\theta + 2\gamma^3\kappa^2(\kappa\tau'' - \kappa''\tau), \tag{3.29}$$

$$\begin{aligned} h_2 = 2\gamma^3\Big\{ &\left[\kappa^2(8\gamma^4\kappa^2\kappa' - \kappa''') + \kappa'(4\kappa\kappa'' - 3\kappa')\right]\sin\theta \\ &+ \gamma\kappa^3(3\kappa\tau'' + \kappa'\tau' - 2\kappa''\tau)\Big\}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} h_3 = 2\gamma^3\kappa\Big\{&\gamma\left[\kappa(11\kappa'\kappa'' - 3\kappa\kappa''') - 8\kappa'^2\right]\sin\theta \\ &+ 2\gamma^2\kappa^3(\kappa\tau'' - \kappa'\tau' - 2\kappa''\tau) + \kappa(\kappa''\tau - \kappa\tau'') + \kappa'(\kappa\tau' - \kappa'\tau)\Big\}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} h_4 = 2\gamma^4\kappa^2\Big\{&2\gamma\left[\kappa(5\kappa'\kappa'' - \kappa\kappa''') - 4\kappa'^2\right]\sin\theta \\ &+ \kappa(2\kappa''\tau + 3\kappa\tau'') + 2\kappa'(\kappa\tau' - \kappa'\tau)\Big\}, \end{aligned} \tag{3.32}$$

$$h_5 = 4\gamma^5\kappa^3\Big[\kappa(2\kappa''\tau + \kappa\tau'') + 2\kappa'(\kappa\tau' - \kappa'\tau)\Big], \tag{3.33}$$

$$q_0 = 6\gamma^3\kappa^2\kappa'\sin\theta, \tag{3.34}$$

$$q_1 = 4\gamma^3\kappa\Big[\kappa(\kappa''\tau - \kappa\tau'') + 2\kappa'(\kappa\tau' - \kappa'\tau)\Big], \tag{3.35}$$

$$\begin{aligned} q_2 = 2\gamma^3\Big\{ &\left[\kappa^2(2\kappa''' - \kappa') + \kappa'(9\kappa'^2 - 10\kappa\kappa'')\right]\sin\theta \\ &+ 2\gamma\kappa^2\left[\kappa(4\kappa''\tau - 5\kappa\tau'') + 7\kappa'(\kappa\tau' - \kappa'\tau)\right]\Big\}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} q_3 = 4\gamma^3\kappa\Big\{&\gamma\left[\kappa^2(4\kappa' + 5\kappa''') + \kappa'(20\kappa'^2 - 23\kappa\kappa'')\right]\sin\theta \\ &+ \kappa(\kappa\tau'' - \kappa''\tau) + 4\gamma^2\kappa^2\left[2\kappa(\kappa''\tau - \kappa\tau'') + \kappa'(3\kappa\tau' - 2\kappa'\tau)\right] \\ &+ 2\kappa'(\kappa'\tau - \kappa\tau')\Big\}, \end{aligned} \tag{3.37}$$

$$\begin{aligned} q_4 = 4\gamma^3\kappa^2\Big\{ &\left[\gamma^2(18\kappa^2\kappa' + 8\kappa^2\kappa''' + 32\kappa'^2 - 39\kappa\kappa'\kappa'') + 2\kappa^2\kappa'\right]\sin\theta \\ &+ 2\gamma^3\kappa^2\left[2\kappa(\kappa''\tau - \kappa\tau'') + \kappa'(5\kappa\tau' - 6\kappa'\tau)\right] \end{aligned} \tag{3.38}$$

$$+\gamma\left[\kappa(7\kappa\tau''-6\kappa''\tau)+8\kappa'(\kappa'\tau-\kappa\tau')\right]\Big\},$$

$$(3.39)\quad q_5=4\gamma^4\kappa\Big\{\left[2\gamma^2\kappa^2(5\kappa^2\kappa'+\kappa^2\kappa'''+6\kappa\kappa'\kappa''+6\kappa'^3)+\kappa(2\kappa^2\kappa'-\kappa''')\right.$$
$$\left.+\kappa'(4\kappa\kappa''-3\kappa'^2)\right]\sin\theta+\gamma\kappa^2\left[\kappa(15\kappa\tau''-13\kappa''\tau)\right.$$
$$\left.+\kappa'(10\kappa'\tau-13\kappa\tau')\right]\Big\},$$

$$(3.40)\quad q_6=4\gamma^4\kappa^2\Big\{\gamma^2\kappa^2\left[\kappa(11\kappa\tau''-18\kappa''\tau)+\kappa'(16\kappa'\tau-15\kappa\tau')+\gamma^2\kappa^2\kappa'^2\tau\right]$$
$$+\gamma\left[\kappa^2(8\gamma^2\kappa^2\kappa'-2\kappa^2\kappa'-7\kappa''')-19\kappa'^3\right]\sin\theta$$
$$+2\left[\kappa(\kappa''\tau-\kappa\tau'')+\kappa'(\kappa\tau'-\kappa'\tau)\right]\Big\},$$

$$(3.41)\quad q_7=4\gamma^3\kappa\Big\{\gamma^3\kappa^2\left[\kappa^2(4\kappa'-7\kappa''')+\kappa'(31\kappa\kappa''-24\kappa'^2)\right]\sin\theta$$
$$+4\gamma^2\kappa^2\left[2\kappa(\kappa''\tau-\kappa\tau'')+\kappa'(3\kappa\tau'-8\kappa'\tau)\right]$$
$$+\kappa(\kappa\tau''-\kappa''\tau)+2\kappa'(\kappa'\tau-\kappa\tau')\Big\},$$

$$(3.42)\quad q_8=4\gamma^6\kappa^4\Big\{2\gamma\left[-\kappa^2(\kappa'+7\kappa''')+\kappa'(5\kappa\kappa''-4\kappa'^2)\right]\sin\theta$$
$$+\kappa(10\kappa''\tau-7\kappa\tau'')+10\kappa'(\kappa\tau'-\kappa'\tau)\Big\}$$

and

$$(3.43)\qquad q_9=8\gamma^7\kappa^5\Big[\kappa(\kappa''\tau-\kappa\tau'')+2\kappa'(\kappa\tau'-\kappa'\tau)\Big].$$

This completes the proofs of Theorems 3.1 and 3.2. □

Now we give three examples of timelike tubular surfaces of Weingarten types along timelike curves.

**Example 3.1.** The curve

$$(3.44)\qquad \alpha(s)=(1+\sinh s,1+\cosh s,1)$$

is a timelike circle in $E_1^3$. Its Frenet frame is

$$(3.45)\qquad \begin{cases}\mathbf{T}(s)=(\cosh s,\sinh s,0)\,,\\ \mathbf{N}(s)=(\sinh s,\cosh s,0),\\ \mathbf{B}(s)=(0,0,1).\end{cases}$$

Then a timelike tubular surface of Weingarten type with the radius $\gamma=1$ around the circle $\alpha(s)=(1+\sinh s,1+\cosh s,1)$ in $E_1^3$ is

$$x(s,\theta)=(x_1(s,\theta),y_1(s,\theta),z_1(s,\theta))\,,$$

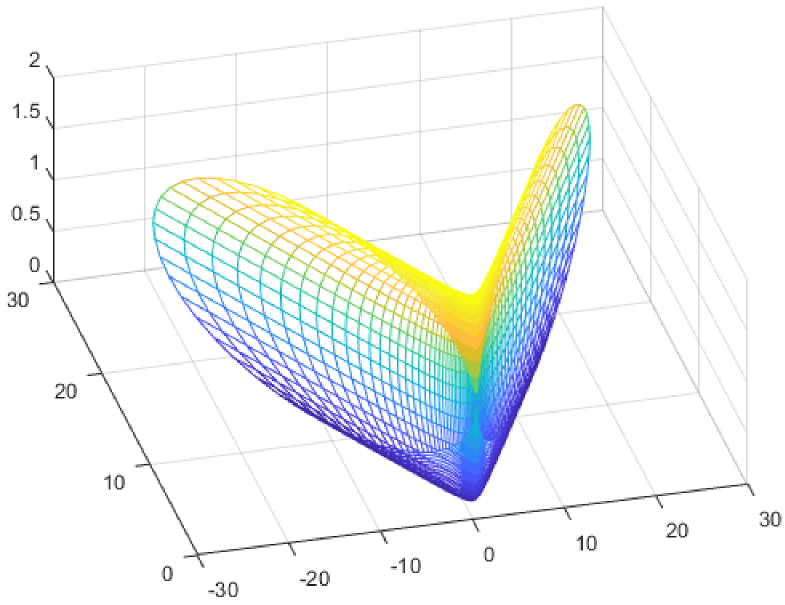


FIGURE 1. Timelike tubular surface of Weingarten type with the radius $\gamma = 1$ around a circle $\alpha(s) = (1+\sinh s, 1+\cosh s, 1)$ in $E_1^3$.

where

$$x_1(s,\theta) = 1 + \sinh s + \cos\theta \sinh s,$$
$$y_1(s,\theta) = 1 + \cosh s + \cos\theta \cosh s$$

and

$$z_1(s,\theta) = 1 + \sin\theta.$$

**Example 3.2.** The curve

$$\alpha(s) = \frac{\kappa}{\kappa^2 - \tau^2}\left(\sinh\xi, \cosh\xi, \frac{\tau}{\kappa}\xi\right) \tag{3.46}$$

is a timelike helix in $E_1^3$, where $\xi = \sqrt{\kappa^2 - \tau^2}s$. Setting $\kappa = 3$ and $\tau = 2$, we have

$$\alpha(s) = \frac{3}{5}\left(\sinh(\sqrt{5}s), \cosh(\sqrt{5}s), \frac{2}{3}\sqrt{5}s\right).$$

We obtain its Frenet frame

$$
\begin{cases}
\mathbf{T}(s) = \dfrac{3}{5}\left(\sinh(\sqrt{5}s), \cosh(\sqrt{5}s), \dfrac{2}{3}\sqrt{5}s\right), \\
\mathbf{N}(s) = \left(\sinh(\sqrt{5}s), \cosh(\sqrt{5}s), 0\right), \\
\mathbf{B}(s) = \dfrac{3}{5}\left(-\dfrac{2\sqrt{5}}{3}\cosh(\sqrt{5}s), \dfrac{2\sqrt{5}}{3}\sinh(\sqrt{5}s), \sqrt{5}\right).
\end{cases} \tag{3.47}
$$

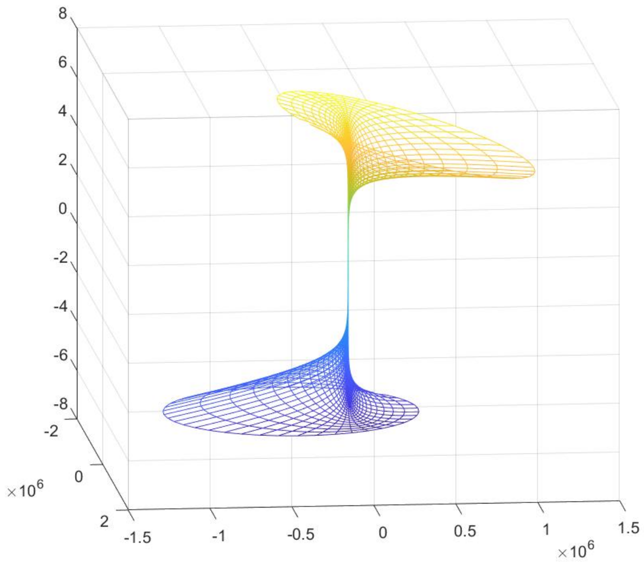


FIGURE 2. Timelike tubular surface of Weingarten type with the radius $\gamma = 1$ along a helix $\alpha(s) = \frac{3}{5}\left(\sinh(\sqrt{5}s), \cosh(\sqrt{5}s), \frac{2}{3}\sqrt{5}s\right)$ in $E_1^3$.

Then a timelike tubular surface of Weingarten type with the radius $\gamma = 1$ along the helix $\alpha(s) = \frac{3}{5}\left(\sinh(\sqrt{5}s), \cosh(\sqrt{5}s), \frac{2}{3}\sqrt{5}s\right)$ in $E_1^3$ is

$$x(s,\theta) = \left(x_2(s,\theta), y_2(s,\theta), z_2(s,\theta)\right),$$

where

$$
\begin{aligned}
x_2(s,\theta) &= \frac{3}{5}\sinh(\sqrt{5}s) + \cos\theta\sinh(\sqrt{5}s) - \frac{2\sqrt{5}}{5}\sin\theta\cosh(\sqrt{5}s), \\
y_2(s,\theta) &= \frac{3}{5}\cosh(\sqrt{5}s) + \cos\theta\cosh(\sqrt{5}s) - \frac{2\sqrt{5}}{5}\sin\theta\sinh(\sqrt{5}s),
\end{aligned}
$$

and

$$z_2(s,\theta) = \frac{2\sqrt{5}}{5} + \frac{3\sqrt{5}}{5}\sin\theta.$$

**Example 3.3.** The curve

$$\alpha(s) = \frac{\kappa}{\tau^2-\kappa^2}\left(\frac{\tau}{\kappa}\xi, \sin\xi, -\cos\xi\right) \tag{3.48}$$

is a timelike helix in $E_1^3$, where $\xi = \sqrt{\tau^2-\kappa^2}s$. Setting $\kappa = \frac{1}{3}$ and $\tau = \frac{1}{2}$, we have

$$\alpha(s) = \left(\frac{3\sqrt{5}}{5}s, \frac{12}{5}\sin(\frac{\sqrt{5}}{6}s), -\frac{12}{5}\cos(\frac{\sqrt{5}}{6}s)\right).$$

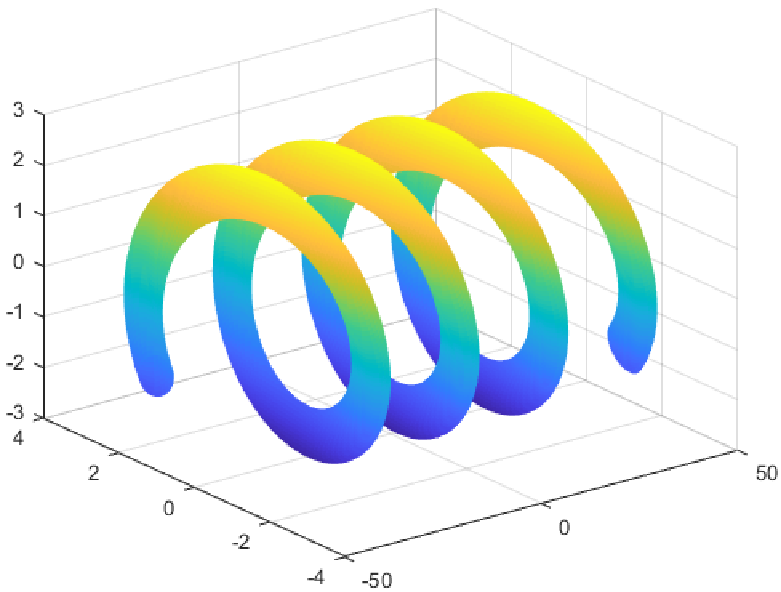


FIGURE 3. Timelike tubular surface of Weingarten type with the radius $\gamma = 1$ along a helix $\alpha(s) = \left(\frac{3\sqrt{5}}{5}s, \frac{12}{5}\sin(\frac{\sqrt{5}}{6}s), -\frac{12}{5}\cos(\frac{\sqrt{5}}{6}s)\right)$ in $E_1^3$.

Therefore, its Frenet frame is

$$\begin{cases} \mathbf{T}(s) = \left(\frac{3\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\cos(\frac{\sqrt{5}}{6}s), \frac{2\sqrt{5}}{5}\sin(\frac{\sqrt{5}}{6}s)\right), \\ \mathbf{N}(s) = \left(0, -\frac{\sqrt{5}}{6}\cos(\frac{\sqrt{5}}{6}s), -\frac{\sqrt{5}}{6}\sin(\frac{\sqrt{5}}{6}s)\right), \\ \mathbf{B}(s) = \left(0, \frac{1}{2}\sin(\frac{\sqrt{5}}{6}s), -\frac{1}{2}\cos(\frac{\sqrt{5}}{6}s)\right). \end{cases} \tag{3.49}$$

Then a timelike tubular surface of Weingarten type with the radius $\gamma = 1$ along the helix $\alpha(s) = \left(\frac{3\sqrt{5}}{5}s, \frac{12}{5}\sin(\frac{\sqrt{5}}{6}s), -\frac{12}{5}\cos(\frac{\sqrt{5}}{6}s)\right)$ in $E_1^3$ is

$$x(s,\theta) = (x_3(s,\theta), y_3(s,\theta), z_3(s,\theta)),$$

where

$$x_3(s,\theta) = \frac{3\sqrt{5}}{5},$$

$$y_3(s,\theta) = \frac{12}{5}\sin(\frac{\sqrt{5}}{6}s) - \frac{\sqrt{5}}{6}\cos\theta\cos(\frac{\sqrt{5}}{6}s) + \frac{1}{2}\sin\theta\sin(\frac{\sqrt{5}}{6}s)$$

and

$$z_3(s,\theta) = \frac{12}{5}\cos(\frac{\sqrt{5}}{6}s) - \frac{\sqrt{5}}{6}\cos\theta\sin(\frac{\sqrt{5}}{6}s) - \frac{1}{2}\sin\theta\cos(\frac{\sqrt{5}}{6}s).$$

## 4. Timelike tubular surfaces of linear Weingarten types in Minkowski 3-space

In this section, we study timelike tubular surfaces of linear Weingarten types along a timelike curve in Minkowski 3-space $E_1^3$. They are $(X, Y)$-linear Weingarten surfaces, $(X, Y, Z)$-linear Weingarten surfaces and $(K, H, K_{II}, H_{II})$-linear Weingarten surfaces, where $(X, Y) \in \Omega_1$ and $(X, Y, Z) \in \Omega_2$. We obtain the following results.

**Theorem 4.1.** *A timelike tubular surface along a timelike curve in Minkowski 3-space must be a $(K, H)$-linear Weingarten surface.*

**Theorem 4.2.** *A timelike tubular surface along a timelike curve in Minkowski 3-space must not be an $(X, Y)$-linear Weingarten surface, where*

$$(X, Y) \in \{(K, K_{II}), (K, H_{II}), (H, K_{II}), (H, H_{II}), (K_{II}, H_{II})\}.$$

**Theorem 4.3.** *A timelike tubular surface along a timelike curve in Minkowski 3-space must not be an $(X, Y, Z)$-linear Weingarten surface, where*

$$(X, Y, Z) \in \{(K, H, K_{II}), (K, H, H_{II}), (K, K_{II}, H_{II}), (H, K_{II}, H_{II})\}.$$

**Theorem 4.4.** *A timelike tubular surface along a timelike curve in Minkowski 3-space must not be a $(K, H, K_{II}, H_{II})$-linear Weingarten surface.*

*Proofs of Theorems 4.1-4.4.* Although there are three types of linear Weingarten surfaces in Minkowski 3-space, we only need to construct a linear combination of $K, H, K_{II}$ and $H_{II}$. That is to say, for a timelike tubular surface $T_\gamma(\alpha)$ with the radius $\gamma$ along a timelike curve $\alpha(s)$, we may consider

$$aK + bH + cK_{II} + dH_{II} = m. \tag{4.1}$$

Without loss of generality, for an $(X, Y)$-linear Weingarten surface, we may assume any two of $a, b, c$ and $d$ are equal to zero. And for an $(X, Y, Z)$-linear Weingarten surface, we let any one of $a, b, c$ or $d$ be zero.

By making some calculations, we obtain the reduced form of (4.1) as follows

$$\begin{aligned} aK+bH+cK_{II}+dH_{II}-m &= \frac{1}{8\gamma^2\kappa^3\cos^3\theta(1+\gamma\kappa\cos\theta)^3}\sum_{i=0}^{6}\lambda_i\cos^i\theta \\ &= 0, \end{aligned} \tag{4.2}$$

where the coefficients are

$$\lambda_0 = 3d\gamma^2\kappa^2, \tag{4.3}$$

$$\lambda_1 = 2d\gamma^2\kappa(\kappa\tau' - \kappa'\tau)\sin\theta + 2\gamma\kappa^3(4d\gamma^2 - c), \tag{4.4}$$

$$\begin{aligned} \lambda_2 = {} & 2d\gamma^3\kappa^2(3\kappa\tau' + 2\kappa'\tau)\sin\theta \\ & + \gamma^2(-2c\kappa^4 + 8d\gamma^2\kappa^2 - d\kappa^2 + 3d\kappa'^2 - 2d\kappa\kappa''), \end{aligned} \tag{4.5}$$

$$\lambda_3 = 4d\gamma^4\kappa^3(\kappa\tau' - 2\kappa'\tau)\sin\theta - 2\gamma\kappa^3(4m\gamma + 2b + 2d + d\gamma^2 + c), \tag{4.6}$$

$$\begin{aligned} \lambda_4 = & -2\gamma\kappa^4\Big[2d\gamma^3 + 12m\gamma^2 + (8b + 8d + 7c)\gamma - 4a\Big] \\ & + 4d\gamma^4\kappa^2(2\kappa'^2 - \kappa\kappa''), \end{aligned} \tag{4.7}$$

$$\lambda_5 = -4\gamma^2\kappa^5\Big[6m\gamma^2 + 5\big(b + c + d\big)\gamma - 4a\Big] \tag{4.8}$$

and

$$\lambda_6 = -8\gamma^3\kappa^6\Big[m\gamma^2 + (b + c + d)\gamma - a\Big]. \tag{4.9}$$

Since (4.2) holds for every $\theta$, there must be $\lambda_i = 0$ for $i = 0, 1, \ldots, 6$. First of all, because $\lambda_0 = 0$, it is observed from (4.3) that $d = 0$. Thus $T_\gamma(\alpha)$ must not be a $(K, H, K_{II}, H_{II})$-linear Weingarten surface. Hence we know that Theorem 4.4 is true.

Moreover, since $d = 0$, it yields from (4.4) that

$$\lambda_1 = -2c\gamma\kappa^3. \tag{4.10}$$

Noticing that $\lambda_1 = 0$, we can find that $c = 0$. Therefore $T_\gamma(\alpha)$ must not be an $(X, Y, Z)$-linear Weingarten surface. Thus Theorem 4.3 is true.

Because $c = d = 0$, $T_\gamma(\alpha)$ must not be any one of $(K, H_{II})$-linear Weingarten surfaces, $(H, H_{II})$-linear Weingarten surfaces or $(K_{II}, H_{II})$-linear Weingarten surfaces.

For the case of $(K, K_{II})$-linear Weingarten surfaces, we ought to assume $b = d = 0$ in (4.2). Remember that we already have $c = d = 0$, thus there is $b = c = d = 0$. Therefore, according to Definition 1.1, it is not possible that $T_\gamma(\alpha)$ is a $(K, K_{II})$-linear Weingarten surface.

For the case of $(H, K_{II})$-linear Weingarten surfaces, we may set $a = d = 0$ in (4.2). But we have known that $c = d = 0$. Thus it implies $a = c = d = 0$. We can find that $T_\gamma(\alpha)$ must not be an $(H, K_{II})$-linear Weingarten surface. Thus Theorem 4.2 is true.

For the case of $(K, H)$-linear Weingarten surfaces, we take $c = d = 0$ in (4.2). Then it yields

$$\lambda_0 = 0, \tag{4.11}$$

$$\lambda_1 = 0, \tag{4.12}$$

$$\lambda_2 = 0, \tag{4.13}$$

$$\lambda_3 = 2\gamma\kappa^3(4m\gamma + 2b), \tag{4.14}$$

$$\lambda_4 = -2\gamma\kappa^4(12m\gamma^2 + 8b\gamma), \tag{4.15}$$

$$\lambda_5 = -4\gamma^2\kappa^5(6m\gamma^2 + 5b\gamma - 4a) \tag{4.16}$$

and

$$\lambda_6 = -8\gamma^3\kappa^6(m\gamma^2 + b\gamma - a). \tag{4.17}$$

Namely the following linear equations with respect to $a$ and $b$ hold

$$\begin{cases} 2\gamma\kappa^3(4m\gamma + 2b) = 0, \\ -2\gamma\kappa^4(12m\gamma^2 + 8b\gamma) = 0, \\ -4\gamma^2\kappa^5(6m\gamma^2 + 5b\gamma - 4a) = 0, \\ -8\gamma^3\kappa^6(m\gamma^2 + b\gamma - a) = 0. \end{cases} \tag{4.18}$$

It is not difficult to find that when $\kappa \neq 0$, the solution of (4.18) is

$$\begin{cases} a = -m\gamma^2, \\ b = -2m\gamma. \end{cases} \tag{4.19}$$

This means that for $m \neq 0$, we can choose two nonzero constants $a$ and $b$ such that $aK + bH = m$. Therefore, $T_\gamma(\alpha)$ is a $(K, H)$-linear Weingarten surface. Thus Theorem 4.1 is true. This finishes the proofs of Theorems 4.1-4.4. □

Using Theorems 3.1 and 4.1, we obtain the following corollary.

**Corollary 4.5.** *A timelike tubular surface along a timelike curve in Minkowski 3-space is a $(K, H)$-Weingarten surface if and only if it is a $(K, H)$-linear Weingarten surface.*

From Theorems 4.1-4.4, we derive the following corollary.

**Corollary 4.6.** *A timelike tubular surface of linear Weingarten type along a timelike curve in Minkowski* 3*-space must be a* $(K, H)$*-linear Weingarten surface.*

## References


[1] B. Brunt and K. Grant, *Potential applications of Weingarten surfaces in CAGD. Part I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Design **13** (1996), no. 6, 569–582. https://doi.org/10.1016/0167-8396(95)00046-1

[2] S. S. Chern, *Some new characterizations of the Euclidean sphere*, Duke Math. J. **12** (1945), 279–290. http://projecteuclid.org/euclid.dmj/1077473104

[3] F. Dillen and W. Kühnel, *Ruled Weingarten surfaces in Minkowski* 3*-space*, Manuscripta Math. **98** (1999), no. 3, 307–320. https://doi.org/10.1007/s002290050142

[4] F. Dillen and W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski* 3*-space*, J. Geom. **83** (2005), no. 1-2, 10–21. https://doi.org/10.1007/s00022-005-0002-4

[5] X. Fu, S. D. Jung, J. Qian, and M. Su, *Geometric characterizations of canal surfaces in Minkowski* 3*-space* $\mathbb{I}$, Bull. Korean Math. Soc. **56** (2019), no. 4, 867–883. https://doi.org/10.4134/BKMS.b180643

[6] J. A. Gálvez, A. Martínez, and F. Milán, *Linear Weingarten surfaces in* $\mathbb{R}^3$, Monatsh. Math. **138** (2003), no. 2, 133–144. https://doi.org/10.1007/s00605-002-0510-3

[7] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. **4** (1951), 232–249. https://doi.org/10.1002/mana.3210040122

[8] M. K. Karacan, D. W. Yoon, and Y. Tunçer, *Tubular surfaces of Weingarten types in Minkowski* 3*-space*, Gen. Math. Notes. **22** (2014), no. 1, 44–56.

[9] Y. H. Kim and D. W. Yoon, *Classification of ruled surfaces in Minkowski* 3*-spaces*, J. Geom. Phys. **49** (2004), no. 1, 89–100. https://doi.org/10.1016/S0393-0440(03)00084-6

[10] W. Kühnel and M. Steller, *On closed Weingarten surfaces*, Monatsh. Math. **146** (2005), no. 2, 113–126. https://doi.org/10.1007/s00605-005-0313-4

[11] R. López, *On linear Weingarten surfaces*, Internat. J. Math. **19** (2008), no. 4, 439–448. https://doi.org/10.1142/S0129167X08004728

[12] R. López, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom. **7** (2014), no. 1, 44–107.

[13] R. López and Á. Pámpano, *Classification of rotational surfaces in Euclidean space satisfying a linear relation between their principal curvatures*, Math. Nachr. **293** (2020), no. 4, 735–753. https://doi.org/10.1002/mana.201800235

[14] M. G. Puopolo and J. D. Jacob, *Velocity control of a cylindrical rolling robot by shape changing*, Adv. Robotics. **30** (2016), no. 23, 1484–1494.

[15] J. H. Qian, M. F. Su, X. S. Fu, and S. D. Jung, *Geometric characterizations of canal surfaces in Minkowski* 3*-space II*, Math. **7** (2019), no. 8, 703.

[16] J. S. Ro and D. W. Yoon, *Tubes of weingarten types in a Euclidean* 3*-space*, J. Chungcheong Math. Soc. **22** (2009), no. 3, 359–366.

[17] H. Rosenberg and R. Sa Earp, *The geometry of properly embedded special surfaces in* $\mathbb{R}^3$*; e.g., surfaces satisfying* $aH + bK = 1$*, where* $a$ *and* $b$ *are positive*, Duke Math. J. **73** (1994), no. 2, 291–306. https://doi.org/10.1215/S0012-7094-94-07314-6

[18] Y. Tunçer, D. W. Yoon, and M. K. Karacan, *Weingarten and linear Weingarten type tubular surfaces in* $\mathbf{E}^3$, Math. Probl. Eng. **2011** (2011), Art. ID 191849, 11 pp. https://doi.org/10.1155/2011/191849

[19] A. Uçum and K. İlarslan, *New types of canal surfaces in Minkowski* 3*-space*, Adv. Appl. Clifford Algebr. **26** (2016), no. 1, 449–468. https://doi.org/10.1007/s00006-015-0556-7

[20] K. Voss, *Über geschlossene Weingartensche Flächen*, Math. Ann. **138** (1959), 42–54. https://doi.org/10.1007/BF01369665

[21] J. Weingarten, *Ueber eine Klasse auf einander abwickelbarer Flächen*, J. Reine Angew. Math. **59** (1861), 382–393. https://doi.org/10.1515/crll.1861.59.382

[22] J. Weingarten, *Ueber die Flächen deren Normalen eine gegebene Fläche berühren*, J. Reine Angew. Math. **62** (1863), 61–63. https://doi.org/10.1515/crll.1863.62.61

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