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THE SECONDARY UPSILON FUNCTION OF L–SPACE KNOTS IS A CONCAVE CONJUGATE

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Abstract. For a knot in the 3–sphere, the Upsilon invariant is a piecewise linear function defined on the interval $[0,2]$. It is known that this invariant of an L–space knot is the Legendre–Fenchel transform (or, convex conjugate) of a certain gap function derived from the Alexander polynomial. To recover an information lost in the Upsilon invariant, Kim and Livingston introduced the secondary Upsilon invariant. In this note, we prove that the secondary Upsilon invariant of an L–space knot is a concave conjugate of a restricted gap function. Also, a similar argument gives an alternative proof of the above fact that the Upsilon invariant of an L–space knot is a convex conjugate of a gap function.

1. Introduction

For a knot K in the 3–sphere, Ozsváth, Stipsicz and Szabó [8] introduced the Upsilon invariant $\Upsilon_K(t)$ which is a piecewise linear function defined on the interval $[0,2]$. It is a smooth concordance invariant, generalizing the τ–invariant [9]. Also, it gives lower bounds for the three–genus, four–genus and concordance genus of a knot.

Although the Upsilon invariant was originally defined through a t–modified knot Floer complex in [8], Livingston [7] later gave an alternative interpretation through the knot Floer complex $\mathrm{CFK}^\infty(K)$. To capture some lost information in $\Upsilon_K(t)$, Kim and Livingston [6] further introduced the secondary Upsilon invariant $\Upsilon^2_{K,t}(s)$ on $[0,2]$, which is also a smooth concordance invariant.

A knot K is called an *L–space knot* if it admits a positive Dehn surgery yielding an L–space. It gives an important class of knots, including (positive) torus knots, Berge knots. For an L–space knot, the complex $\mathrm{CFK}^\infty(K)$ is always represented as a staircase diagram $\mathrm{St}(K)$, which is determined only by the Alexander polynomial (see Section 2). Hence the Upsilon invariant is also determined by the Alexander polynomial, and [8] gives a concise formula of

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$\Upsilon_K(t)$. Based on this fact, Borodzik and Hedden [3] showed that the Upsilon invariant of an L–space knot is the Legendre–Fenchel transform (or, convex conjugate) of a gap function $G(x)$ which is defined from the Alexander polynomial. This immediately implies that the Upsilon invariant $\Upsilon_K(t)$ of an L–space knot K is necessarily a convex function. In fact, $\Upsilon'_K(t)$ is piecewise constant and has only finitely many singularities (see [7, Theorem 8.1] or [8]).

In this paper, we examine only L–space knots. Roughly speaking, the secondary Upsilon invariant $\Upsilon^2_{K,t}(s)$ has an essential meaning at each singularity t of $\Upsilon'_K(t)$. Except the singularities, $\Upsilon^2_{K,t}(s)$ is identically ∞. For a fixed singularity t_0, the staircase diagram $\mathrm{St}(K)$ determines two vertices $p^-_{t_0}$ and $p^+_{t_0}$, called the *negative* and *positive pivot points*, which are grading 0 bifiltered generators of $\mathrm{CFK}^\infty(K)$ ([13]). The secondary Upsilon invariant $\Upsilon^2_{K,t_0}(s)$ is defined by examining the part of $\mathrm{St}(K)$ cut out by these two pivot points.

For our purpose, we define the convex conjugate and concave conjugate of a continuous function as follows. For a continuous function $G(x)$ defined on a closed interval I, its *convex conjugate* is given by

$$G_*(t) = \max_{x\in I}(tx - G(x)),$$

and its *concave conjugate* is given by

$$G^*(s) = \min_{x\in I}(sx - G(x)).$$

In this paper, we always choose $t, s \in [0,2]$. (In our argument, the gap function $G(x)$ is a piecewise linear continuous function defined on $\mathbb{R}$, whose graph consists of segments with slope 0 or 2. This is the reason why we do not use supremum and infimum in the above definition, unlike the standard convex analysis.)

Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $\Phi(x,y) = (x-y, 2x)$. In other words, Φ is a composition of a $\pi/2$ counterclockwise rotation, a doubling vertically, and a map $(x,y) \mapsto (x+y/2, y)$. For a point $p \in \mathbb{R}^2$, $\Phi_1(p)$ denotes the first coordinate of $\Phi(p)$. We use this Φ to transform $\mathrm{St}(K)$ into the graph of the gap function $G(x)$ of K.

Theorem 1.1. *Let K be an L–space knot and $\Upsilon_K(t)$ its Upsilon invariant. Let $t_0 \in (0,2)$ be a singularity of $\Upsilon'_K(t)$, and let p^- and p^+ be the corresponding negative and positive pivot points on the staircase diagram* $\mathrm{St}(K)$*. Then the secondary Upsilon invariant $\Upsilon^2_{K,t_0}(s)$ at t_0 is given by*

$$\Upsilon^2_{K,t_0}(s) = G^*(s) - \Upsilon_K(t_0),$$

where $G^*(s)$ is the concave conjugate of the restriction of the gap function $G(x)$ on the interval $[\Phi_1(p^-), \Phi_1(p^+)]$.

Remark 1.2. If we add $\Upsilon_K(t_0)$ to $G(x)$ on $[\Phi_1(p^-), \Phi_1(p^+)]$ beforehand, then the secondary Upsilon invariant coincides with its concave conjugate.

By a similar argument, we can give an alternative proof that the Upsilon invariant of an L–space knot is the convex conjugate $G_*(t)$ of the gap function $G(x)$. See Section 4.

We should mention that the secondary Upsilon invariants for some torus knots and L–space knots are computed in [2, 13].

2. Gap function

In this section, we suppose that K is an L–space knot with genus g. Its Alexander polynomial has the form of

$$\Delta_K(t) = \sum_{k=0}^{2n} (-1)^k t^{a_k} \tag{2.1}$$

for some strictly increasing sequence of integers $a_0, a_1, \ldots, a_{2n}$, where $a_0 = 0$ and $a_{2n} = 2g$ [11]. By the symmetry of Alexander polynomial, $a_k - g = -(a_{2n-k} - g)$, and we know that $a_1 = 1$ and $a_{2n-1} = 2g - 1$ by [5]. Then the knot Floer complex $\mathrm{CFK}^\infty(K)$ can be represented as a diagram $\mathrm{St}(K)$ in the $(\mathrm{alg}, \mathrm{Alex})$–plane, where each element x of $\mathrm{CFK}^\infty(K)$ has an algebraic and an Alexander filtration, denoted by $\mathrm{alg}(x)$ and $\mathrm{Alex}(x)$, respectively ([4, 9–11]). It is well known that $\mathrm{St}(K)$ has the form of staircase specified by

$$[a_1 - a_0, a_2 - a_1, \ldots, a_{2n} - a_{2n-1}], \tag{2.2}$$

where the indices alternate between horizontal and vertical steps. More precisely, we start at the black vertex at $(0, g)$ on the plane $\mathbb{R}^2$. Go right $a_1 - a_0$ and put a white vertex, go down $a_2 - a_1$ and put a black vertex. Repeat this procedure until we reach the black vertex $(g, 0)$. Draw an arrow from each white vertex to the two adjacent black vertices to represent the boundary map. In this paper, we do not use boundary maps. All black vertices have homological (Maslov) grading 0, but all white ones have grading 1.

The full complex $\mathrm{CFK}^\infty(K)$ is obtained by taking all integer diagonal translations of $\mathrm{St}(K)$, where the action of the formal variable U decreases both algebraic and Alexander filtrations by one, and homological grading by two. Thus $\mathrm{CFK}^\infty(K) \cong \mathrm{St}(K) \otimes \mathbb{Z}_2[U, U^{-1}]$ as modules. Also, $H_0(\mathrm{CFK}^\infty(K)) \cong \mathbb{Z}_2$ is generated by any one of black vertex of $\mathrm{St}(K)$.

By the symmetry of Alexander polynomial as mentioned before, (2.2) and its reverse

$$[a_{2n} - a_{2n-1}, a_{2n-1} - a_{2n-2}, \ldots, a_1 - a_0] \tag{2.3}$$

coincide. This immediately implies the symmetry of $\mathrm{St}(K)$ along the line $y = x$ on $\mathbb{R}^2$.

The Alexander polynomial also has the form of

$$\Delta_K(t) = 1 + (t-1)(t^{b_1} + t^{b_2} + \cdots + t^{b_g}),$$

where $b_1, \dots, b_g$ are positive integers ([4]). The strictly increasing sequence $b_1, b_2, \dots, b_g$ is called the *gap sequence*. Consider the set

$$\mathcal{G} = \mathbb{Z}_{<0} \cup \{b_1, \dots, b_g\},$$

where $\mathbb{Z}_{<0}$ is the set of negative integers. This is called the *gap set* of K. In fact, $\mathbb{Z} - \mathcal{G}$ coincides with the *formal semigroup* $\mathcal{S}$ introduced in [12]. As a formal power series, we expand

$$\frac{\Delta_K(t)}{1-t} = \sum_{s \in \mathcal{S}} t^s.$$

Then $\mathcal{S} = \{0, c_1, c_2, \dots, c_{g-1}\} \cup \mathbb{Z}_{\geq 2g}$, where $\mathbb{Z}_{\geq 2g} = \{m \in \mathbb{Z} \mid m \geq 2g\}$. By the symmetry of Alexander polynomial, we see that $x \in \mathcal{S}$ if and only if $2g - 1 - x \notin \mathcal{S}$. (Indeed, this observation implies that the gap sequence consists of exactly g integers.)

Define a function on $\mathbb{Z}$ by

$$I(m) = \#\{x \in \mathbb{Z} \mid x \geq m, x \in \mathcal{G}\},$$

and set $J(m) = I(m+g)$ as in [3]. Then we extend $J(m)$ linearly to obtain a piecewise linear function on $\mathbb{R}$. That is, for $k \in \mathbb{Z}$, if $J(k) = J(k+1)$, then $J(x) = J(k)$ on $[k, k+1]$, and if $J(k+1) = J(k) - 1$, then $J(k+x) = J(k) - x$ for $0 \leq x \leq 1$. Borodzik and Hedden [3] showed that the Upsilon invariant of an L–space knot K is the Legendre–Fenchel transform, or convex conjugate, of the function $2J(-x)$. We set $G(x) = 2J(-x)$, and call it the *gap function* of K.

Example 2.1. Let $K = T(3,7)$ be the torus knot of type $(3,7)$. It admits a (positive) lens space surgery, so K is an L–space knot. Also, it has genus 6, and the Alexander polynomial $\Delta_K(t)$ is $1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}$. Hence the staircase diagram $\mathrm{St}(K)$ is specified by the sequence

$$[a_1 - a_0, a_2 - a_1, \dots, a_8 - a_7] = [1, 2, 1, 2, 2, 1, 2, 1].$$

See Figure 1.

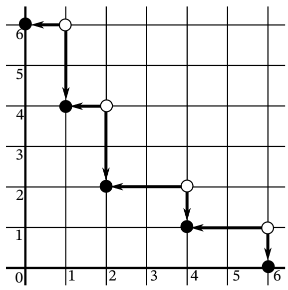

FIGURE 1. The staircase diagram of $K = T(3,7)$.

On the other hand, $\mathcal{G} = \mathbb{Z}_{<0} \cup \{1, 2, 4, 5, 8, 11\}$. Tables 1 and 2 show some values of $I(m)$ and $2J(-m)$.

TABLE 1. The values of $I(m)$ for $T(3, 7)$.

m	≥ 12	11	8	5	4	2	1	0	-1	-2	$\cdots$
$I(m)$	0	1	2	3	4	5	6	6	7	8	$\cdots$

TABLE 2. The values of $2J(-m)$ for $T(3, 7)$.

m	≤ -6	-5	-2	1	2	4	5	6	7	8	$\cdots$
$2J(-m)$	0	2	4	6	8	10	12	12	14	16	$\cdots$

Figure 2 shows the graph of the gap function $G(x) = 2J(-x)$ and its convex hull (dotted line). Here, the convex hull $f(x)$ of the gap function is given by

$$f(x) = \begin{cases} 0 & \text{for } x \leq -6, \\ \frac{2}{3}x + 4 & \text{for } -6 \leq x \leq 0, \\ \frac{4}{3}x + 4 & \text{for } 0 \leq x \leq 6, \\ 2x & \text{for } 6 \leq x. \end{cases}$$

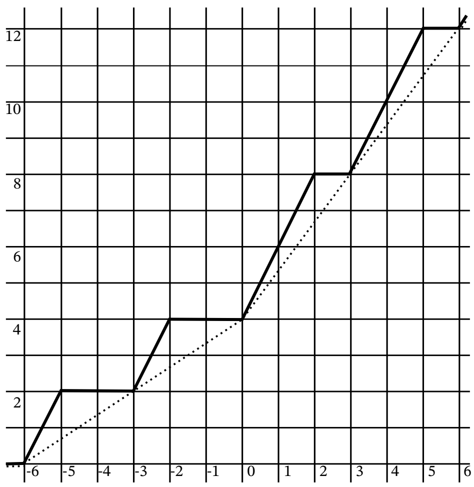

FIGURE 2. The graph of the gap function $G(x) = 2J(-x)$ for $T(3, 7)$ and its convex hull (dotted line).

Then the convex conjugate $G_*(t)$ gives the Upsilon invariant

$$\Upsilon_K(t) = \begin{cases} -6t & \text{for } 0 \le t \le \frac{2}{3}, \\ -4 & \text{for } \frac{2}{3} \le t \le \frac{4}{3}, \\ 6t-12 & \text{for } \frac{4}{3} \le t \le 2, \end{cases}$$

by [3]. Thus $t = 2/3$ and $4/3$ are the singularities of $\Upsilon'_K(t)$.

We remark that $G_*(t)$ $(= \Upsilon_K(t))$ is convex, so the convex conjugate of $G_*(t)$ returns back to the convex hull of $G(x)$.

In general, the gap function of an L–space knot has a specific property: the slope of each segment of the graph is 0 or 2. To be precise, the next lemma describes the form of the graph. Consider the vectors $\boldsymbol{u} = (1, 2)$ and $\boldsymbol{v} = (1, 0)$ on $\mathbb{R}^2$.

Lemma 2.2. *The graph of $G(x)$ restricted on $[-g, g]$ has a form of staircase specified by* (2.3)*. More precisely, we start at the point $(-g, 0)$, and move along $\boldsymbol{u}$ $a_{2n}-a_{2n-1}$ times, along $\boldsymbol{v}$ $a_{2n-1}-a_{2n-2}$ times, and so on. Finally, we reach the point $(g, 2g)$.*

Proof. Recall that

$$\Delta_K(t) = (t^{a_0} - t^{a_1}) + (t^{a_2} - t^{a_3}) + (t^{a_4} - t^{a_5}) + \cdots + (t^{a_{2n-2}} - t^{a_{2n-1}}) + t^{a_{2n}},$$

where $a_0 = 0$, $a_1 = 1$, $a_{2n-1} = 2g-1$ and $a_{2n} = 2g$. Then

$$\begin{aligned} \frac{\Delta_K(t)}{1-t} &= 1 + (t^{a_2} + t^{a_2+1} + \cdots + t^{a_3-1}) + (t^{a_4} + t^{a_4+1} + \cdots + t^{a_5-1}) + \cdots \\ &\quad + (t^{a_{2n-2}} + t^{a_{2n-2}+1} + \cdots + t^{a_{2n-1}-1}) + (t^{a_{2n}} + t^{a_{2n}+1} + \cdots) \end{aligned}$$

as a formal power series. (Here, we expand $1/(1-t) = 1 + t + t^2 + \cdots$.) Thus the gap sequence is

$$\{a_1, a_1+1, \dots, a_2-1\} \cup \{a_3, a_3+1, \dots, a_4-1\} \cup \cdots \cup \{a_{2n-1}, a_{2n-1}+1, \dots, a_{2n}-1\}.$$

We input the integers $2g, 2g-1, \dots, 1, 0$ to $I(m)$ in descending order. First, $I(2g) = I(a_{2n}) = 0$, so $J(g) = 0$. Then we have $G(-g) = 0$, which gives the starting point $(-g, 0)$ of the graph. Second, $I(2g-1) = I(a_{2n-1}) = I(a_{2n}-1) = 1$. If $m \in \{a_{2n}-1, a_{2n}-2, \dots, a_{2n-1}\}$, then $I(m)$ increases by one. This implies that $G(g-m)$ increases by two $a_{2n} - a_{2n-1}$ times.

Next, $I(m)$ does not change if $a_{2n-1} > m \ge a_{2n-2}$, so $G(m-g)$ records the same values as $G(g - a_{2n-1})$ $a_{2n-1} - a_{2n-2}$ times. Repeat this process. Finally, if $m \in \{a_2-1, a_2-2, \dots, a_1\}$, then $I(m)$ increases by one, and if $m \in \{a_1-1, a_1-2, \dots, a_0\}$ $(= \{0\})$, then $I(m)$ does not change. Since $I(a_0) = I(0) = g$, $J(-g) = g$ and $G(g) = 2g$. □

Recall that $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by $\Phi(x, y) = (x - y, 2x)$. Also, for $p \in \mathbb{R}^2$, $\Phi_1(p)$ denotes the first coordinate of $\Phi(p)$.

Lemma 2.3. *Let K be an L–space knot, and let* $\mathrm{St}(K)$ *be the staircase diagram representing* $\mathrm{CFK}^\infty(K)$*. Then* $\Phi(\mathrm{St}(K))$ *coincides with the graph of* $G(x)$ *on* $[-g, g]$.

Moreover, if S is the part of $\mathrm{St}(K)$ *between two black vertices p^- and p^+, where p^- is located higher, then $\Phi(S)$ coincides with the graph of $G(x)$ restricted on* $[\Phi_1(p^-), \Phi_1(p^+)]$.

Proof. First, $\Phi((0, g)) = (-g, 0)$ and $\Phi((g, 0)) = (g, 2g)$. A horizontal vector $(a_{2i+1} - a_{2i}, 0)$ maps to $(a_{2i+1} - a_{2i}, 2(a_{2i+1} - a_{2i}))$, and a vertical vector $-(0, a_{2i+2} - a_{2i+1})$ maps to $(a_{2i+2} - a_{2i+1}, 0)$. Thus Lemma 2.2 shows that the image of $\mathrm{St}(K)$ gives the graph of $G(x)$ on $[-g, g]$.

The second conclusion easily follows from a similar argument. □

3. Upsilon and secondary Upsilon invariants

We quickly review the Upsilon invariant along [7], and the secondary Upsilon invariant introduced by Kim and Livingston [6] (see also [1, 13]). For our purpose, we consider only L–space knots.

Let K be an L–space knot. For any $t \in [0, 2]$ and $s \in \mathbb{R}$, let the subcomplex $\mathcal{F}_{t,s} \subset \mathrm{CFK}^\infty(K)$ be the one generated by basis elements x satisfying

$$\frac{t}{2}\,\mathrm{Alex}(x) + \left(1 - \frac{t}{2}\right)\mathrm{alg}(x) \leq s$$

on the $(\mathrm{alg}, \mathrm{Alex})$–plane. The subcomplex $\mathcal{F}_{t,s}$ is represented by the half plane to the left-lower of the line $L_{t,s}$ with slope $1 - 2/t$ and the Alex–intercept $2s/t$, which is called the *support line* of $\mathcal{F}_{t,s}$.

Put

$$\gamma_K(t) = \min\{s \mid H_0(\mathcal{F}_{t,s}) \to H_0(\mathrm{CFK}^\infty(K))\,\text{is surjective}\},$$

and $\Upsilon_K(t) = -2\gamma_K(t)$. Intuitively, the line $L_{t,s}$ moves upward as s increases with a fixed t. Then $\gamma_K(t)$ outputs the first moment when this line meets $\mathrm{St}(K)$ (at some grading 0 vertex).

Fix a singularity $t \in (0, 2)$ of $\Upsilon_K(t)$. Let $L = L_{t,\gamma_K(t)}$. Then this line L meets the staircase complex $\mathrm{St}(K)$ in at least two points ([7, Theorem 8.1]). Among such intersection points, the top most point p_t^- is called the *negative pivot point*, and the bottom most point p_t^+ is called the *positive pivot point.* In Figure 1, $p_t^- = (0, 6)$ and $p_t^+ = (2, 2)$ for $t = 2/3$.

We consider the part S_t of $\mathrm{St}(K)$ between p_t^- and p_t^+. For $s \in [0, 2]$, $\gamma^2_{K,t}(s)$ is the minimum value of r such that p_t^- and p_t^+ represent the same homology class in $H_0(\mathcal{F}_{t,\gamma_K(t)} + \mathcal{F}_{s,r})$. In other words, $\gamma^2_{K,t}(s)$ is the maximum value of r such that $L_{s,r}$ meets S_t. Define $\Upsilon^2_{K,t}(s) = -2\gamma^2_{K,t}(s) - \Upsilon_K(t)$. This is the *secondary Upsilon invariant* at the singularity t. Except singularities, $\Upsilon^2_{K,t}(s)$ is identically ∞.

Example 3.1. Let $K = T(3,7)$ as in Example 2.1. First, put $t = 2/3$. Then we see

$$\gamma^2_{K,2/3}(s) = \begin{cases} s+2 & \text{if } 0 \le s \le \frac{2}{3}, \\ \frac{5}{2}s+1 & \text{if } \frac{2}{3} \le s \le 2, \end{cases}$$

from Figure 1. Since $\Upsilon_K(2/3) = -4$, we have

$$\Upsilon^2_{K,2/3}(s) = -2\gamma^2_{K,2/3}(s) - \Upsilon_K(2/3) = \begin{cases} -2s & \text{if } 0 \le s \le \frac{2}{3}, \\ -5s+2 & \text{if } \frac{2}{3} \le s \le 2. \end{cases}$$

Next, put $t = 4/3$. Then we see again

$$\gamma^2_{K,4/3}(s) = \begin{cases} -\frac{5}{2}s+6 & \text{if } 0 \le s \le \frac{4}{3}, \\ -s+4 & \text{if } \frac{4}{3} \le s \le 2, \end{cases}$$

and

$$\Upsilon^2_{K,4/3}(s) = -2\gamma^2_{K,4/3}(s) - \Upsilon_K(4/3) = \begin{cases} 5s-8 & \text{if } 0 \le s \le \frac{4}{3}, \\ 2s-4 & \text{if } \frac{4}{3} \le s \le 2. \end{cases}$$

4. Proof of Theorem 1.1

We are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Fix a singularity $t_0 \in (0,2)$ of $\Upsilon_K(t)$. Let S_{t_0} be the part of $\mathrm{St}(K)$ between the negative and positive pivot points p^- and p^+ of t_0.

Under Φ used in Section 2, S_{t_0} maps to the part of the graph of $G(x)$ restricted on the interval $[\Phi_1(p^-), \Phi_1(p^+)]$ by Lemma 2.3.

Recall that $r = \gamma^2_{K,t_0}(s)$ is determined by the line $L_{s,r}$. Since $L_{s,r}$ has slope $1 - 2/s$ and intercept $2r/s$, $\Phi(L_{s,r})$ gives the line $y = sx + 2r$ on $\mathbb{R}^2$. Hence $G^*(s) = -2r = -2\gamma^2_{K,t_0}(s)$. Thus $\Upsilon^2_{K,t_0}(s) = -2\gamma^2_{K,t_0}(s) - \Upsilon_K(t_0) = G^*(s) - \Upsilon_K(t_0)$. $\square$

By a similar argument, we can give an alternative proof that $\Upsilon_K(t)$ of an L–space knot K is the convex conjugate $G_*(t)$ of $G(x)$.

Set $s = \gamma_K(t)$. Then the line $L_{t,s}$ gives the support line for $\mathrm{St}(K)$ with slope $1-2/t$. Again, Φ maps $L_{t,s}$ to the line $y = tx+2s$. Thus $G^*(t) = -2s = \Upsilon_K(t)$.

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