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WAVELET CHARACTERIZATIONS OF VARIABLE HARDY-LORENTZ SPACES

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Abstract. In this paper, let $q \in (0,1]$. We establish the boundedness of intrinsic g-functions from the Hardy-Lorentz spaces with variable exponent $H^{p(\cdot),q}(\mathbb{R}^n)$ into Lorentz spaces with variable exponent $L^{p(\cdot),q}(\mathbb{R}^n)$. Then, for any $q \in (0,1]$, via some estimates on a discrete Littlewood-Paley g-function and a Peetre-type maximal function, we obtain several equivalent characterizations of $H^{p(\cdot),q}(\mathbb{R}^n)$ in terms of wavelets.

1. Introduction

As a generalization of $L^p(\mathbb{R}^n)$, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ was introduced by Orlicz [22] in 1930's. Lorentz spaces on $\mathbb{R}^n$ were studied by Lorentz in the early 1950's. Lorentz spaces, as generalizations of $L^p(\mathbb{R}^n)$, are known to be the intermediate spaces of Lebesgue spaces in the real interpolation method; see [1, 18]. Over the past couple of years, the study of Hardy-Lorentz spaces has always been an interesting topic. For example, the real interpolation of the Hardy-Lorentz space $H^{p,q}(\mathbb{R}^n)$ was investigated by Fefferman, Riviére, and Sagher [4]; the space $H^{1,\infty}(\mathbb{R}^n)$ was considered by Fefferman and Soria [5].

Nowadays, due to the development of variable Lebesgue spaces, there has been a lot of research on the study of Hardy spaces with variable exponents in harmonic analysis. A major breakthrough on Lebesgue spaces with variable exponent is that under some regularity assumptions on $p(\cdot)$, the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ [3]. Moreover, Nakai and Sawano [21] made a lot of progress on variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$. They established the atomic decompositions and the dual spaces of $H^{p(\cdot)}(\mathbb{R}^n)$ in [3]. Later, Sawano [23] extended the atomic characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ and improved the corresponding results in [21]. Recently, Jiao et al. [14] established some real-variable characterizations of variable Hardy-Lorentz spaces. As applications of

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the atomic decompositions, they developed a theory of real interpolation and formulated the dual space of the variable Hardy-Lorentz space with $0 < p_- \leq p_+ \leq 1$ and $0 < q < \infty$.

In the 1990s, the wavelet theory was established involving different Hardy-type spaces. Precisely, several equivalent wavelet characterizations of $H^1(\mathbb{R}^n)$ were established by Meyer [20]; some equivalent wavelet characterizations of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ were studied by Liu [19]; a wavelet area integral characterization of the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ for any $p \in (0,1]$ was established by Wu [25]; and independently, via the vector-valued Calderón-Zygmund theory, a characterization of $H^p_\omega(\mathbb{R}^n)$ for $p \in (0,1]$ in terms of wavelets without compact supports was established by García-Cuerva and Martell [10]. Later, the wavelet inequalities of Lebesgue spaces with variable exponents were introduced by Kopaliani [16] and Izuki [12] independently. In addition, the wavelet characterization for weighted Lebesgue spaces with variable exponents was established by Izuki, Nakai, and Sawano [13].

Recently, via wavelets, several equivalent characterizations of the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ were established by Fu and Yang [8]. Later, via wavelets, several equivalent characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ were established by Fu [7], which extends the wavelet characterizations of the classical Hardy space in [20, Theorems 5.1, 6.4]. In addition, when $(\mathcal{X}, d, \mu)$ is a metric measure space of homogeneous type in the sense of R. R. Coifman and G. Weiss and $H^1_{at}(\mathcal{X})$ is the atomic Hardy space, Fu and Yang [9] established several equivalent characterizations of $H^1_{at}(\mathcal{X})$ in terms of wavelets.

Motivated by the above results, especially by [8, 14], we establish several equivalent characterizations of $H^{p(\cdot),q}(\mathbb{R}^n)$ in terms of wavelets where $q \in (0,1]$.

We describe how we organize this paper. In Section 2, we first recall some known notions and notation. Then, recall the atomic characterizations of $H^{p(\cdot),q}(\mathbb{R}^n)$ from [14, Theorem 5.4] (see Lemma 2.12 below). In Section 3, for any $q \in (0,1]$, we establish the boundedness of intrinsic g-functions from the Hardy-Lorentz spaces with variable exponent $H^{p(\cdot),q}(\mathbb{R}^n)$ into Lorentz spaces with variable exponent $L^{p(\cdot),q}(\mathbb{R}^n)$ (see Theorem 3.3 below), and get some estimates on a discrete Littlewood-Paley g-function and a Peetre-type maximal function (see Propositions 3.5 and 3.6, respectively, below). In Section 4, we prove Theorem 4.1. Via the estimate on the Peetre-type maximal function, the wavelet characterizations of Lebesgue spaces from [20] and some standard arguments on the wavelet characterizations of the classical Hardy spaces, we complete the proof of Theorem 4.1.

Notation. In this paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant depending on the parameters $\alpha, \beta, \ldots$. The symbol $f \lesssim g$ means $f \leq Cg$ for a positive constant C, and $f \sim g$ amounts to $f \gtrsim g \gtrsim f$. For any $a \in \mathbb{R}$, the symbol $\lfloor a \rfloor$ denotes the largest integer m such that $m \leq a$. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$. For any

$p \in [1,\infty]$, p' denotes its conjugate number, namely, $1/p + 1/p' = 1$. For any subset E of $\mathbb{R}^n$, we use χ_E to denote its characteristic function. Moreover, $\langle \cdot, \cdot \rangle$ and $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$ represent the duality relation and the $L^2(\mathbb{R}^n)$ inner product, respectively.

2. Preliminaries

In this section, we first recall some notions and notation. For any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let

$$B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$$

denote the open ball. Let $\mathcal{A}(\mathbb{R}^n)$ be the set of all Lebesgue measurable functions on $\mathbb{R}^n$.

A measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ is called a variable exponent. Denote by $\mathcal{P}(\mathbb{R}^n)$ the collection of all variable exponents $p(\cdot)$ satisfying

$$0 < p_- := \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x) \le \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty.$$

In the following, let

$$\underline{p} = \min\{p_-, 1\}.$$

Definition 2.1 ([2, Definition 2.16])**.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the Lebesgue space with variable exponent is defined by setting

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{f \in \mathcal{A}(\mathbb{R}^n) : \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \rho_{p(\cdot)}\left(\frac{|f|}{\lambda}\right) \leqslant 1\right\},\ \rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx.$$

Definition 2.2 ([15, Definition 2.2])**.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $0 < q \le \infty$. Then the Lorentz space with variable exponent is defined by setting

$$L^{p(\cdot),q}(\mathbb{R}^n) := \left\{f \in \mathcal{A}(\mathbb{R}^n) : \|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{L^{p(\cdot)q}(\mathbb{R}^n)} := \begin{cases} \left(\int_0^\infty \lambda^q \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{d\lambda}{\lambda}\right)^{1/q}, & \text{if } 0<q<\infty,\\ \sup_{\lambda>0} \lambda \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, & \text{if } q=\infty. \end{cases}$$

Lemma 2.3 ([14, Lemma 2.8])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $0 < q \le \infty$. Then, for all $f \in L^{p(\cdot),q}(\mathbb{R}^n)$ and $s \in (0,\infty)$, it holds true that*

$$\||f|^s\|_{L^{p(\cdot),q}(\mathbb{R}^n)} = \|f\|^s_{L^{sp(\cdot),sq}(\mathbb{R}^n)}.$$

A function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the globally log-Hölder continuous condition, denoted by $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, if there exist positive constants $C_{p(\cdot)}$, C_∞ and p_∞ such that, for all $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{p(\cdot)}}{\log(e + 1/|x-y|)} \tag{2.1}$$

and

$$|p(x)-p_\infty| \le \frac{C_\infty}{\log(e+|x|)}. \tag{2.2}$$

For $r\in(0,\infty)$, we denote $L^r_{\rm loc}(\mathbb{R}^n)$ to be the set of all r-locally integrable functions on $\mathbb{R}^n$. Recall that the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by setting, for all $f\in L^1_{\rm loc}(\mathbb{R}^n)$,

$$\mathcal{M}(f)(x) := \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy, \quad x\in\mathbb{R}^n, \tag{2.3}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

We denote $S(\mathbb{R}^n)$ to be the space of all Schwartz functions and $S'(\mathbb{R}^n)$ to be its topological dual space equipped with the weak-$*$ topology. For $N\in\mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^n) := \left\{\psi\in\mathcal{S}(\mathbb{R}^n): \sum_{\beta\in\mathbb{Z}^n_+,|\beta|\le N}\sup_{x\in\mathbb{R}^n}\left[(1+|x|)^N\left|D^\beta\psi(x)\right|\right]\le 1\right\},$$

where, for any $\beta:=(\beta_1,\ldots,\beta_n)\in\mathbb{Z}^n_+$, $|\beta|=\beta_1+\cdots+\beta_n$ and $D^\beta := \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$. Then for all $f\in\mathcal{S}'(\mathbb{R}^n)$, the radial grand maximal function $f^*_{N,+}$ of f is defined by setting, for all $x\in\mathbb{R}^n$,

$$f^*_{N,+}(x) := \sup\{|f*\psi_t(x)| : t\in(0,\infty) \text{ and } \psi\in\mathcal{F}_N(\mathbb{R}^n)\},$$

where, for all $t\in(0,\infty)$ and $\xi\in\mathbb{R}^n$, $\psi_t(\xi):=t^{-n}\psi(\xi/t)$. We simply use f^* to denote $f^*_{N,+}$.

Definition 2.4 ([14, Definition 2.14])**.** Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, let $0<q\le\infty$ and let $N\in\left(\frac{n}{\underline{p}}+n+1,\infty\right)$ be a positive integer. The variable Hardy-Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$ is defined by setting

$$H^{p(\cdot),q}(\mathbb{R}^n) := \left\{f\in S'(\mathbb{R}^n) : \|f^*\|_{L^{p(\cdot),q}(\mathbb{R}^n)}<\infty\right\},$$

equipped with the quasi-norm

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} := \|f^*\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$

We denote $\mathbb{P}^s(\mathbb{R}^n)$ to be the set of all polynomials having degree at most s. For a locally integrable function f, a ball B and a nonnegative integer s, there exists a unique polynomial P such that for any polynomial $R\in\mathbb{P}^s(\mathbb{R}^n)$,

$$\int_B (f(x)-P(x))R(x)dx = 0.$$

Denote this unique polynomial P by $P^s_B f$.

Definition 2.5 ([21])**.** Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and let $1\le r<\infty$. The BMO space $BMO_{p(\cdot),r}(\mathbb{R}^n)$ is defined by setting

$$BMO_{p(\cdot),r}(\mathbb{R}^n) := \left\{f\in L^r_{\rm loc}(\mathbb{R}^n) : \|f\|_{BMO_{p(\cdot),r}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{BMO_{p(\cdot),r}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left(\frac{1}{|B|}\int_B |f(x)-P_B^s f(x)|^r\,dx\right)^{1/r},$$
$$(1\le r<\infty)$$

where $\mathcal{B}$ is the set of all balls.

Remark 2.6. Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, let d be an integer satisfying

$$d \geqslant d_{p(\cdot)} := \min\{d\in\mathbb{Z}_+ : p_-(n+1+d)>n\}, \tag{2.4}$$

and let $r\in(\max\{p_+,1\},\infty]$ and $q\in(0,1]$. By [14, Theorems 5.4 and 7.2], we find that

(i) $\left(H^{p(\cdot),q}(\mathbb{R}^n)\right)^* = BMO_{p(\cdot),r}(\mathbb{R}^n)$;
(ii) $f\in H^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f\in\left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$ and $f^*\in L^{p(\cdot),q}(\mathbb{R}^n)$.

Indeed, for any $q\in(0,1]$, if $f\in H^{p(\cdot),q}(\mathbb{R}^n)$, then, by (i), we deduce that

$$f\in\left(H^{p(\cdot),q}(\mathbb{R}^n)\right)^{**} = \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*.$$

It follows from Definition 2.4 that $f^*\in L^{p(\cdot),q}(\mathbb{R}^n)$. On the other hand, for any $q\in(0,1]$, if $f\in\left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$ and $f^*\in L^{p(\cdot),q}(\mathbb{R}^n)$, then, by [26, Lemma 2.8], we deduce that $S(\mathbb{R}^n)\subset BMO_{p(\cdot),r}(\mathbb{R}^n)$ and hence $f\in S'(\mathbb{R}^n)$, which, together with Definition 2.4, implies that $f\in H^{p(\cdot),q}(\mathbb{R}^n)$.

In the following, we recall the following notion of the multiresolution analysis on $\mathbb{R}$ (see [20, 24] for more details).

Definition 2.7. An increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces in $L^2(\mathbb{R})$ is called a multiresolution analysis (MRA) on $\mathbb{R}$ if

(i) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j=\{\theta\}$, where θ denotes the zero function;
(ii) for any $j\in\mathbb{Z}$ and $f\in L^2(\mathbb{R})$, $f\in V_j$ if and only if $f\left(2^{-j}\cdot\right)\in V_0$;
(iii) for any $k\in\mathbb{Z}$ and $f\in L^2(\mathbb{R})$, $f\in V_0$ if and only if $f(\cdot-k)\in V_0$;
(iv) there exists a function $\phi\in L^2(\mathbb{R})$ (called father wavelet) such that $\{\phi_k(\cdot)\}_{k\in\mathbb{Z}} := \{\phi(\cdot-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_0.

For any fixed $s\in\mathbb{Z}_+$, according to [24, Theorem 1.61(ii)], we choose the father and the mother wavelets $\phi,\psi\in C_c^{s+1}(\mathbb{R})$, the set of all functions with compact supports having continuous derivatives up to order $s+1$, such that $\widehat{\phi}(0)=(2\pi)^{-1/2}$ and, for any $l\in\{0,\dots,s+1\}$, $\int_{\mathbb{R}} x^l\psi(x)dx=0$, where $\widehat{\phi}$ denotes the Fourier transform of ϕ; namely, for any $\xi\in\mathbb{R}$,

$$\widehat{\phi}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\phi(y)e^{-i\xi y}dy.$$

In the following, we assume that

$$\operatorname{supp}\phi,\operatorname{supp}\psi\subset 1/2+m(-1/2,1/2), \tag{2.5}$$

namely, $x \in [(1-m)/2, (1+m)/2]$ if and only if $\left|x - \frac{1}{2}\right| \leqslant m/2$. Here $m \in [1, \infty)$ is a positive constant independent of the main parameters involved in the whole paper.

By the standard procedure of tensor products, we can extend the above considerations from 1-dimension to n-dimension. More precisely, let

$$\vec{\theta}_n := (\overbrace{0, \ldots, 0}^{n \text{ times}}) \quad \text{and} \quad E := \{0,1\}^n \backslash \left\{\vec{\theta}_n\right\}.$$

Assume that $\mathcal{D}$ is the set of all dyadic cubes in $\mathbb{R}^n$, i.e., for any $Q \in \mathcal{D}$, there exist $j \in \mathbb{Z}_+$ and $k := \{k_1, \ldots, k_n\} \in \mathbb{Z}^n$ such that

$$Q = Q_{j,k} := \left\{x \in \mathbb{R}^n : k_i \le 2^j x_i < k_i + 1 \text{ for any } i \in \{1, \ldots, n\}\right\}. \tag{2.6}$$

Let mQ be the m dilation of Q with the same center as Q and m as in (2.5). According to the tensor product in [20, p. 108], for any $\lambda := (\lambda_1, \ldots, \lambda_n) \in E$, $Q := Q_{j,k}$ with $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $j \in \mathbb{Z}$, and $x = (x_1, \ldots, x_n)$, let

$$\psi_Q^\lambda(x) := 2^{jn/2} \psi^{\lambda_1}\left(2^j x_1 - k_1\right) \cdots \psi^{\lambda_n}\left(2^j x_n - k_n\right),$$
$$\phi_Q(x) := 2^{jn/2} \phi\left(2^j x_1 - k_1\right) \cdots \phi\left(2^j x_n - k_n\right),$$

where $\psi^0 := \phi$ and $\psi^1 := \psi$.

A family $\left\{\psi_Q^\lambda\right\}_{Q \in \mathcal{D}, \lambda \in E} \subset C^{s+1}(\mathbb{R}^n)$ (the set of all functions having continuous derivatives up to order $s+1$) is called an s-order wavelet system (see [8, p. 6]) if $\left\{\psi_Q^\lambda\right\}_{Q \in \mathcal{D}, \lambda \in E}$ satisfy

(i) $\left\{\psi_Q^\lambda\right\}_{Q \in \mathcal{D}, \lambda \in E}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$;

(ii) ψ_Q^λ are compactly supported, namely,

$$\operatorname{supp} \psi_Q^\lambda \subset mQ;$$

(iii) there exists a positive constant C, depending on s, such that, for any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $|\beta| := \beta_1 + \cdots + \beta_n \le s+1$,

$$\left|\partial^\beta \psi_Q^\lambda(x)\right| \le C 2^{j|\beta|} 2^{jn/2}, \quad \forall x \in \mathbb{R}^n; \tag{2.7}$$

(iv) for any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $|\beta| \le s$, $\int_{\mathbb{R}^n} x^\beta \psi_Q^\lambda(x) dx = 0$, here and hereafter, for any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$.

See more details in [20, p. 108].

Hence, for any $f \in L^2(\mathbb{R}^n)$, we find that

$$f = \sum_{\lambda \in E} \sum_{Q \in \mathcal{D}} (f, \psi_Q^\lambda) \psi_Q^\lambda = \sum_{\lambda \in E} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^n} (f, \psi_{j,k}^\lambda) \psi_{j,k}^\lambda \quad \text{in } L^2(\mathbb{R}^n),$$

and for any $k \in \mathbb{Z}^n$, $j \in \mathbb{Z}_+$ with $Q = Q_{j,k} \in \mathcal{D}$ as in (2.6) and $\lambda \in E$,

$$\psi_{j,k}^\lambda := \psi_Q^\lambda.$$

By [20, p. 142], for any $\lambda \in E$, we assume that there exists some set $W^\lambda \subset [0,1)^n$ such that $0 < \lambda \leq \left|W^\lambda\right|$ and $c_0 \chi_{W^\lambda} \leq \left|\psi^\lambda\right|$ for some fixed positive constants γ and c_0, where

$$\psi^\lambda := \psi^\lambda_{[0,1)^n}. \tag{2.8}$$

For every $j \in \mathbb{Z}$, $\lambda \in E$, $k \in \mathbb{Z}^n$, and $Q := Q_{j,k}$, let

$$W^\lambda_{j,k} := \left\{x \in \mathbb{R}^n : 2^j x - k \in W^\lambda\right\} =: W^\lambda_Q. \tag{2.9}$$

Then, for each $\lambda \in E$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we obtain

$$W^\lambda_{j,k} \subset Q_{j,k}, \quad \left|W^\lambda_{j,k}\right| \geq \gamma \left|Q_{j,k}\right| \tag{2.10}$$

and

$$\left|\psi^\lambda_{j,k}\right| \geq c_0 \frac{\chi_{W^\lambda_{j,k}}}{\left|Q_{j,k}\right|}. \tag{2.11}$$

In the following, let

$$\Lambda := \left\{(\lambda, j, k) : \lambda \in E, (j,k) \in \mathbb{Z} \times \mathbb{Z}^n\right\}. \tag{2.12}$$

Further, for any $j \in \mathbb{Z}$, let V_j be the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{\phi_Q\}_{|Q|=2^{-jn}}$. It is known that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA on $\mathbb{R}^n$, whose definition extends MRA on $\mathbb{R}$ in Definition 2.7 (see [20, Chapter 2] for more details).

Next, we show that the wavelets belong to Campanato spaces with variable exponent.

For any $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$. Denote by $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ the class of all functions $\eta \in C^s(\mathbb{R}^n)$, the set of all functions having continuous derivatives up to order s, such that, for any $\nu \in \mathbb{Z}^n_+$, with $|\nu| \leq s$, and for any $x \in \mathbb{R}^n$,

$$|\partial^\nu \eta(x)| \leq (1+|x|)^{-n-\epsilon} \tag{2.13}$$

and, for any $\nu \in \mathbb{Z}^n_+$, with $|\nu| = s$, and for any $x_1, x_2 \in \mathbb{R}^n$,

$$|\partial^\nu \eta(x_1) - \partial^\nu \eta(x_2)| \leq |x_1 - x_2|^\alpha \left[(1+|x_1|)^{-n-\epsilon} + (1+|x_2|)^{-n-\epsilon}\right]. \tag{2.14}$$

In the following, let $\mathcal{C}_{\epsilon,s}(\mathbb{R}^n) := \mathcal{C}_{(1,\epsilon),s}(\mathbb{R}^n)$.

The following results are [7, Proposition 1 and Corollary 2], respectively.

Proposition 2.8. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $s \in \mathbb{Z}_+$.*

(i) *if $\alpha \in (0,1]$, $\varepsilon \in (\alpha + s, \infty)$ and $p_- \in (n/(n+\alpha+s), 1]$, then*

$$\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n) \subset BMO_{p(\cdot),r}(\mathbb{R}^n);$$

(ii) *if $\varepsilon \in (1+s,\infty)$ and $p_- \in (n/(n+1+s), 1]$, then*

$$\mathcal{C}_{\epsilon,s}(\mathbb{R}^n) \subset BMO_{p(\cdot),r}(\mathbb{R}^n).$$

Corollary 2.9. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $s \in \mathbb{Z}_+$ and $p_- \in (n/(n+1+s), 1]$. Then, for any $(\lambda, j, k) \in \Lambda$ with Λ as in (2.12), $\psi^\lambda_{j,k} \in BMO_{p(\cdot),r}(\mathbb{R}^n)$.*

We now recall the definition of $(p(\cdot), r, s)$-atom introduced in [21].

Definition 2.10. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $1 < r \le \infty$. Fix an integer

$$d \in \left(\frac{n}{p_-} - n - 1, \infty\right) \cap \mathbb{Z}_+. \tag{2.15}$$

A measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot), r, d)$-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$;

(ii) $\|a\|_{L^r(\mathbb{R}^n)} \le \frac{|B|^{1/r}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;

(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le d$.

We recall the notion of the variable atomic Hardy-Lorentz space, which is taken from [14, Definition 5.2].

Definition 2.11. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let $0 < q \le \infty$, $1 < r \le \infty$ and let d be as (2.15). The variable atomic Hardy-Lorentz space $H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be decomposed as

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} \quad \text{in} \quad S'(\mathbb{R}^n), \tag{2.16}$$

where $\{a_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$ is a sequence of $(p(\cdot), r, d)$-atoms, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$, satisfying that, for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \le A$ with A being a positive constant independent of x and i; and for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := \tilde{A} 2^i \left\|\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with $\tilde{A}$ being a positive constant independent of i and j. Moreover, for $f \in H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)$, we define

$$\|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)} := \inf \left(\sum_{i\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{N}} \left(\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\left\|\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{q}},$$

where the infimum is taken over all decompositions of f as (2.16).

Then, we give the atomic characterization of $H^{p(\cdot),q}(\mathbb{R}^n)$ from [14, Theorem 5.4].

Lemma 2.12. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, let $0 < q \le \infty$, let $r \in (\max\{p_+, 1\}, \infty]$ and let d be as in* (2.15). *Then $H^{p(\cdot),q}(\mathbb{R}^n) = H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)$ with equivalent quasi-norms.*

In the following, we introduce some notation, let

$$\mathcal{U}_{\psi,(\mathrm{i})} f := \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi^\lambda_{j,k} \right\rangle\right|^2 \left|\psi^\lambda_{j,k}\right|^2 \right]^{1/2},$$

$$\mathcal{U}_{\psi,(\mathrm{ii})}f := \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \frac{\chi_{Q_{j,k}}}{|Q_{j,k}|}\right]^{1/2},$$

$$\mathcal{U}_{\psi,(\mathrm{iii})}f := \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \frac{\chi_{W_{j,k}^{\lambda}}}{|Q_{j,k}|}\right]^{1/2},$$

where Λ is as in (2.12). By Corollary 2.9, we know that $\mathcal{U}_{\psi,(\mathrm{i})}f$, $\mathcal{U}_{\psi,(\mathrm{ii})}f$, and $\mathcal{U}_{\psi,(\mathrm{iii})}f$ are well defined.

Now, we recall the following definition of atoms introduced in [10, Definition 4.17].

Definition 2.13. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $p_+ < r$ and $r \in (1,\infty)$, and let ψ be the mother wavelet. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p(\cdot), r, \psi)$-atom if there exists a dyadic cube R such that

$$a = \sum_{\lambda\in E} \sum_{Q\subset R, Q\in\mathcal{D}} \left(a, \psi_Q^{\lambda}\right) \psi_Q^{\lambda}$$

supported in mR with m as in (2.5) and

$$\begin{aligned}\left\|\mathcal{U}_{\psi,(\mathrm{ii})}a\right\|_{L^r(\mathbb{R}^n)} &= \left\|\left[\sum_{\lambda\in E} \sum_{Q\subset R, Q\in\mathcal{D}} \left|\left(a, \psi_Q^{\lambda}\right)\right|^2 \frac{\chi_Q}{|Q|}\right]^{1/2}\right\|_{L^r(\mathbb{R}^n)} \\ &\leqslant \frac{|mR|^{1/r}}{\|\chi_{mR}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.\end{aligned}$$

Remark 2.14. By [8, Remark 2.15], we know that Definition 2.13 is well defined.

The following lemma is just [7, Lemma 1].

Lemma 2.15. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $p_+ < r$ and $r \in (1,\infty)$, and let ψ be the mother wavelet. If a is a $(p(\cdot), r, \psi)$-atom related to a cube R, then there exists a positive harmless constant c, independent of a, such that a/c is a $(p(\cdot), r, d)$-atom.*

Lemma 2.16 ([21, Lemma 2.4])**.** *Let $1 < u < \infty$. Suppose $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $p_- > 1$. Then there exists a positive constant C such that, for any sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$,*

$$\left\|\left(\sum_{j=1}^{\infty} [\mathcal{M}f_j]^u\right)^{1/u}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \left\|\left(\sum_{j=1}^{\infty} |f_j|^u\right)^{1/u}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator as in (2.3).

Lemma 2.17 ([14, Theorem 3.4]). *Let $1 < u < \infty$ and let $q \in (0,\infty]$. Suppose $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $p_- > 1$. Then there exists a positive constant C such that, for any sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$,*

$$\left\| \left(\sum_{j=1}^{\infty} [\mathcal{M} f_j]^u \right)^{1/u} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{1/u} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)},$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator as in (2.3).

3. Intrinsic g-function characterization of $H^{p(\cdot),q}(\mathbb{R}^n)$

In this section, firstly, we recall the definition of intrinsic g-functions from [17]. For any $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{T}_{\alpha,s}(\mathbb{R}^n)$ be the class of all functions $\eta \in C^s(\mathbb{R}^n)$ such that $\operatorname{supp} \eta \subset \{x \in \mathbb{R}^n : |x| \le 1\}$,

$$\int_{\mathbb{R}^n} \eta(x) x^{\gamma} dx = 0 \quad \text{for any } \gamma \in \mathbb{Z}_+^n \text{ with } |\gamma| \le s,$$

and there exists a positive constant C, depending on s, such that, for any $\nu \in \mathbb{Z}_+^n$, with $|\nu| = s$, and any $x_1, x_2 \in \mathbb{R}^n$,

$$|\partial^{\nu} \eta(x_1) - \partial^{\nu} \eta(x_2)| \le C |x_1 - x_2|^{\alpha}.$$

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\eta \in \mathcal{T}_{\alpha,s}(\mathbb{R}^n)} |f * \eta_t(y)|,$$

where $\eta_t(\cdot) := t^{-n} \eta(\frac{\cdot}{t})$ for any $t \in (0,\infty)$. Then the intrinsic g-function from [17] is defined by setting, for any $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) := \left\{ \int_0^{\infty} [A_{\alpha,s}(f)(x,t)]^2 \frac{dt}{t} \right\}^{1/2}.$$

Recall that, for all $f \in S'(\mathbb{R}^n)$, the Littlewood-Paley g-function is defined by setting, for all $x \in \mathbb{R}^n$,

$$g(f)(x) := \left(\int_0^{\infty} |f * \phi_t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where, $\phi \in S(\mathbb{R}^n)$ is a radial function satisfying [14, (8.1), (8.2) and (8.3)] and, for any $t \in (0,\infty)$, $\phi_t(\cdot) := \frac{1}{t^n} \phi\left(\frac{\cdot}{t}\right)$.

Recall that $f \in S'(\mathbb{R}^n)$ is said to vanish weakly at infinity if, for every $\psi \in S(\mathbb{R}^n)$, $f * \psi_t \to 0$ in $S'(\mathbb{R}^n)$ as $t \to \infty$ (see [6, p. 50]).

The following result follows from [14, Theorem 8.2].

Lemma 3.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and let $0 < q \le \infty$. Then $f \in H^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f \in S'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, for all $f \in H^{p(\cdot),q}(\mathbb{R}^n)$,*

$$C^{-1} \|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C \|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

The following lemma is a special case of [17, Proposition 3.2].

Lemma 3.2. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $r \in (1,\infty)$. Then there exists a positive constant C such that, for any $f \in \mathcal{A}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} [g_{\alpha,s}(f)(x)]^r \, dx \le C \int_{\mathbb{R}^n} |f(x)|^r dx.$$

Now we are ready to state and prove the main result of this section.

Theorem 3.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $p_+ \in (0,1]$ and $q \in (0,1]$. Suppose that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $p_- \in (n/n+\alpha+s, 1]$. Then $f \in H^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f \in \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$, the dual space of $BMO_{p(\cdot),r}(\mathbb{R}^n)$, f vanishes weakly at infinity and $g_{\alpha,s}(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$; moreover, it holds true that*

$$\frac{1}{C} \|g_{\alpha,s}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C \|g_{\alpha,s}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

Proof. Let $q \in (0,1]$, $f \in \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$ vanish weakly at infinity and $g_{\alpha,s}(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Then, by Proposition 2.8, we find that $f \in \mathcal{S}'(\mathbb{R}^n)$. Notice that, for all $x \in \mathbb{R}^n$, $g(f)(x) \lesssim g_{\alpha,s}(f)(x)$, it follows that $g(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. From this and Lemma 3.1, we find that there exists a distribution $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widetilde{f} = f$ in $\mathcal{S}'(\mathbb{R}^n)$, $\widetilde{f} \in H^{p(\cdot),q}(\mathbb{R}^n)$ and $\|\widetilde{f}\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$, which, together with [14, Lemma 8.4] and the fact that f vanishes weakly at infinity, implies that $f = \widetilde{f}$ in $\mathcal{S}'(\mathbb{R}^n)$ and hence

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \sim \|\widetilde{f}\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|g_{\alpha,s}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.$$

This finishes the proof of the sufficiency of Theorem 3.3.

It remains to prove the necessity. For $q \in (0,1]$, let $f \in H^{p(\cdot),q}(\mathbb{R}^n)$. Then, by [14, Lemma 8.4], we see that f vanishes weakly at infinity and, by Lemma 2.12 and [14, Theorem 7.2], we have $f \in \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$. By Lemma 2.12, there exist sequences of $(p(\cdot),\infty,d)$-atoms $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and nonnegative numbers $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that the series $\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j}$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$ and $\lambda_{i,j} \approx 2^i \left\|\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. For $i_0 \in \mathbb{Z}$, let

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} =: f_1 + f_2.$$

Hence, we get

$$\begin{aligned}
&\left\|\chi_{\{x\in\mathbb{R}^n: g_{\alpha,s}(f)(x)>2^{i_0}\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\lesssim\ & \left\|\chi_{\{x\in\mathbb{R}^n: g_{\alpha,s}(f_1)(x)>2^{i_0-1}\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|\chi_{\left\{x\in A_{i_0}: g_{\alpha,s}(f_2)(x)>2^{i_0-1}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&+ \left\|\chi_{\left\{x\in A_{i_0}^{\complement}: g_{\alpha,s}(f_2)(x)>2^{i_0-1}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$=: J_1 + J_2 + J_3,$$

where $A_{i_0} := \bigcup_{i=i_0}^{\infty} \bigcup_{j\in\mathbb{N}} (4B_{i,j})$ and $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ are the balls as in [14, Theorem 5.4].

Let $\delta_1, \delta_2, \delta_3$ and δ be the same as in [14, Theorem 5.4]. We first estimate J_1. It is obvious that

$$\begin{aligned} J_1 &\lesssim \left\| \chi_{\left\{x\in\mathbb{R}^n: \sum\limits_{i=-\infty}^{i_0-1} \sum\limits_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\chi_{4B_{i,j}} > 2^{i_0-2}\right\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\quad + \left\| \chi_{\left\{x\in\mathbb{R}^n: \sum\limits_{i=-\infty}^{i_0-1} \sum\limits_{j\in\mathbb{N}} \lambda_{i,j} g_{\alpha,s}(a_{i,j})(x)\chi_{(4B_{i,j})^{\complement}} > 2^{i_0-2}\right\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &=: J_{1,1} + J_{1,2}. \end{aligned}$$

We now estimate $J_{1,1}$. By Lemma 3.2 and the proof of [14, (5.8)], we obtain

$$J_{1,1} \lesssim 2^{-i_0\delta_1} \left(\sum_{i=-\infty}^{i_0-1} 2^{iq\delta_1} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}$$

and

$$\sum_{i_0=-\infty}^{\infty} 2^{i_0 q} \left\| \chi_{\{x\in\mathbb{R}^n: g_{\alpha,s}(f_1)\chi_{4B_{i,j}}(x) > 2^{i_0}\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \lesssim \|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)}. \tag{3.1}$$

Next, we deal with $J_{1,2}$. By the similar argument that used in [26, p. 1564], for all $i \in \mathbb{Z}$, $j \in \mathbb{N}$ and $x \in (4\chi_{B_{i,j}})^{\complement}$, we obtain

$$|g_{\alpha,s}(a_{i,j})(x)| \lesssim \left\|\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \left(\mathcal{M}\left(\chi_{B_{i,j}}\right)(x)\right)^{\frac{n+s+\alpha}{n}}. \tag{3.2}$$

Then by the Hölder inequality and a similar argument that used in the proof of [14, (5.10)], we get

$$J_{1,2} \lesssim 2^{-i_0\delta_2} \left(\sum_{i=-\infty}^{i_0-1} 2^{iq\delta_2} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}$$

and

$$\sum_{i_0=-\infty}^{\infty} 2^{i_0 q} \left\| \chi_{\left\{x\in\mathbb{R}^n: g_{\alpha,s}(f_1)\chi_{(4B_{i,j})^{\complement}}(x) > 2^{i_0}\right\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \lesssim \|f\|^q_{H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)}. \tag{3.3}$$

For I_2, by an argument similar to that used in the proof of [14, (5.11)], we get

$$J_2 \leq \left\|\chi_{A_{i_0}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 2^{-i_0\delta_3}\left(\sum_{i=i_0}^{\infty} 2^{i\delta_3 q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\right)^{\frac{1}{q}}$$

and

$$\sum_{i_0=-\infty}^{\infty} 2^{i_0 q}\left\|\chi_{\{x\in A_{i_0}:g_{\alpha,s}(f_2)(x)>2^{i_0}\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \lesssim \|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)}^q. \tag{3.4}$$

For J_3, by (3.2), Lemma 2.16 and an argument similar to that used in the proof of [14, (5.12)], we find that

$$J_3 \lesssim 2^{-i_0\delta}\left(\sum_{i=i_0}^{\infty} 2^{i\delta q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right\|_{L^{p(\cdot)}}^q\right)^{\frac{1}{q}}$$

and

$$\sum_{i_0=-\infty}^{\infty} 2^{i_0 q}\left\|\chi_{\left\{x\in\left(A_{i_0}\right)^{\complement}:g_{\alpha,s}(f_2)(x)>2^{i_0}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \lesssim \|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,d}(\mathbb{R}^n)}^q. \tag{3.5}$$

Finally, combining (3.1), (3.3), (3.4) and (3.5), we obtain

$$\|g_{\alpha,s}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}.$$

The proof is complete. □

Now, we recall a discrete variant of the Littlewood-Paley g-function $\widetilde{g}^\lambda(f)$ from [8]. For any $\lambda \in E$, $f \in L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, let

$$\widetilde{g}^\lambda(f)(x) := \left[\sum_{j\in\mathbb{Z}}\left|f * \psi^\lambda_{2^{-j}}(x)\right|^2\right]^{1/2},$$

where ψ^λ is as in (2.8).

Lemma 3.4 ([17, Theorem 2.6])**.** *Let $\lambda \in E$, $s \in \mathbb{Z}_+$, $\alpha \in (0,1]$, and $\varepsilon \in (\max\{s,\alpha\},\infty)$. Then there exists a positive constant C such that, for any f satisfying*

$$|f(\cdot)|(1+|\cdot|)^{-n-\varepsilon} \in L^1(\mathbb{R}^n), \tag{3.6}$$

it holds that

$$\widetilde{g}^\lambda(f)(x) \leqslant C g_{\alpha,s}(f)(x), \quad \forall x \in \mathbb{R}^n.$$

The following conclusion follows from Theorem 3.3 and Lemma 3.4, the details being omitted.

Proposition 3.5. *Let $\lambda \in E$, $s \in \mathbb{Z}_+$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $p_- \in (n/n+\alpha+s, 1]$ and let $0 < q \leq 1$. If $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widetilde{g}^\lambda(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$; moreover, there exists a positive constant $C_{(\lambda)}$, depending on λ, such that, for any $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,*

$$\left\|\widetilde{g}^\lambda(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leqslant C_{(\lambda)}\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}.$$

For any $\lambda \in E$, $j \in \mathbb{Z}$, $\nu \in (0,\infty)$, $f \in L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, we recall a variant of the Peetre type maximal functions from [8] defined by setting,

$$\psi_{j,\nu}^{\lambda,**}(f)(x) := \sup_{y\in\mathbb{R}^n} \frac{\left|f * \psi_{2^{-j}}^\lambda(x-y)\right|}{[1+2^j|y|]^\nu}.$$

From some arguments similar to those used in the proof of [10, Proposition 4.8], we obtain the following result.

Proposition 3.6. *Let $s \in \mathbb{Z}_+$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (1,\infty)$, $\nu \in (\max\{1/2, 1/p_-\}, \infty)$ and $q \in (0,1]$. Then there exists a positive constant $C_{(\lambda,\nu)}$, depending on λ and ν, such that, for any $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$,*

$$\left\|\left\{\sum_{j=-\infty}^{\infty}\left|\psi_{j,\nu}^{\lambda,**}(f)\right|^2\right\}^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leqslant C_{(\lambda,\nu)}\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}.$$

Proof. By some arguments similar to those used in the proof of [11, p. 271] in 1-dimensional case, for any $j \in \mathbb{Z}$, $f \in L^r(\mathbb{R}^n)$, $\nu \in (0,\infty)$, and $x \in \mathbb{R}^n$, we find that,

$$\psi_{j,\nu}^{\lambda,**}f(x) \lesssim \left[\mathcal{M}\left(\left|f * \psi_{2^{-j}}^\lambda\right|^{1/\nu}\right)(x)\right]^\nu$$

with the implicit positive constant depending only on λ, ν, and n.

Combining this, the Fefferman-Stein vector-valued maximal inequality, Theorem 3.3 and Lemma 3.4, for any $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\left\|\left[\sum_{j\in\mathbb{Z}}\left|\psi_{j,\nu}^{\lambda,**}(f)\right|^2\right]^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \\
\lesssim &\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\left|\psi_{2^{-j}}^\lambda * f\right|^{1/\nu}\right)\right]^{2\nu}\right\}^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \\
\sim &\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\left|\psi_{2^{-j}}^\lambda * f\right|^{1/\nu}\right)\right]^{2\nu}\right\}^{1/(2\nu)}\right\|_{L^{\nu p(\cdot),\nu q}(\mathbb{R}^n)}^{\nu}
\end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left|\psi^{\lambda}_{2^{-j}} * f\right|^2 \right\}^{1/(2\nu)} \right\|^{\nu}_{L^{\nu p(\cdot),\nu q}(\mathbb{R}^n)}$$
$$\sim \left\|\widetilde{g}^{\lambda}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$$
$$\lesssim \|g_{\alpha,s}(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}$$
$$\lesssim \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)},$$

where $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $(p\nu)_- = p_-\nu > 1$. The proof is complete. □

4. Wavelet characterizations of $H^{p(\cdot),q}(\mathbb{R}^n)$

In this section, we provide several equivalent characterizations of the variable Hardy-Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$ via wavelets.

Let d be an integer satisfying (2.4). By Remark 2.6(ii), it follows that if $f \in H^{p(\cdot),q}(\mathbb{R}^n)$ for $0 < q \le 1$, then $f \in \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$. Further, from Corollary 2.9, it follows that $\psi^{\lambda}_{j,k} \in BMO_{p(\cdot),r}(\mathbb{R}^n)$. Let $\langle\cdot,\cdot\rangle$ denote the dual relation between $BMO_{p(\cdot),r}(\mathbb{R}^n)$ and $\left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$. Hence, following an idea used in [20, p. 177], it follows that $\left\langle f, \psi^{\lambda}_{j,k}\right\rangle$ is well defined in the sense of the duality between $BMO_{p(\cdot),r}(\mathbb{R}^n)$ and $\left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$.

Now we state the main results of this section.

Theorem 4.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0,1]$ and $1 \le r < \infty$. Let d be an integer satisfying*

$$\frac{n}{n+1+d} < p_- \leqslant p_+ \leqslant 1$$

and suppose $\left\{\psi^{\lambda}_{j,k}\right\}_{(\lambda,j,k)\in\Lambda}$ is a d-order wavelet system.

For any $f \in \left(BMO_{p(\cdot),r}(\mathbb{R}^n)\right)^*$, assume that

$$f = \sum_{(\lambda,j,k)\in\Lambda} \left\langle f, \psi^{\lambda}_{j,k}\right\rangle \psi^{\lambda}_{j,k}. \tag{4.1}$$

Then the following statements are mutually equivalent:

(i) $f \in H^{p(\cdot),q}(\mathbb{R}^n)$;

(ii)
$$\|f\|_{(\mathrm{i})} := \left\| \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi^{\lambda}_{j,k}\right\rangle\right|^2 \left|\psi^{\lambda}_{j,k}\right|^2 \right]^{1/2} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty;$$

(iii)
$$\|f\|_{(\mathrm{ii})} := \left\| \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi^{\lambda}_{j,k}\right\rangle\right|^2 \frac{\chi_{Q_{j,k}}}{|Q_{j,k}|} \right]^{1/2} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty;$$

(iv)

$$\|f\|_{(\mathrm{iii})} := \left\| \left[\sum_{(\lambda,j,k)\in\Lambda} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \frac{\chi_{W_{j,k}^{\lambda}}}{|Q_{j,k}|} \right]^{1/2} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty,$$

where, for any $(\lambda, j, k) \in \Lambda$, $W_{j,k}^{\lambda} \subset Q_{j,k}$ *is as in* (2.9) *and*

$$\left|W_{j,k}^{\lambda}\right| \sim |Q_{j,k}| \tag{4.2}$$

with the implicit positive constants independent of (λ, j, k).

Moreover, all the quasi-norms $\|\cdot\|_{(\mathrm{i})}$, $\|\cdot\|_{(\mathrm{ii})}$, *and* $\|\cdot\|_{(\mathrm{iii})}$ *are equivalent to* $\|\cdot\|_{H^{p(\cdot),q}(\mathbb{R}^n)}$.

Proof. We observe that (4.2) follows from (2.10). Then, we only need to prove that (i) through (iv) of Theorem 4.1 are mutually equivalent. Indeed, we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Notice that $H^{p(\cdot),q}(\mathbb{R}^n)$ is a quasi-Banach space and $H^{p(\cdot),q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with r as in Lemma 2.12 is dense in $H^{p(\cdot),q}(\mathbb{R}^n)$. Hence, it suffices to prove that, for any $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$,

$$\left\|\mathcal{U}_{\psi,(\mathrm{i})} f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}.$$

Indeed, from the proof of [8, Theorem 1.9], for any $(\lambda, j, k) \in \Lambda$ and $f \in H^{p(\cdot),q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, we have that,

$$\left|\left(f, \psi_{j,k}^{\lambda}\right)\right| \lesssim 2^{-jn/2} \sup_{y\in Q_{j,k}^{\lambda}} \left|\widetilde{\psi}_{2^{-j}}^{\lambda} * f(y)\right|$$

with $\widetilde{\psi}(x) := \overline{\psi(-x)}$ for any $x \in \mathbb{R}^n$ and, for almost every $x \in \mathbb{R}^n$,

$$\sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \left|\psi_{j,k}^{\lambda}(x)\right|^2 \lesssim \left[\psi_{j,\nu}^{\lambda,**} f(x)\right]^2,$$

where the implicit positive constants depend only on ν, m, and n with m as in (2.5).

By some arguments similar to [7, (17)], we select $\nu \in (\max\{1/2, 1/p_-\}, \infty)$. Combining these facts and Proposition 3.6, we get

$$\begin{aligned} &\left\|\mathcal{U}_{\psi,(\mathrm{i})} f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{p_-} \\ &\lesssim \left\| \sum_{\lambda\in E} \left\{ \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \left|\psi_{j,k}^{\lambda}\right|^2 \right\}^{1/2} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{p_-} \\ &\lesssim \sum_{\lambda\in E} \left\| \left\{ \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}^n} \left|\left\langle f, \psi_{j,k}^{\lambda}\right\rangle\right|^2 \left|\psi_{j,k}^{\lambda}\right|^2 \right\}^{1/2} \right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{p_-} \end{aligned} \tag{4.3}$$

$$\lesssim \sum_{\lambda\in E}\left\|\left\{\sum_{j=-\infty}^{\infty}\left|\psi_{j,\nu}^{\lambda,**}(f)\right|^2\right\}^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^{p_-}$$
$$\lesssim \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}^{p_-},$$

which completes the proof for (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iv) It is an easy consequence of the fact (2.11). Moreover, we obtain

$$\left\|\mathcal{U}_{\psi,(\mathrm{iii})}f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \leqslant \left\|\mathcal{U}_{\psi,(\mathrm{i})}f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{4.4}$$

(iv) $\Rightarrow$ (iii) By [8, (3.3)], we have that, for any $s\in(0,\infty)$ and $(\lambda,j,k)\in\Lambda$,

$$\chi_{Q_{j,k}} \lesssim \left[\mathcal{M}\left(\chi_{W_{j,k}^{\lambda}}\right)\right]^{1/s}. \tag{4.5}$$

Moreover, choosing $\frac{1}{e}\in(\max\{1/2,1/p_-\},\infty)$, combining (4.5) and the Fefferman-Stein vector valued maximal inequality (see Lemma 2.17) with u replaced by $2/e$ and $\frac{1}{e}\in(\max\{1/2,1/p_-\},\infty)$, we get

$$\begin{aligned}
&\|f\|_{(\mathrm{ii})}\\
&\lesssim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\frac{\left|\left\langle f,\psi_{j,k}^{\lambda}\right\rangle\right|^2}{|Q_{j,k}|}\left[\mathcal{M}\left(\chi_{W_{j,k}^{\lambda}}\right)\right]^{2/e}\right\}^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\\
&\sim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left[\mathcal{M}\left(\left[\frac{\left|\left\langle f,\psi_{j,k}^{\lambda}\right\rangle\right|}{|Q_{j,k}|^{1/2}}\chi_{W_{j,k}^{\lambda}}\right]^{e}\right)\right]^{2/e}\right\}^{e/2}\right\|_{L^{p(\cdot)/e,q/e}(\mathbb{R}^n)}^{1/e}\\
&\lesssim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left[\frac{\left|\left\langle f,\psi_{j,k}^{\lambda}\right\rangle\right|}{|Q_{j,k}|^{1/2}}\chi_{W_{j,k}^{\lambda}}\right]^{2}\right\}^{e/2}\right\|_{L^{p(\cdot)/e,q/e}(\mathbb{R}^n)}^{1/e}\\
&\sim \left\|\left\{\sum_{(\lambda,j,k)\in\Lambda}\left|\left\langle f,\psi_{j,k}^{\lambda}\right\rangle\right|^2\frac{\chi_{W_{j,k}^{\lambda}}}{|Q_{j,k}|}\right\}^{1/2}\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\\
&\sim \|f\|_{(\mathrm{iii})},
\end{aligned} \tag{4.6}$$

where

$$\left(\frac{p}{e}\right)_- = \frac{1}{e}p_- > 1.$$

This shows that (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) From some arguments similar to those used in the proof of (iii) $\Rightarrow$ (i) in [8, Theorem 1.9], we only need to prove that, for any $f\in L^2(\mathbb{R}^n)$ with

$\mathcal{U}_{\psi,(\mathrm{ii})} f \in L^{p(\cdot),q}(\mathbb{R}^n)$,

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \left\|\mathcal{U}_{\psi,(\mathrm{ii})} f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{4.7}$$

For any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{U}_{\psi,(\mathrm{ii})} f \in L^{p(\cdot),q}(\mathbb{R}^n)$, we aim to show

$$f = \sum_{Q \in \mathcal{D}} \left(f, \psi_Q^\lambda\right) \psi_Q^\lambda = \sum_{k \in \mathbb{Z}} \sum_{i \in \Delta_k} b(k,i), \tag{4.8}$$

where $\{b(k,i) : k \in \mathbb{Z}, i \in \Delta_k\}$ are some multiples of $(p(\cdot), r, \psi)$-atoms with $p(\cdot)$ and r as in Lemma 2.12, and ψ is the mother wavelet, which will be determined later. For any $k \in \mathbb{Z}$, let

$$\Omega_k := \left\{x \in \mathbb{R}^n : \mathcal{U}_{\psi,(\mathrm{ii})} f(x) > 2^k\right\},$$

$$\mathcal{D}_k := \left\{Q \in \mathcal{D} : |Q \cap \Omega_k| \geqslant \frac{1}{2}|Q|, |Q \cap \Omega_{k+1}| < \frac{1}{2}|Q|\right\},$$

and $\widetilde{\mathcal{D}} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$.

From the proof of [8, Theorem 1.7], we have that, for any $Q \in \widetilde{\mathcal{D}}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{D}_k$ and, for any $f \in L^2(\mathbb{R}^n)$ such that $\mathcal{U}_{\psi,(\mathrm{ii})} f \in L^{p(\cdot),q}(\mathbb{R}^n)$, and $Q \in \mathcal{D} \backslash \widetilde{\mathcal{D}}$,

$$\left\langle f, \psi_Q^\lambda \right\rangle = 0. \tag{4.9}$$

Observe that, due to the nesting property of dyadic cubes, for any $Q \in \mathcal{D}_k$, there exists a unique maximal dyadic cube $\widetilde{Q} \in \mathcal{D}_k$ such that $Q \subset \widetilde{Q}$. Let $\left\{\widetilde{Q}_k^i \in \mathcal{D}_k : i \in \Delta_k\right\}$ be the collection of all such maximal dyadic cubes in $\mathcal{D}_k$. Then

$$\widetilde{\mathcal{D}} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k = \bigcup_{k \in \mathbb{Z}} \bigcup_{i \in \Delta_k} \left\{Q \in \mathcal{D}_k : Q \subset \widetilde{Q}_k^i\right\}.$$

By (4.8) and (4.9), we find that, for any $f \in L^2(\mathbb{R}^n)$ with $\mathcal{U}_{\psi,(\mathrm{ii})} f \in L^{p(\cdot),q}(\mathbb{R}^n)$,

$$\begin{aligned} f &= \sum_{\lambda \in E} \sum_{Q \in \mathcal{D}} \left(f, \psi_Q^\lambda\right) \psi_Q^\lambda \\ &= \sum_{\lambda \in E} \sum_{Q \in \widetilde{\mathcal{D}}} \left(f, \psi_Q^\lambda\right) \psi_Q^\lambda \\ &= \sum_{\lambda \in E} \sum_{k \in \mathbb{Z}} \sum_{i \in \Delta_k} \left\{ \sum_{\left\{Q \subset \widetilde{Q}_k^i, Q \in \mathcal{D}_k\right\}} \left(f, \psi_Q^\lambda\right) \psi_Q^\lambda \right\} \\ &=: \sum_{k \in \mathbb{Z}} \sum_{i \in \Delta_k} b(k,i), \end{aligned}$$

where, for any $k \in \mathbb{Z}$ and $i \in \Delta_k$,

$$b(k,i) := \sum_{\lambda \in E} \sum_{\left\{Q \subset \widetilde{Q}_k^i, Q \in \mathcal{D}_k\right\}} \left(f, \psi_Q^\lambda\right) \psi_Q^\lambda.$$

According to the estimate in [7, p. 754], we know that $b(k,i)$ is a multiple of $(p(\cdot),r,\psi)$-atom.

Let $m\in\mathbb{N}$ satisfy $2m\geqslant r$. According to the estimate in [7, (24)], we have

$$\left\|\mathcal{U}_{\psi,\mathrm{(ii)}}b(k,i)\right\|_{L^r(\mathbb{R}^n)}\lesssim 2^k\left|\widetilde{Q}_k^i\right|^{1/r}. \tag{4.10}$$

By some arguments similar to (4.5) and the fact that

$$\left|m\widetilde{Q}_k^i\right|\sim\left|\widetilde{Q}_k^i\right|\lesssim\left|\widetilde{Q}_k^i\cap\Omega_k\right|$$

for any $t\in(0,\infty)$, we deduce that,

$$\chi_{m\widetilde{Q}_{j,k}}\lesssim\left[\mathcal{M}\left(\chi_{\widetilde{Q}_k^i\cap\Omega_k}\right)\right]^{1/t}. \tag{4.11}$$

This, together with (4.10), the Fefferman-Stein vector-valued maximal inequality, some arguments similar to those used in the estimate of (4.6), and the disjointness of $\{\widetilde{Q}_k^i\}_{i\in\Delta_k}$, implies that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left\|\left[\sum_{i\in\Delta_k}\left(\frac{|\lambda(k,i)|}{\left\|\chi_{m\widetilde{Q}_k^i}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right)^{\underline{p}}\chi_{m\widetilde{Q}_k^i}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&=\sum_{k\in\mathbb{Z}}\left\|\left[\sum_{i\in\Delta_k}\left(\frac{\left\|\mathcal{U}_{\psi,\mathrm{(ii)}}b(k,i)\right\|_{L^r(\mathbb{R}^n)}}{\left|m\widetilde{Q}_k^i\right|^{1/r}}\right)^{\underline{p}}\chi_{m\widetilde{Q}_k^i}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&\lesssim\sum_{k\in\mathbb{Z}}\left\|\left[\sum_{i\in\Delta_k}2^{k\underline{p}}\chi_{m\widetilde{Q}_k^i}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&\lesssim\sum_{k\in\mathbb{Z}}\left\|\left[\sum_{i\in\Delta_k}2^{k\underline{p}}\chi_{\widetilde{Q}_k^i\cap\Omega_k}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&\lesssim\sum_{k\in\mathbb{Z}}\left\|\left[2^{k\underline{p}}\chi_{\Omega_k}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&\sim\sum_{k\in\mathbb{Z}}2^{kq}\left\|\chi_{\Omega_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\\
&\sim\left\|\mathcal{U}_{\psi,\mathrm{(ii)}}f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}^q.
\end{aligned}$$

Hence, by Lemmas 2.12 and 2.15, we get

$$
\begin{aligned}
&\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \\
(4.12) \qquad &\lesssim \left\{\sum_{k\in\mathbb{Z}}\left\|\left[\sum_{i\in\Delta_k}\left(\frac{|\lambda(k,i)|}{\left\|\chi_{m\widetilde{Q}_k^i}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right)^{\underline{p}}\chi_{m\widetilde{Q}_k^i}\right]^{1/\underline{p}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{q}} \\
&\lesssim \left\|\mathcal{U}_{\psi,(\mathrm{ii})}f\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.
\end{aligned}
$$

This implies (iii) $\Rightarrow$ (i).

Hence, (i)-(iv) are mutually equivalent.

Moreover, by (4.3), (4.4), (4.7), and (4.12), we find that each of $\|\cdot\|_{(\mathrm{i})}$, $\|\cdot\|_{(\mathrm{ii})}$, and $\|\cdot\|_{(\mathrm{iii})}$ is equivalent to $\|\cdot\|_{H^{p(\cdot),q}(\mathbb{R}^n)}$. This completes the proof of Theorem 4.1. □

From Theorem 4.1, we conclude the following result, see the proof of [8, Corollary 1.10] for more details.

Corollary 4.2. *Replacing the assumption* (4.1) *in Theorem 4.1 by* $f \in L^2(\mathbb{R}^n)$, *then all the conclusions in Theorem 4.1 still hold true.*

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