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USING ROTATIONALLY SYMMETRIC PLANES TO ESTABLISH TOPOLOGICAL FINITENESS OF MANIFOLDS

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Abstract. Let (M, p) denote a noncompact manifold M together with arbitrary basepoint p. In [7], Kondo-Tanaka show that (M, p) can be compared with a rotationally symmetric plane M_m in such a way that if M_m satisfies certain conditions, then M is proved to be topologically finite. We substitute Kondo-Tanaka's condition of finite total curvature of M_m with a weaker condition and show that the same conclusion can be drawn. We also use our results to show that when M_m satisfies certain conditions, then M is homeomorphic to $\mathbb{R}^n$.

1. Introduction

Let (M, p) denote a complete, noncompact Riemannian manifold M with arbitrarily chosen basepoint p. Let (M_m, o) denote a rotationally symmetric plane M_m together with its origin o, where M_m equals $\mathbb{R}^2$ equipped with a smooth, complete Riemannian metric $g_m := dr^2 + m^2(r)d\theta^2$ with $m(0) = 0$ and $m'(0) = 1$.

Let G be the sectional curvature function for M, and for any meridian $\mu(t)$ emanating from $o = \mu(0)$, let $G_m(\mu(t))$ be the curvature at $\mu(t)$. We say that (M, p) has *radial curvature bounded below* by that of M_m if, along every unit-speed minimal geodesic $\gamma : [0, a) \to M$ emanating from $p = \gamma(0)$, we have $G(\sigma_t) \geq G_m(\mu(t))$ for all $t \in [0, a)$ and all 2-dimensional subspaces σ_t spanned by $\gamma'(t)$ and an element of $T_{\gamma(t)}M$.

Given a rotationally symmetric plane M_m, we define a *sector of angular measure* δ, $V(\delta)$, as

$$V(\delta) := \{q \in M_m \,|\, 0 < \theta(q) < \delta\}.$$

Likewise we define a *closed sector of angular measure* δ, $\overline{V}(\delta)$, as

$$\overline{V}(\delta) := \{q \in M_m \,|\, 0 \leq \theta(q) \leq \delta\}.$$

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The Toponogov comparison theorem was extended in [5] to open complete manifolds with radial sectional curvature bounded below by the curvature of a von Mangoldt plane, leading to various applications in [6,8,12] and generalizations in [7,9,10].

We present below the main result of [7], which is foundational to this paper; recall that a manifold M has *finite topological type*, or is *topologically finite*, if it is homeomorphic to the interior of a compact set with boundary.

Theorem 1.1 (Main Theorem of [7])**.** *Let (M,p) be a complete open Riemannian n-manifold whose radial curvature at basepoint p is bounded below by that of a noncompact rotationally symmetric plane M_m with finite total curvature and a sector $V(\delta)$, $\delta > 0$ free of cut points. Then M is of finite topological type.*

Theorem 1.1 uses the so-called Isotopy Lemma, which is a part of the the critical point theory of distance functions by Grove-Shiohama [4], [3], [2, Lemma 3.1], [11, Section 11.1]; recall that given (M,p), a point $q \in M$ is a *critical point of* $d(\cdot,p)$ (the distance function to p) if, given any $v \in T_qM$, there exists a minimal geodesic γ emanating from q to p such that $\measuredangle(\dot\gamma(0),v) \leq \frac{\pi}{2}$.

Theorem 1.2 (Isotopy Lemma)**.** *Given (M,p), suppose that for R_1, R_2 with $0 < R_1 < R_2 \leq \infty$, $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ has no critical point of $d(\cdot,p)$. Then $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$ is homeomorphic to $\partial B_{R_1}(p) \times [R_1,R_2]$.*

The authors of [7] prove Theorem 1.1 by showing that if the conditions are satisfied, then the critical points of $d(\cdot,p)$ are confined to $B_R(p)$, $R < \infty$.

We modify Theorem 1.1 by replacing the condition of finite total curvature with the condition that $m'(r)$ be bounded. We state our result formally below.

Theorem 1.3. *Let (M,p) be a complete open Riemannian n-manifold whose radial curvature at basepoint p is bounded below by that of a noncompact rotationally symmetric plane M_m with m' bounded above and a sector $V(\delta)$, $\delta > 0$, free of cut points. Then M is of finite topological type.*

Note that the condition of m' being bounded above is more general than the condition of finite total curvature. Indeed, if M_m admits total curvature, then we have

$$c(M_m) = \int_0^{2\pi}\int_0^{\infty} G_m(r)m(r)\,dr\,d\theta = -2\pi\int_0^{\infty} m'' = 2\pi(1-m'(\infty)) \in [-\infty, 2\pi].$$

So, $c(M_m) > -\infty$ implies $m'(\infty) \in [0,\infty)$. Hence, $m'(r)$ must be bounded above on all r.

On the other hand, there exists a rotationally symmetric plane such that total curvature is not admitted but $m'(r)$ is bounded above on all r: Define $m(r)$ as $m(r) = r$ on $[0,2\pi]$ and $m(r) = r - \frac{1}{2}\sin r$ on $(2\pi,\infty)$. Next, smooth out $m(r)$ on a neighborhood σ of 2π such that $m(r) > 0$ on σ. Then $m(r)$ is a smooth function on $[0,\infty)$ that can be extended to a smooth odd function around 0 with

$m(r) > 0$ for all r, $m(0) = 0$, and $m'(0) = 1$. Hence the metric $dr^2 + m^2(r)d\theta^2$ describes a rotationally symmetric plane. Since $m'(r) = 1 - \frac{1}{2}\cos r$ does not converge to a limit as $r \to \infty$, M_m does not admit total curvature. However, $m'(r) = 1 - \frac{1}{2}\cos r$ is bounded above on all r.

The theorem below is a special case of Theorem 1.3.

Theorem 1.4. *Let the radial curvature of (M, p) be bounded below by that of M_m satisfying the following conditions*:

1) $\sup\{m' \mid r \geq 0\} = 1$.

2) *There exists a cut-point-free sector $V(\delta)$ with $\delta > \frac{\pi}{2}$.*

3) *Given any r_0, there exists $r \geq r_0$ such that $m' < 1$.*

Then if p is a critical point of infinity, then M is homeomorphic to $\mathbb{R}^n$, where n is the dimension of M.

Remark 1.5. If M_m is von Mangoldt, has nonnegative curvature, and is not isometric to $\mathbb{R}^2$, then the conditions in Theorem 1.4 are satisfied.

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Remark 1.6. This paper reflects changes in the corresponding content in the author's Ph.D thesis. The changes are in the statements and/or proofs of Theorem 1.4 and Lemmas 4.4, 4.5, and 4.7.

2. Notations, conventions, and definitions

All geodesics are parametrized by arclength. The term *segments* refers to minimizing geodesics. Let ∂_r, ∂_θ denote the vector fields dual to dr, $d\theta$ on $\mathbb{R}^2$. Given $q \neq o$, denote its polar coordinates by θ_q, r_q. Let γ_q, μ_q, τ_q denote the geodesics defined on $[0, \infty)$ that start at q in the direction of ∂_θ, ∂_r, $-\partial_r$, respectively. We refer to $\tau_q|_{(r_q,\infty)}$ as the *meridian opposite* q; note that $\tau_q(r_q) = o$. Also set $\kappa_{\gamma(t)} := \measuredangle(\dot\gamma(t), \partial_r)$.

A geodesic is called *escaping* if its image is unbounded; for example, any ray is escaping.

We write $\dot r$, $\dot\theta$, and $\dot\gamma$ for the derivatives of $r_{\gamma(t)}$, $\theta_{\gamma(t)}$, and $\gamma(t)$ by t, while m' denotes $\frac{dm}{dr}$, and proceed similarly for higher derivatives.

3. Turn angle formula for geodesics

A geodesic γ in $M_m - \{o\}$ is called *counterclockwise* if $\frac{d}{dt}\theta_{\gamma(t)} > 0$ and *clockwise* if $\frac{d}{dt}\theta_{\gamma(t)} < 0$ for some (or equivalently any) t. A geodesic in M_m is clockwise, counterclockwise, or can be extended to a geodesic through o. If

γ is clockwise, then it can be mapped to a counterclockwise geodesic by an isometric involution of M_m.

Convention. *Unless stated otherwise, any geodesic in M_m that we consider is either tangent to a meridian or counterclockwise.*

Due to this convention the Clairaut constant and the turn angle defined below are nonnegative, which will simplify notations.

For a geodesic $\gamma\colon (t_1, t_2) \to M_m$ that does not pass through o, we define the *turn angle T_γ of γ* as

$$T_\gamma := \int_\gamma d\theta = \int_{t_1}^{t_2} \dot{\theta}_{\gamma(t)} ds = \theta_{\gamma(t_2)} - \theta_{\gamma(t_1)}.$$

Note that $T_\gamma \in [0, \infty]$ as $\dot{\theta} = c/m^2 \geq 0$. Since γ is unit speed, we have $(\dot{r})^2 + m^2\dot{\theta}^2 = 1$. Combining this with $\dot{\theta} = c/m^2$ gives $\dot{r} = \operatorname{sign}(\dot{r})\sqrt{1 - \frac{c^2}{m^2}}$, which yields a useful formula for the turn angle: if γ is never tangent to a meridian or a parallel, so that $\operatorname{sign}(\dot{r}_{\gamma(t)})$ is a nonzero constant, then

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \operatorname{sign}(\dot{r}_{\gamma(t)})\, F_c(r) \quad \text{where} \quad F_c := \frac{c}{m\sqrt{m^2 - c^2}}.$$

Thus if (t_1, t_2) is the domain of γ and $r_i := r_{\gamma(t_i)}$, then

$$T_\gamma = \operatorname{sign}(\dot{r}) \int_{r_1}^{r_2} F_c(r) dr.$$

Since $c^2 \leq m^2$, this integral is finite except possibly when some r_i is in the set $\{\infty, m^{-1}(c)\}$, in which cases it converges at $r_i = \infty$ if and only if $\int_1^\infty m^{-2} dr$ converges, and converges at $m(r_i) = c$ if and only if $m'(r_i) \neq 0$.

4. Proof of Theorems 1.1 and 1.4

Below we state what is called the generalized Toponogov Comparison Theorem, developed in [7]:

Theorem 4.1. *Let the radial curvature of (M, p) be bounded below by that of M_m. Assume that M_m admits a sector $V(\delta)$ for some $\delta \in (0, \pi)$ that has no pair of cut points. Then, for every geodesic triangle $\triangle(pxy)$ in M with $\measuredangle(xpy) < \delta$, there exists a geodesic triangle $\triangle(o\tilde{x}\tilde{y})$ in $V(\delta)$ such that*

$$d(o, \tilde{x}) = d(p, x), \quad d(o, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y)$$

and that

$$\measuredangle(xpy) \geq \measuredangle(\tilde{x}o\tilde{y}), \quad \measuredangle(pxy) \geq \measuredangle(o\tilde{x}\tilde{y}), \quad \measuredangle(pyx) \geq \measuredangle(o\tilde{y}\tilde{x}).$$

Lemma 4.2 (Lemma 4.11, [7])**.** *Let the radial curvature of (M, p) be bounded below by that of M_m. Assume that M_m admits a sector $V(\delta)$ for some $\delta \in (0, \pi)$ that has no pair of cut points. If a geodesic triangle $\triangle pxy$ in M admits a geodesic triangle $\triangle o\tilde{x}\tilde{y}$ in $V(\delta)$ satisfying*

$$d(o, \tilde{x}) = d(p, x), \quad d(o, \tilde{y}) = d(p, y), \quad d(\tilde{x}, \tilde{y}) = d(x, y),$$

then

$$\measuredangle(pxy) \geq \measuredangle(o\tilde{x}\tilde{y}) \quad \textit{and} \quad \measuredangle(pyx) \geq \measuredangle(o\tilde{y}\tilde{x}).$$

Lemma 4.3 (Lemma 3.9, [1])**.** *If $\gamma : [0,\infty) \to M_m$ is a geodesic with finite turn angle, then γ is escaping.*

Note. From this point on, set $N := \sup\{m'(r) \mid r \geq 0\}$. We will always assume that $N < \infty$.

Lemma 4.4. *Given $q \in M_m$, $\gamma_q : [0,\infty) \to M_m$ has turn angle $\geq \frac{\pi}{2N}$. If there exists $r \geq r_q$ where $m' < N$, then the turn angle of $\gamma_q > \frac{\pi}{2N}$.*

Proof. If γ_q is not an escaping geodesic, then it must have infinite turn angle by Lemma 4.3. So assume γ_q is escaping. Let c be the Clairaut constant of γ_q, and let ρ be the value at which $N\rho = c = m(r_q)$. Since $N \geq m'(r)$ for all $r \geq r_q$, we have

$$\sum_{n=1}^{M} \frac{c\Delta r}{m(r_q+n\Delta r)\sqrt{m^2(r_q+n\Delta r)-c^2}} \geq \sum_{n=1}^{M} \frac{c\Delta r}{N(\rho+n\Delta r)\sqrt{(N(\rho+n\Delta r))^2-c^2}}.$$

This implies

$$T_{\gamma_q} = \int_{r_q}^{\infty} \frac{cdr}{m(r)\sqrt{m^2(r)-c^2}} \geq \int_{\rho}^{\infty} \frac{cdr}{Nr\sqrt{(Nr)^2-c^2}}.$$

Now we show that the second integral equals $\frac{\pi}{2N}$. Applying the change of variables $r := \frac{ct}{N}$, we have

$$\int_1^{\infty} \frac{c\frac{c}{N}dt}{ct\sqrt{(ct)^2-c^2}} = \int_1^{\infty} \frac{dt}{N\,t\,\sqrt{t^2-1}} = -\frac{1}{N}\operatorname{arccot}(\sqrt{t^2-1})|_1^{\infty} = \frac{\pi}{2N}.$$

It follows that if $m' < N$ for some $r \geq r_q$, then $T_{\gamma_q} > \frac{\pi}{2N}$. □

Lemma 4.5. *Given $q \in M_m$, assume that there exists a sector $V(\delta)$ free of cut points. If σ is a ray emanting from q such that $\kappa_\sigma \geq \frac{\pi}{2}$, then $T_\sigma \geq \min(\frac{\pi}{2N}, \delta)$. If furthermore $m' < N$ for some $r \geq r_q$ and $\delta > \frac{\pi}{2N}$, then $T_\sigma > \frac{\pi}{2N}$.*

Proof. If γ_q is not escaping, then it has infinite turn angle by Lemma 4.3. If γ_q is escaping, then $T_{\gamma_q} \geq \frac{\pi}{2N}$ by Lemma 4.4. Choose $\epsilon < \min(\frac{\pi}{2N}, \delta)$ and assume $q \in \partial\overline{V}(\epsilon)$. Now γ_q and $\overline{V}(\epsilon)$ determine a bounded region. For small $t > 0$, because $\kappa_\sigma \geq \frac{\pi}{2}$, $\sigma(t)$ lies in this region. In order for σ to escape this region, either $T_\sigma > \epsilon$ or it must intersect γ_q within $\overline{V}(\epsilon)$. But the latter is impossible, so $T_\sigma > \epsilon$. Since ϵ was arbitrary, we have $T_\sigma \geq \min(\frac{\pi}{2N}, \delta)$.

Suppose $m' < N$ for some $r \geq r_q$ and $\delta > \frac{\pi}{2N}$. Then $T_{\gamma_q} > \frac{\pi}{2N}$ by Lemma 4.4. Hence, γ_q and $\overline{V}(\frac{\pi}{2N})$ determine a bounded region, and for small $t > 0$, because $\kappa_\sigma \geq \frac{\pi}{2}$, $\sigma(t)$ lies in this region. In order for σ to escape this region, either $T_\sigma > \frac{\pi}{2N}$ or it must intersect γ_q within $\overline{V}(\frac{\pi}{2N})$. But the latter is impossible. □

Lemma 4.6. *Let the radial curvature of (M, p) be bounded below by that of M_m with a cut-point-free sector $V(\delta)$, let q be a critical point of $d(\cdot, p)$, and let $\gamma : [0, \infty) \to M$ be a ray emanating from p. Let α be a minimal geodesic connecting $p = \alpha(0)$ to q such that $\measuredangle(\dot\gamma(0), \dot\alpha(0)) =: \theta < \delta$. Then there exists a ray $\tilde\eta \subset M_m$ with $T_{\tilde\eta} \le \theta$ and $\kappa_{\tilde\eta} \ge \frac{\pi}{2}$.*

Proof. If q is a critical point of $d(\cdot, p)$, then we can always construct a triangle $\subset M$ with q a vertex and one of the sides $\subset \gamma$. Note that γ cannot pass through q; indeed, if it did, then $\gamma|_{[0,d(p,q)]}$ would be the only minimal geodesic joining q to p, which is impossible since q is a critical point of $d(\cdot, p)$.

Let η_j be a minimal geodesic joining q to $\gamma(t_j)$, where $t_j \to \infty$ as $j \to \infty$. Consider the sequence of triangles $\triangle(pq\gamma(t_j))$, consisting of edges α, η_j, and $\gamma|_{[0,t_j]}$. Since $\measuredangle(qp\gamma(t_j)) = \theta$ for each j, the generalized Toponogov theorem implies that there exists a sequence of comparison triangles $\triangle o\tilde q\tilde\gamma(t_j) \subset M_m$ with corresponding sides (all minimal geodesics) of equal length and corresponding angles dominated by those in $\triangle pq\gamma(t_j)$. In particular, $\triangle o\tilde q\tilde\gamma(t_j) \subset \overline{V}(\theta)$.

Since $\ell(\eta_j) \to \infty$ as $j \to \infty$, we have $\ell(\tilde\eta_j) \to \infty$ as $j \to \infty$. Hence $\{\tilde\eta_j\}$ must subconverge to a ray $\tilde\eta$. Since $T_{\tilde\eta_j} \le \theta$ for each j, we have $T_{\tilde\eta} \le \theta$.

Since q is a critical point of $d(\cdot, p)$, there exists a minimal geodesic σ emanating from p to q such that $\measuredangle(-\dot\sigma(d(p, q)), \dot\eta_j(0)) \le \frac{\pi}{2}$. Let $\triangle p\sigma(d(p, q))\gamma(t_j)$ denote the triangle consisting of the edges σ, η_j, and $\gamma|_{[0,t_j]}$. Since

$$\triangle p\sigma(d(p, q))\gamma(t_j)$$

has the same side lengths as $\triangle pq\gamma(t_j)$ (with edges α, η_j, and $\gamma|_{[0,t_j]}$), it admits the triangle $\triangle o\tilde q\tilde\gamma(t_j)$ satisfying the angle inequalities in Lemma 4.2. In particular, $\measuredangle(o\tilde q\tilde\gamma(t_j)) \le \measuredangle(-\dot\sigma(d(p, q)), \dot\eta_j(0)) \le \frac{\pi}{2}$. Since the segment joining o to $\tilde q$ is a subarc of a meridian, we have $\kappa_{\tilde\eta_j} \ge \frac{\pi}{2}$ for each j. Hence, in the limit, $\kappa_{\tilde\eta} \ge \frac{\pi}{2}$. □

Lemma 4.7. *Let the radial curvature of (M, p) be bounded below by that of M_m with $V(\delta)$ free of cut points, let q be a critical point of $d(\cdot, p)$, let γ be a ray emanating from p, and let α be a minimal geodesic joining $p = \alpha(0)$ to q. Then $\measuredangle(\dot\gamma(0), \dot\alpha(0)) \ge \min(\frac{\pi}{2N}, \delta)$. Furthermore, if $m' < N$ for some $r \ge r_{\tilde q}$, where $\tilde q \in M_m$ satisfies $d(p, q) = d(o, \tilde q)$, and if $\delta > \frac{\pi}{2N}$, then $\measuredangle(\dot\gamma(0), \dot\alpha(0)) > \frac{\pi}{2N}$.*

Proof. Suppose $\measuredangle(\dot\gamma(0), \dot\alpha(0)) < \min(\frac{\pi}{2N}, \delta)$. Lemma 4.6 implies that there exists a ray $\tilde\eta \subset M_m$ with $T_{\tilde\eta} < \min(\frac{\pi}{2N}, \delta)$ and $\kappa_{\tilde\eta} \ge \frac{\pi}{2N}$. But Lemma 4.5 implies $T_{\tilde\eta} \ge \min(\frac{\pi}{2N}, \delta)$, a contradiction.

Now suppose $m' < N$ for some $r \ge r_{\tilde q}$ and $\delta > \frac{\pi}{2N}$, and assume $\measuredangle(\dot\gamma(0), \dot\alpha(0)) \le \frac{\pi}{2N}$. Lemma 4.6 implies that there exists a ray $\tilde\eta \subset M_m$ with $T_{\tilde\eta} \le \frac{\pi}{2N}$. But Lemma 4.5 implies $T_{\tilde\eta} > \frac{\pi}{2N}$, a contradiction. □

Proof of Theorem 1.3. We prove the claim by showing that $\{q_i\}$, the set of critical points of $d(\cdot, p)$, is bounded. Suppose the set is unbounded. Let α_i be a minimal geodesic emanating from p to q_i. Since $\ell(\alpha_i) \to \infty$, $\{\alpha_i\}$ must subconverge to a ray γ emanating from p. In particular, there exists α_i such that $\measuredangle(\dot{\gamma}(0), \dot{\alpha}_i(0)) < \min(\delta, \frac{\pi}{2N})$. But this is impossible by Lemma 4.7. □

Proof of Theorem 1.4. We prove the claim by showing that M has no critical point of $d(\cdot, p)$. Suppose q were a critical point of $d(\cdot, p)$, and let α be a minimal geodesic joining $p = \alpha(0)$ to q. For any ray γ emanating from $p = \gamma(0)$, Lemma 4.7 implies $\measuredangle(\dot{\alpha}(0), \dot{\gamma}(0)) > \frac{\pi}{2N} = \frac{\pi}{2}$, the last equality holding because the conditions of the theorem give $N = 1$. But since p is a critical point of infinity, $\measuredangle(\dot{\alpha}(0), \dot{\gamma}(0)) \leq \frac{\pi}{2}$ for some ray γ emanating from p, a contradiction. □

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