A GENERALIZATION OF THE SYMMETRY PROPERTY OF A RING VIA ITS ENDOMORPHISM

FATMA KAYNARCA AND HALISE MELIS TEKIN AKCIN

Abstract. Lambek introduced the concept of symmetric rings to expand the commutative ideal theory to noncommutative rings. In this study, we propose an extension of symmetric rings called strongly $\alpha$-symmetric rings, which serves as both a generalization of strongly symmetric rings and an extension of symmetric rings. We define a ring $R$ as strongly $\alpha$-symmetric if the skew polynomial ring $R[x;\alpha]$ is symmetric. Consequently, we provide proofs for previously established outcomes regarding symmetric and strongly symmetric rings, directly derived from the results we have obtained. Furthermore, we explore various properties and extensions of strongly $\alpha$-symmetric rings.

1. Introduction

Let $R$ be an associative ring with identity and $\alpha$ be a non-zero and non-identity endomorphism of $R$. We denote the polynomial ring with an indeterminate $x$ over $R$ by $R[x]$ and the degree of $f(x) \in R[x]$ by the notation $\deg f$.

Recall that a ring $R$ is called reduced if it has no nonzero nilpotent elements. Reversible rings are defined as a generalization of commutative rings by Cohn [4] as follows: A ring $R$ is called reversible if whenever $a, b \in R$ satisfies $ab = 0$, then $ba = 0$. These classes of rings have also found application in one of the most famous conjectures in ring theory known as K"othe's Conjecture. Cohn [4] showed that reversible rings satisfy K"othe's Conjecture.

According to Lambek [23], a ring $R$ is called symmetric if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. It is obvious that a commutative ring $R$ is symmetric. Also, if $R$ is a symmetric ring, then it is reversible. But, in general, the other aspects need not have to be satisfied (see [24, Example 5] and [1, Example II.5]).
In [19], Krempa introduced rigid endomorphisms. An endomorphism $\alpha$ of a ring $R$ is called rigid if $ao(a) = 0$ implies $a = 0$, where $a \in R$. For a ring $R$ if there exists a rigid endomorphism $\alpha$, then $R$ is called $\alpha$-rigid. It is well-known that any rigid endomorphism of a ring is a monomorphism and it is showed in [9, Proposition 5] that $\alpha$-rigid rings are reduced.

Armendariz rings are defined by Rege and Chhawchharia in [29] as a generalization of reduced rings. Let $p(x) = p_0 + p_1 x + \cdots + p_m x^m$ and $q(x) = q_0 + q_1 x + \cdots + q_n x^n \in R[x]$. If $p(x)q(x) = 0$ implies $p_i q_j = 0$ for each $i$ and $j$, then $R$ is called an Armendariz ring. Also, note that Armendariz rings can be used to establish an association between the annihilators of $R$ and its polynomial extension $R[x]$.

According to [7], a skew polynomial ring over a coefficient ring $R$ (also called an Ore extension of endomorphism type) is defined as the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \alpha(r)x$, where $r \in R$ and it is denoted by $R[x; \alpha]$. This property makes the study of Ore extensions of endomorphism type more difficult than that of the polynomial rings. Note that for any skew polynomial ring $R[x; \alpha]$ of $R$ we have $\alpha(1) = 1$, since $1 \cdot x = x \cdot 1 = \alpha(1)x$.

Armendariz property of a ring is extended to skew polynomial rings by considering the polynomials in $R[x; \alpha]$ instead of $R[x]$. $R$ is called $\alpha$-Armendariz (resp., $\alpha$-skew Armendariz) if for $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j$ in $R[x; \alpha]$, $p(x)q(x) = 0$ implies $p_i q_j = 0$ (resp., $p_i \alpha^j(q_j) = 0$) for all $i$, $j$ (see [10] and [13] for more details).

In [18, Example 2.1], Kim and Lee showed that polynomial rings over reversible rings need not be reversible. Following [31], Yang and Liu consider reversible rings over which polynomial rings are reversible and called them strongly reversible. According to Bell [3], a one-sided ideal $I$ of a ring $R$ is said to have the insertion-of factors-principle (or simply IFP), if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. A ring $R$ is called an IFP ring if the zero ideal of $R$ has the IFP. Also, note that polynomial rings over IFP rings need not be IFP by [15, Example 2]. Following [21], Kwak et al. called a ring $R$ strongly IFP if $R[x]$ has IFP. Huh et al. proved in [14, Example 3.1] that polynomial rings over symmetric rings need not be symmetric. In [6], Eltiyeb and Ayoub investigated strongly symmetric rings over which polynomial rings are symmetric and a ring $R$ is called strongly symmetric if whenever polynomials $p(x), q(x), r(x)$ in $R[x]$ satisfy $p(x)q(x)r(x) = 0$, then $p(x)r(x)q(x) = 0$. Recall that a ring $R$ is called abelian if every idempotent element of $R$ is central.

Another approach to generalize reversible and IFP properties is obtained by considering the properties on Ore extensions of endomorphism type. Following Jin et al. [16], a ring $R$ is called strongly $\alpha$-skew reversible if the skew polynomial ring $R[x; \alpha]$ is reversible and in [2], Başer et al. called a ring $R$ strongly $\alpha$-IFP if the skew polynomial ring $R[x; \alpha]$ has IFP.
Motivated by the above, in this paper, we introduce a new class of rings which is called strongly $\alpha$-symmetric ring to extend the symmetry property on skew polynomials. The following diagram describes all known implications. Also, note that no other implications in the diagram hold in general.

In Section 2, we examine the relationships between several classes of rings and strongly $\alpha$-symmetric rings and prove some statements about the links given in the above diagram. We also provide some examples of strongly $\alpha$-symmetric rings and counterexamples to several naturally raised situations.

In Section 3, as suggested by the literature, there is a considerable interest whether strongly $\alpha$-symmetric property is preserved under extensions. For this aim, we examine whether some ring extensions over strongly $\alpha$-symmetric ring $R$ again possess this property, where $\alpha$ is an endomorphism of the ring $R$. With this generalization, several known results relating to symmetric rings can be obtained as corollaries of our results.

2. Properties of strongly $\alpha$-symmetric rings

In this section, we introduce the concept of a strongly $\alpha$-symmetric ring for an endomorphism $\alpha$ and investigate its ring-theoretical properties. Firstly, we begin with the following example which illustrates the need to introduce the symmetric property of skew polynomial rings.

Example 2.1. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Since $R$ is reduced, $R$ is symmetric. Let $\alpha : R \rightarrow R$ be an endomorphism of $R$ defined by $\alpha((a, b)) = (b, a)$. If we consider $p(x) = (1, 1)$, $q(x) = (1, 0)$ and $r(x) = (0, 1)x$ in $R[x; \alpha]$, then we have $p(x)q(x)r(x) = 0$. On the other hand, $p(x)r(x)q(x) = (1, 1)(0, 1)x(1, 0) = (1, 1)(0, 1)\alpha((1, 0))x = (0, 1)x \neq (0, 0)$. Hence, $R[x; \alpha]$ is not symmetric.

Inspired by this example, we give the following definition.

Definition. Let $R$ be a ring and $\alpha$ be an endomorphism of $R$. Then $R$ is called strongly $\alpha$-symmetric if $R[x; \alpha]$ is symmetric.

Every strongly $\alpha$-symmetric ring is symmetric, but the converse is not true by Example 2.1. Any $\alpha$-rigid ring $R$ (i.e., $R[x; \alpha]$ is reduced) is clearly strongly
α-symmetric. However, there exists a strongly α-symmetric ring which is not α-rigid by [25, Example 3.4]. It is clear that any domain $R$ with a monomorphism $\alpha$ is strongly α-symmetric since $R$ is α-rigid. Note that every subring $S$ of a strongly α-symmetric ring with $\alpha(S) \subseteq S$ is also strongly α-symmetric. Any strongly α-symmetric ring is clearly strongly α-IFP, but the converse may not be true.

**Example 2.2.** Let $R = \mathbb{Z}[x]$. Consider the endomorphism

$$\alpha : R \rightarrow R$$

defined by $\alpha(p(x)) = p(0)$, where $p(x) \in \mathbb{Z}[x]$. Then by [2, Example 2.3], we know that $R$ is strongly α-IFP. On the other hand, for the polynomials $p(y) = 1 + x$, $q(y) = xy$ and $r(y) = x \in \mathbb{Z}[x][y; \alpha]$ we have $p(y)q(y)r(y) = 0$, but $p(y)r(y)q(y) \neq 0$. Therefore, $R$ is not strongly α-symmetric.

Following Kwak [20, Definition 2.1], an endomorphism $\alpha$ of a ring $R$ is called right (resp., left) symmetric if whenever $abc = 0$ for $a, b, c \in R$, then $ac\alpha(b) = 0$ (resp., $\alpha(b)ac = 0$). A ring $R$ is called right (resp., left) α-symmetric if there exists a right (resp., left) symmetric endomorphism $\alpha$ of $R$. $R$ is called α-symmetric if it is both right and left α-symmetric.

**Proposition 2.3.** Let $R$ be a ring with an endomorphism $\alpha$. If $R$ is strongly α-symmetric, then $R$ is both symmetric and α-symmetric.

**Proof.** Let $R$ be a strongly α-symmetric ring. Then it is clear that $R$ is symmetric. Assume now that $abc = 0$, where $a, b, c \in R$ and consider the polynomials $p(x) = a, q(x) = b$ and $r(x) = cx$ in $R[x; \alpha]$. Then $p(x)q(x)r(x) = 0$. Since $R$ is strongly α-symmetric, we have $p(x)r(x)q(x) = 0$ and so, $ac\alpha(b) = 0$. Hence, we obtain that $R$ is right α-symmetric. Moreover, we have $\alpha(b)ac = 0$ since $R$ is reversible. Therefore, $R$ is left α-symmetric. \(\square\)

Let $R$ be a ring with an endomorphism $\alpha$. Then by [11, Definition 2.1], $R$ is called α-skew quasi Armendariz if whenever $p(x)R[x; \alpha]q(x) = 0$ for $p(x) = \sum_{i=0}^{n}p_{i}x^{i}$ and $q(x) = \sum_{j=0}^{n}q_{j}x^{j} \in R[x; \alpha]$, then $p_{i}R\sigma^{i}(q_{j}) = 0$ for all $i, j$. Also, note that any α-skew Armendariz ring is α-skew quasi Armendariz, when $\alpha$ is an epimorphism by [11]. But, α-skew quasi Armendariz rings need not be α-skew Armendariz even if $\alpha$ is an automorphism by [11, Example 2.2(1)]. In the following theorem, our aim is to show that over strongly α-symmetric rings these concepts are equivalent.

**Theorem 2.4.** Let $R$ be a ring with an endomorphism $\alpha$. If $R$ is strongly α-symmetric, then $R$ is α-skew Armendariz if and only if $R$ is α-skew quasi Armendariz.

**Proof.** It is enough to show that $R$ is α-skew Armendariz when $R$ is α-skew quasi Armendariz. Let $p(x)q(x) = 0$, where $p(x) = \sum_{i=0}^{n}p_{i}x^{i}$ and $q(x) = \sum_{j=0}^{n}q_{j}x^{j} \in R[x; \alpha]$. Then $p(x)q(x)r(x) = 0$ for each $r(x) \in R[x; \alpha]$. By the assumption, we have $p(x)r(x)q(x) = 0$ for each $r(x) \in R[x; \alpha]$. Hence,
p(x)R[x; α]q(x) = 0 and since R is α-skew quasi Armendariz, \( p_iRα(q_j) = 0 \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Therefore, \( p_iα(q_j) = 0 \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \) as required.

Note that for a ring \( R \) being “α-skew Armendariz” and “strongly α-symmetric” are independent of each other by the following examples.

**Example 2.5.** (1) Consider \( R = \mathbb{Z}_2[x] \) with an the endomorphism \( α \), where \( α \) is defined by \( α(p(x)) = p(0) \) for \( p(x) \in \mathbb{Z}_2[x] \). Then \( R \) is α-skew Armendariz by [10, Example 5], but is not strongly \( α \)-symmetric. Indeed, for \( p(y) = 1 \), \( q(y) = (\bar{1} + x)y \), \( r(y) = x \in \mathbb{Z}_2[x][y; α] \), we have \( p(y)q(y)r(y) = 0 \) and \( p(y)r(y)q(y) = x(\bar{1} + x)y \neq 0 \).

(2) Let \( S = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{pmatrix} \mid \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\} \). By [10, Example 14], \( S \) is not \( I_\mathbb{S} \)-skew Armendariz, where \( I_\mathbb{S} \) denotes the identity map of \( S \). On the other hand, it can be seen that \( S \) is strongly \( I_\mathbb{S} \)-symmetric.

**Theorem 2.6.** Let \( R \) be an \( α \)-Armendariz ring. Then the followings are equivalent:

1. \( R \) is right \( α \)-symmetric.
2. \( R \) is symmetric.
3. \( R \) is strongly \( α \)-symmetric.

**Proof.** (1) ⇒ (2) Assume that \( R \) is right \( α \)-symmetric. Let \( a, b, c \in R \) such that \( abc = 0 \). Since \( R \) is right \( α \)-symmetric, we have \( acα(b) = 0 \). Then \( acb = 0 \) by [13, Proposition 1.3(ii)] since \( R \) is \( α \)-Armendariz. Hence, \( R \) is symmetric.

(2) ⇒ (3) It is clear by [13, Theorem 3.6(i)] since \( R \) is \( α \)-Armendariz.

(3) ⇒ (1) Assume that \( R \) is strongly \( α \)-symmetric and \( abc = 0 \), where \( a, b, c \in R \). Let \( p(x) = a, q(x) = b \) and \( r(x) = cx \) in \( R[x; α] \). Then \( p(x)q(x)r(x) = 0 \) and by the assumption, we get \( p(x)r(x)q(x) = 0 \). Therefore, \( acα(b) = 0 \) as required.

Note that the condition of \( R \) being “\( α \)-Armendariz” in Theorem 2.6 is not superfluous by Example 2.5(1). In [13, Example 1.9], it is proved that \( R = \mathbb{Z}_2[x] \) is not \( α \)-Armendariz. Since \( R \) is a commutative domain, then \( R \) is symmetric and right \( α \)-symmetric for any endomorphism.

**Corollary 2.7 ([14, Proposition 3.4]).** Let \( R \) be an Armendariz ring. Then \( R \) is symmetric if and only if \( R[x] \) is symmetric.

The following lemma can be seen by using [16, Lemma 2.3], since any strongly \( α \)-symmetric ring is strongly \( α \)-skew reversible. For the sake of completeness, we include the statements.

**Lemma 2.8.** Let \( R \) be a strongly \( α \)-symmetric ring. Then we have the following results:

1. \( R \) is symmetric.
2. \( α \) is a monomorphism.
(3) For any \( a, b \in R \) and nonnegative integer \( m \) and \( n \), we have \( a\alpha^m(b) = 0 \Leftrightarrow ab = 0 \Leftrightarrow ba = 0 \Leftrightarrow \alpha^m(b)\alpha^n(a) = 0 \Leftrightarrow \alpha^n(a)\alpha^m(b) = 0 \).

(4) \( R \) is abelian and \( \alpha(e) = e \) for any \( e^2 = e \in R \).

Following [8], a ring \( R \) with an endomorphism \( \alpha \) is called \( \alpha \)-compatible if for each \( a, b \in R \), \( ab = 0 \Leftrightarrow a\alpha(b) = 0 \). It is a well-known fact that if \( R \) is a \( \alpha \)-compatible ring, then \( \alpha \) is a monomorphism.

**Lemma 2.9.** Let \( R \) be a ring with an endomorphism \( \alpha \). If \( R \) is strongly \( \alpha \)-symmetric, then \( R \) is \( \alpha \)-compatible.

**Proof.** If \( R \) is strongly \( \alpha \)-symmetric, then \( R \) is strongly \( \alpha \)-skew reversible and by using [16, Corollary 2.4(1)], we obtain that \( R \) is \( \alpha \)-compatible. \( \square \)

On the other hand, \( \alpha \)-compatible rings need not be strongly \( \alpha \)-symmetric by [16, Example 2.11]. In the following theorem, we show the relation between \( \alpha \)-compatible rings and strongly \( \alpha \)-symmetric rings.

**Theorem 2.10.** Let \( R \) be a ring with an endomorphism \( \alpha \). Assume that \( R \) is \( \alpha \)-skew Armendariz. Then \( R \) is strongly \( \alpha \)-symmetric if and only if \( R \) is symmetric and \( \alpha \)-compatible.

**Proof.** By using Lemma 2.9 and the fact that strongly \( \alpha \)-symmetric property is inherited by its subrings, it suffices to show the necessity. Suppose that \( R \) is symmetric and \( \alpha \)-compatible. Let \( p(x)q(x)r(x) = 0 \), where \( p(x) = \sum_{i=0}^{m} p_i x^i \), \( q(x) = \sum_{j=0}^{n} q_j x^j \) and \( r(x) = \sum_{k=0}^{l} r_k x^k \) in \( R[x; \alpha] \). Then we have \( p_i \alpha^i(q_j)\alpha^{i+j}(r_k) = 0 \) for all \( 0 \leq i \leq m, 0 \leq j \leq n \) and \( 0 \leq k \leq l \). Since \( R \) is \( \alpha \)-compatible, we obtain \( p_i q_j r_k = 0 \). By symmetric property, we get \( p_i r_k q_j = 0 \). Hence, \( p_i \alpha^i(r_k)\alpha^{i+j}(q_j) = 0 \) for all \( 0 \leq i \leq m, 0 \leq j \leq n, 0 \leq k \leq l \) and this implies that \( p(x)r(x)q(x) = 0 \). \( \square \)

Note that the statements of \( R \) being “an \( \alpha \)-compatible ring” and “an \( \alpha \)-skew Armendariz ring” in Theorem 2.10 can not be dropped by the following examples.

**Example 2.11.** Consider the polynomial ring \( R = \mathbb{Z}_2[x] \) and the endomorphism \( \alpha : R \to R \) defined by \( \alpha(p(x)) = p(0) \), where \( p(x) \in \mathbb{Z}_2[x] \). Since \( R \) is a domain, it is symmetric. Also by [10, Example 5], \( R \) is \( \alpha \)-skew Armendariz. On the other hand, \( R \) is not strongly \( \alpha \)-symmetric by Example 2.5(1) and since \( \alpha \) is not a monomorphism, we get that \( R \) is not \( \alpha \)-compatible.

**Example 2.12.** Inspired by [14, Example 3.1], let us construct a ring \( R \) as follows:

Let \( A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c] \) be the free algebra of polynomials with noncommuting indeterminates \( a_0, a_1, a_2, b_0, b_1, b_2, c \) over \( \mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) denotes the ring of integers modulo 2. Let \( \alpha \) be an automorphism of \( A \) defined by:

\[ a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c, \]

where \( \mathbb{Z}_2 \) is the integers modulo 2. Let \( \alpha \) be an automorphism of \( A \) defined by:

\[ a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c, \]

and \( \mathbb{Z}_2 \) as the integers modulo 2.
respectively. Let $B$ be the set that consists of all polynomials in $A$ with zero constant terms and $I$ be the ideal of $A$ generated by
\begin{align*}
& a_0b_0, b_0a_0, a_2b_2, b_2a_2, a_0a_0, a_2a_2, b_0b_0, b_2b_2 \\
& a_0r_b_0, b_0a_0, a_2r_2b_2, b_2r_2a_2, a_0r_a_0, a_2r_a_2, b_0r_b_0, b_2r_b_2, r_1r_2r_3r_4 \\
& a_0b_0 + a_1b_0, b_0a_1 + b_0a_0, a_2b_2 + a_2b_1, b_1a_2 + b_2a_1, a_0a_0 + a_1a_0, b_0b_1 + b_1b_0 \\
& a_1a_2 + a_2a_1, b_1b_2 + b_2b_1, a_0a_2 + a_1b_1 + a_2b_0, b_0a_2 + b_1a_1 + a_2b_0, a_0a_2 + a_1a_1 + a_2a_0 \\
& b_0b_2 + b_1b_1 + b_2b, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (a_0 + b_1 + b_2)r(a_0 + a_1 + a_2) \\
& (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2)
\end{align*}
for $r,r_1,r_2,r_3,r_4 \in B$. It can be seen that $B^I \subseteq I$. Now set $R = A/I$. Since $\alpha(I) \subseteq I$, an automorphism $\alpha$ of $R$ can be induced by defining $\alpha(a + I) = \alpha(a) + I$, where $a \in A$. Also, note that $\alpha^2 = I_R$. By [16, Example 2.5], $R$ is $\alpha$-compatible, but not strongly $\alpha$-skew reversibe. Therefore, $R$ is not strongly $\alpha$-symmetric. Moreover, by [14, Example 2.10(1)], $R$ is not $\alpha$-skew Armendariz.

Now we show that $R$ is symmetric. Recall that $p \in A$ is a monomial of degree $n$ if it is a product of exactly $n$ number of indeterminates. The set of all linear combinations of monomials of degree $n$ over $Z_2$ is denoted by $H_n$. Then $H_n$ is finite for any $n$. In addition, the ideal $I$ is homogeneous (i.e., if $\sum_{i=1}^{l} r_i \in I$ with $r_i \in H_i$, then every $r_i$ is in $I$) by the construction.

**Claim.** If $p_1q_1s_1 \in I$ where $p_1, q_1, s_1 \in H_1$, then $p_1q_1s_1 \in I$.

**Proof of Claim.** Based on the construction of $I$ we have the following cases; 

**Case 1:**

\begin{align*}
(p_1 = a_0, q_1 = b_0, s_1 = r), & \quad (p_1 = a_0, q_1 = r, s_1 = b_0), \quad (p_1 = b_0, q_1 = a_0, s_1 = r) \\
(p_1 = b_0, q_1 = r, s_1 = a_0) & \quad (p_1 = a_0, q_1 = r, s_1 = a_0) \\
(p_1 = r, q_1 = b_0, s_1 = a_0) & \quad (p_1 = a_2, q_1 = b_2, s_1 = r) & \quad (p_1 = a_2, q_1 = r, s_1 = b_2) \\
(p_1 = b_2, q_1 = a_2, s_1 = r) & \quad (p_1 = b_2, q_1 = r, s_1 = a_2) & \quad (p_1 = b_2, q_1 = r, s_1 = b_2) \\
(p_1 = r, q_1 = b_2, s_1 = a_2) & \quad (p_1 = r, q_1 = b_2, s_1 = a_2) & \quad (p_1 = r, q_1 = a_2, s_1 = a_2) \\
(p_1 = r, q_1 = a_2, s_1 = a_2) & \quad (p_1 = r, q_1 = a_2, s_1 = a_2) & \quad (p_1 = r, q_1 = a_2, s_1 = a_2) \\
(p_1 = a_0 + a_1 + a_2, q_1 = r, s_1 = b_0 + b_2) & \quad (p_1 = a_0 + a_1 + a_2, q_1 = b_0 + b_2, s_1 = r) \\
(p_1 = r, q_1 = a_0 + b_1 + b_2, s_1 = b_0 + b_1 + b_2) & \quad (p_1 = r, q_1 = b_0 + b_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = r, q_1 = b_0 + b_2, s_1 = a_0 + a_1 + a_2) \\
(p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = a_0 + a_1 + a_2, q_1 = a_0 + a_1 + a_2, s_1 = r) & \quad (p_1 = a_0 + a_1 + a_2, q_1 = r, s_1 = a_0 + a_1 + a_2) \\
(p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) \\
(p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2) & \quad (p_1 = r, q_1 = a_0 + a_1 + a_2, s_1 = a_0 + a_1 + a_2)
\end{align*}

**Case 2:**

- If $(p_1 = a_0, q_1 = b_1, s_1 = a_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_0b_0a_0 = a_0b_1a_0 + a_1b_0a_0 = (a_0b_1 + a_1b_0)a_0 \in I$ and $a_0a_0b_1 \in I$.
- If $(p_1 = b_0, q_1 = a_1, s_1 = b_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_0a_1b_0 = b_0a_1b_0 + b_2a_0b_0 = (b_0a_1 + b_1a_0)b_0 \in I$ and $b_0b_0a_1 \in I$.
If \((p_1 = b_2, q_1 = a_1, s_1 = b_2)\), then \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\), since \(b_2a_1b_2 = b_2a_1b_2 + b_2a_2b_1 = b_2(a_1b_2 + a_2b_1) \in I\) and \(b_2b_2a_1 \in I\).

• If \((p_1 = a_2, q_1 = b_1, s_1 = a_2)\), then \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\), since \(a_2b_1a_2 = a_2b_1a_2 + a_2b_2a_2 = (a_1a_2 + a_2a_1) \in I\) and \(a_2b_2a_1 \in I\).

• If \((p_1 = b_0, q_1 = a_1, s_1 = a_0)\), then \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\), since \(b_0a_1b_0 = b_0a_1b_0 + b_0a_0a_1 = b_0(a_0a_1 + a_1a_0) \in I\) and \(b_0a_0a_1 \in I\).

• If \((p_1 = a_0, q_1 = a_1, s_1 = b_0)\), then \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\), since \(a_0a_1b_0 = a_0a_1b_0 + a_1a_0b_0 = (a_0a_1 + a_1a_0)b_0 \in I\) and \(a_0b_0a_1 \in I\).

Thus, we obtain that \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\) for each case and it proves our claim. Now let \(p = p_1 + p_2 + p_3 + p_4\), \(q = q_1 + q_2 + q_3 + q_4\), \(s = s_1 + s_2 + s_3 + s_4\) with \(p_i, q_i, s_i \in H_i\) for \(i = 1, 2, 3\) and \(p_4, q_4, s_4 \in I\). Let \(pq_s \in I\), where \(p, q, s \in A\). We want to see that \(R\) is symmetric. Since each monomial of degree \(\geq 4\) is contained in \(I\), then we have \(pq_s = p_1q_1s_1 + s' \in I\), where \(s' \in I\). Hence, \(p_1q_1s_1 \in I\) and \(p_1s_1q_1 \in I\). Therefore, we have \(psq \in I\) as required.

Recall that an ideal \(I\) of \(R\) is called an \(\alpha\)-ideal if \(\alpha(I) \subseteq I\), where \(\alpha\) is an endomorphism of \(R\).

**Definition** ([12, Definition 1.1]). Let \(R\) be a ring with an automorphism \(\alpha\). For an \(\alpha\)-ideal \(I\) of \(R\), \(I\) is called strongly \(\alpha\)-semiprime ideal of \(R\) if \(\alpha a \alpha(a) \subseteq I\) implies \(a \in I\) for any \(a \in R\). \(R\) is called a strongly \(\alpha\)-semiprime ring if the zero ideal is strongly \(\alpha\)-semiprime.

For an automorphism \(\alpha\) of \(R\), every \(\alpha\)-rigid ring is strongly \(\alpha\)-semiprime. Also, recall that any \(\alpha\)-rigid ring is strongly \(\alpha\)-symmetric. The following proposition shows when strongly \(\alpha\)-symmetric rings are \(\alpha\)-rigid.

**Proposition 2.13.** Let \(R\) be a ring with an automorphism \(\alpha\). Then \(R\) is \(\alpha\)-rigid if and only if \(R\) is strongly \(\alpha\)-semiprime and \(R\) is strongly \(\alpha\)-symmetric.

**Proof.** Suppose that \(R\) is strongly \(\alpha\)-semiprime and strongly \(\alpha\)-symmetric ring and let \(\alpha a \alpha(a) = 0\), where \(a \in R\). Now, consider the polynomials \(p(x) = ax\) and \(q(x) = a\) in \(R[x; \alpha]\). Then we have \(p(x)q(x)r(x) = 0\) for any \(r(x) \in R[x; \alpha]\).

Since \(R\) is strongly \(\alpha\)-symmetric, we obtain \(p(x)r(x)q(x) = 0\) for any \(r(x) \in R[x; \alpha]\).
R[x; α]. This implies that \( aR_\alpha(a) = 0 \) since \( α \) is onto. By using the fact that \( R \) is strongly \( α \)-semiprime, we get \( a = 0 \).

The following examples show that the concepts of strongly \( α \)-symmetric ring and strongly \( α \)-semiprime ring are independent of each other.

**Example 2.14.** (1) Let \( F \) be a field with \( \text{char}(F) \neq 2 \). Consider the ring \( R = M_2(F) \) and let \( α : R \rightarrow R \) be an endomorphism defined by \( α \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \). Then it is proved in [12, Example 2.1] that \( R \) is strongly \( α \)-symmetric. On the other hand, for the polynomials \( p(x) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), \( q(x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x \) and \( r(x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) in \( R[x; α] \), we have \( p(x)q(x)r(x) = 0 \), but

\[
p(x)r(x)q(x) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \neq O.
\]

Hence, \( R \) is not strongly \( α \)-symmetric. Moreover, \( R \) is not \( α \)-rigid since \( \alpha \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = 0 \) and \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) \( \neq 0 \). This example shows that in Theorem 2.13, being strongly \( α \)-symmetric is not superfluous.

(2) Consider the ring \( R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, \ t \in \mathbb{Q} \right\} \) and let \( α \) be the identity endomorphism of \( R \). \( R \) is not strongly \( α \)-semiprime since

\[
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0
\]

for any \( a \in \mathbb{Z} \) and \( t \in \mathbb{Q} \). On the other hand, suppose that \( p(x)q(x)r(x) = 0 \), where \( p(x) = \sum_{i=0}^{m} \begin{pmatrix} a_i & t_i \\ 0 & a_i \end{pmatrix} x^i \), \( q(x) = \sum_{j=0}^{n} \begin{pmatrix} b_j & t'_j \\ 0 & b_j \end{pmatrix} x^j \) and \( r(x) = \sum_{k=0}^{t} \begin{pmatrix} c_k & t''_k \\ 0 & c_k \end{pmatrix} x^k \) in \( R[x; α] \). Then we obtain

\[
0 = p(x)q(x)r(x) = \sum_{t=0}^{m+n+i} \left( \sum_{i+j+k=s} \begin{pmatrix} a_i & t_i \\ 0 & a_i \end{pmatrix} \alpha^i \begin{pmatrix} b_j & t'_j \\ 0 & b_j \end{pmatrix} \alpha^j \begin{pmatrix} c_k & t''_k \\ 0 & c_k \end{pmatrix} \right)x^t = \sum_{t=0}^{m+n+i} \sum_{i+j+k=s} a_i b_j c_k + a_i b_j t''_k + a_i t'_j c_k + t_i b_j c_k x^t.
\]

Therefore, \( R \) is strongly \( α \)-symmetric.

**Corollary 2.15.** Let \( R \) be a ring with an automorphism \( α \). Then the following statements are equivalent:
Proposition 2.17. Let $\lambda$ ideal of $R$ (Proposition 2.7(1)), Corollary 2.16, we have $R/I$.

Proof. (1) $\Leftrightarrow$ (2) is obtained by Theorem 2.13. It can be seen that (2) $\Leftrightarrow$ (3) follows from [28, Theorem 3].

By using Proposition 2.13, we obtain a generalization of the following result.

Corollary 2.16 ([14, Proposition 2.7(1)]). $R$ is reduced if and only if $R$ is semiprime and $R$ is symmetric.

Proposition 2.17. Let $R$ be a ring with an endomorphism $\alpha$ and $I_\lambda$ be an ideal of $R$ with $\alpha(I_\lambda) \subseteq I_\lambda$ for all $\lambda \in \Lambda$. Let $\alpha_\lambda : R/I_\lambda \to R/I_\lambda$ be the induced endomorphism of $R/I_\lambda$. If $R$ is a subdirect sum of strongly $\alpha_\lambda$-symmetric rings for all $\lambda \in \Lambda$, then $R$ is a strongly $\alpha$-symmetric ring.

Proof. Since $R$ is a subdirect sum of strongly $\alpha_\lambda$-symmetric rings, by [27, Theorem 3], we have $R/I_\lambda$ is a strongly $\alpha_\lambda$-symmetric ring for all $\lambda \in \Lambda$ and $\cap_{\lambda \in \Lambda} I_\lambda = 0$. Suppose that $p(x)q(x)r(x) = 0$, where $p(x) = \sum_{i=0}^{m} p_ix^i$, $q(x) = \sum_{j=0}^{n} q_jx^j$ and $r(x) = \sum_{k=0}^{n} r_kx^k$ in $R[x; \alpha]$. Then $p(x)q(x)r(x) = 0$ in $(R/I_\lambda)[x; \alpha_\lambda]$ for all $\lambda \in \Lambda$. Since $R/I_\lambda$ is strongly $\alpha_\lambda$-symmetric for all $\lambda \in \Lambda$, we can deduce that $p(x)q(x)r(x) = 0$. Then $\sum_{t=s+j}^{t=j+k} \sum_{s=i+k}^{r_i\alpha^i(r_k)\alpha^i(q_j)} = 0$, we obtain that

$$\sum_{t=s+j}^{t=j+k} \sum_{s=i+k}^{r_i\alpha^i(r_k)\alpha^i(q_j)} = 0.$$ 

Therefore, $p(x)r(x)q(x) = 0$ as required.

Let $\alpha_i$ be an endomorphism of a ring $R_i$ for each $i \in I$. Then the map $\alpha : \prod_{i \in I} R_i \to \prod_{i \in I} R_i$ defined by $\alpha((a_i)) = (\alpha_i(a_i))$, where $(a_i) \in \prod_{i \in I} R_i$, is an endomorphism. The proof of the following lemma can be obtained by routine computations.

Lemma 2.18. Let $R_i$ be a ring with an endomorphism $\alpha_i$, for each $i \in I$. Then the followings are equivalent:

1. $R_i$ is strongly $\alpha_i$-symmetric for each $i \in I$.
2. The direct product $\prod_{i \in I} R_i$ is strongly $\alpha$-symmetric.
3. The direct sum $\bigoplus_{i \in I} R_i$ is strongly $\alpha$-symmetric.

Recall that a ring $R$ is called local if $R/J(R)$ is a division ring, where $J(R)$ denotes the Jacobson radical of $R$. $R$ is called semilocal if $R/J(R)$ is semisimple Artinian and $R$ is called semiperfect if $R$ is semilocal and idempotents can be lifted modulo $J(R)$ (see [23] for more details). Also, note that local rings are abelian and semilocal.

Proposition 2.19. Let $R$ be a ring with an endomorphism $\alpha$. Then we have the followings:
(i) $R$ is a strongly $\alpha$-symmetric and semiperfect ring if and only if $R = \bigoplus_{i=1}^{n} R_i$ such that $R_i$ is local and strongly $\alpha_i$-symmetric ring, where $\alpha_i$ is an endomorphism of $R_i$ for all $i = 0, 1, \ldots, n$.

(ii) Let $R$ be a ring and $e$ be a central idempotent of $R$. Then $eR$ and $(1-e)R$ are strongly $\alpha$-symmetric if and only if $R$ is strongly $\alpha$-symmetric.

Proof. (i) Suppose that $R$ is strongly $\alpha$-symmetric and semiperfect. Since $R$ is semiperfect, by using \cite[Corollary 3.7.2]{22}, $R$ has a finite orthogonal set \{ $e_1, \ldots, e_n$ \} of local idempotents whose sum is 1. Then $R = \sum_{i=1}^{n} e_iR$ such that $e_iR$ is a local ring for all $i = 1, \ldots, n$. Since $R$ is strongly $\alpha$-symmetric, then $R$ is abelian and $e_iR = e_iR$. Also, by Lemma 2.8(iv), $\alpha_i(e_iR) \subseteq e_iR$ for all $i = 1, \ldots, n$. Let $\alpha_i$ be an endomorphism of $e_iR$ induced by $\alpha$. Then $e_iR$ is strongly $\alpha_i$-symmetric and local subring of $R$.

Conversely, let $R$ be a finite direct sum of strongly $\alpha_i$-symmetric local rings $R_i$ for all $i = 0, 1, \ldots, n$. Since local rings are semiperfect and $R$ is strongly $\alpha$-symmetric by Lemma 2.18, then we get $R$ is semiperfect.

(ii) It can be seen by using the fact $R \cong eR \oplus (1-e)R$ and Lemma 2.18. \(\square\)

3. Extensions of strongly $\alpha$-symmetric rings

In this section, we investigate properties of strongly $\alpha$-symmetric rings and their extensions. First, we deal with the polynomial extensions of strongly $\alpha$-symmetric rings. Note that an endomorphism $\alpha$ of a ring $R$ can be extended to an endomorphism $\bar{\alpha}$ of the polynomial ring $R[x]$, where

$$\bar{\alpha} \left( \sum_{i=0}^{m} a_i x^i \right) = \sum_{i=0}^{m} \alpha(a_i) x^i.$$ 

It is obvious that the polynomial rings over commutative (resp., reduced) rings are commutative (resp., reduced). But, this is not true for IFP, reversible and symmetric rings by \cite[Example 2]{15}, \cite[Example 2.1]{18} and \cite[Example 3.1]{14}, respectively.

Based on these results, one may ask whether a polynomial ring over a strongly $\alpha$-symmetric ring is strongly $\bar{\alpha}$-symmetric. We remark that the idea of the following proof is similar to \cite[Theorem 6]{10}.

**Theorem 3.1.** Let $R$ be a ring and let $\alpha$ be an endomorphism of $R$ with $\alpha^t = I_R$ for some positive integer $t$, where $I_R$ denotes the identity endomorphism of $R$. Then $R$ is strongly $\alpha$-symmetric if and only if $R[x]$ is strongly $\bar{\alpha}$-symmetric.

**Proof.** It is enough to prove that $R[x]$ is strongly $\bar{\alpha}$-symmetric when $R$ is strongly $\alpha$-symmetric. Assume that

$$p(y) = p_0 + p_1 y + \cdots + p_m y^m,$$

$$q(y) = q_0 + q_1 y + \cdots + q_n y^n,$$

$$r(y) = r_0 + r_1 y + \cdots + r_l y^l.$$
in \( R[x][y;\bar{a}] \) such that \( p(y)q(y)r(y) = 0 \). We also let
\[
\begin{align*}
p_i &= p_i x + \cdots + p_{n_i} x^{n_i}, \\
q_j &= q_j x + \cdots + q_{j+1} x^{j+1}, \\
r_k &= r_k x + \cdots + r_{k+s_k} x^{s_k}
\end{align*}
\]
in \( R[x] \), where \( u_i, v_j, w_k \geq 0 \) for each \( 0 \leq i \leq m, 0 \leq j \leq n \) and \( 0 \leq k \leq l \). We want to show that \( p(y)r(y)q(y) = 0 \). Take a positive integer \( s \) such that
\[
s > \max\{\deg p_i, \deg q_j, \deg r_k\}
\]
for any \( i, j \) and \( k \), where the degree is considered as polynomials in \( R[x] \). Also, note that we assume the degree of zero polynomial is zero. Let
\[
\begin{align*}
p(x^{ts+1}) &= p_0 + p_1 x^{ts+1} + \cdots + p_m x^{mts+m}, \\
q(x^{ts+1}) &= q_0 + q_1 x^{ts+1} + \cdots + q_n x^{nts+n}, \\
r(x^{ts+1}) &= r_0 + r_1 x^{ts+1} + \cdots + r_l x^{ts+l}.
\end{align*}
\]
Then the set of coefficients of the \( p_i \) (resp., \( q_j \) and \( r_k \)) equals the set of coefficients of \( p(x^{ts+1}) \) (resp., \( q(x^{ts+1}) \) and \( r(x^{ts+1}) \)). Since \( p(y)q(y)r(y) = 0 \), \( x \) commutes with elements of \( R \) in the polynomial ring \( R[x] \) and \( \alpha^t = I_R \), we get \( p(x^{ts+1})q(x^{ts+1})r(x^{ts+1}) = 0 \in R[x,\alpha] \). Then \( p(x^{ts+1})r(x^{ts+1})q(x^{ts+1}) = 0 \in R[x,\alpha] \) since \( R \) is strongly \( \alpha \)-symmetric. Thus, we obtain \( p(y)r(y)q(y) = 0 \) as required.
\[
\square
\]
In the following example, we show that there exists a non-identity endomorphism \( \alpha \) of a strongly \( \alpha \)-symmetric ring \( R \) such that \( \alpha^t = I_R \) for some positive integer \( t \).

**Example 3.2.** Consider the ring
\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z}_4 \right\}
\]
and let \( \alpha \) be an endomorphism defined by \( \alpha \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \). By [13, Example 1.10], \( R \) is \( \alpha \)-Armendariz and we also have \( R \) is symmetric since it is commutative. Then by using Theorem 2.6, we obtain \( R \) is strongly \( \alpha \)-symmetric and \( \alpha^2 = I_R \).

Let \( R \) be a ring and \( u \) be an element of \( R \). Recall that if \( ur = 0 \) implies \( r = 0 \) for \( r \in R \), then the element \( u \) is called right regular and left regular elements is defined, similarly. An element is called regular if it is both left and right regular. Following [7], a ring \( R \) is called left Ore for given \( a, b \in R \) with \( b \) regular, there exist \( a_1, b_1 \in R \) with \( b_1 \) regular such that \( b_1 a = a_1 b \). In [26, Theorem 2.1.12], it is showed that the classical left quotient ring \( S^{-1}R \) exists if and only if \( S \) is a left Ore set and the set \( S = \{ s + ass(S) \in R/ass(S) \mid s \in S \} \) consists of regular elements, where \( ass(S) := \{ r \in R \mid sr = 0 \text{ for some } s \in S \} \). In [14, Theorem 4.1]), it is proved that \( R \) is symmetric if and only if \( Q(R) \) is symmetric. In
the following theorem, we consider the classical left quotient rings of strongly α-symmetric rings. Let R be a ring with the classical left quotient ring Q(R). Then each automorphism α of R can be extended to an endomorphism α of Q(R) with α(b^{-1}a) = α(b)^{-1}α(a), where a, b ∈ R with b regular.

**Theorem 3.3.** Let R be a ring with an automorphism α. If the classical left quotient ring Q(R) of R exists, then R is strongly α-symmetric if and only if Q(R) is strongly α-symmetric.

**Proof.** It is enough to prove that Q(R) is strongly α-symmetric whenever R is strongly α-symmetric. Let \( p(x) = \sum_{i=0}^{m} v_{1}^{-1}p_{i}x^{i} \), \( q(x) = \sum_{j=0}^{n} v_{1}^{-1}q_{j}x^{j} \) and \( r(x) = \sum_{k=0}^{i} w_{1}^{-1}r_{k}x^{k} \) \in Q(R)[x; \bar{\alpha}] \) such that \( p(x)q(x)r(x) = 0 \), where \( p_{i}, q_{j}, r_{k} \in R \) and \( u, v, w \) are regular elements in R for \( 0 \leq i \leq m, 0 \leq j \leq n \) and \( 0 \leq k \leq l \). Then we obtain

\[
p = p(x)q(x)r(x)
\]

\[
= v_{1}^{-1}\left(\sum_{i=0}^{m} p_{i}x^{i}v_{1}^{-1}\right)\left(\sum_{j=0}^{n} q_{j}x^{j}v_{1}^{-1}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right)
\]

\[
= \left(\sum_{i=0}^{m} p_{i}\alpha^{i}(v_{1})^{-1}x^{i}\right)\left(\sum_{j=0}^{n} q_{j}\alpha^{j}(w_{1})^{-1}x^{j}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right).
\]

There exist \( p'_{i}, q'_{j} \in R \) and regular elements \( v_{2}, w_{2} \in R \) such that

1. \( p_{i}\alpha^{i}(v_{1})^{-1} = v_{2}^{-1}p'_{i}, \)
2. \( q_{j}\alpha^{j}(w_{1})^{-1} = w_{2}^{-1}q'_{j} \)

for \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Thus, we have

\[
0 = \left(\sum_{i=0}^{m} v_{2}^{-1}p'_{i}x^{i}\right)\left(\sum_{j=0}^{n} w_{2}^{-1}q'_{j}x^{j}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right)
\]

\[
= v_{2}^{-1}\left(\sum_{i=0}^{m} p'_{i}\alpha^{i}(w_{2})^{-1}x^{i}\right)\left(\sum_{j=0}^{n} q'_{j}x^{j}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right).
\]

There exist \( p''_{i} \in R \) and regular element \( w_{3} \in R \) such that

3. \( p'_{i}\alpha^{i}(w_{2})^{-1} = w_{3}^{-1}p''_{i} \)

for \( 0 \leq i \leq m \). Then we have

\[
\left(\sum_{i=0}^{m} p''_{i}x^{i}\right)\left(\sum_{j=0}^{n} q'_{j}x^{j}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right) = 0.
\]

Using strongly α-symmetric property of R, we deduce that

4. \( \left(\sum_{i=0}^{m} p''_{i}x^{i}\right)\left(\sum_{k=0}^{i} r_{k}x^{k}\right)\left(\sum_{j=0}^{n} q'_{j}x^{j}\right) = 0. \)
Since $R$ is strongly $\alpha$-IFP, we obtain that
\[ 0 = \left( \sum_{i=0}^{m} p_i' x^i \right) w_2 v_1 \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right) \]
(5)
\[ = \left( \sum_{i=0}^{m} p_i' \alpha^i (w_2 v_1) x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right). \]
By multiplying (5) on the left hand side with $(w_3 v_2)^{-1}$ and using (1) and (3), we obtain
\[ 0 = \left( \sum_{i=0}^{m} (w_3 v_2)^{-1} p_i' \alpha^i (w_2 v_1) x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right) \]
\[ = \left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right). \]
Thus, we get
\[ \left( \sum_{j=0}^{n} q_j x^j \right) w_1 \left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) = 0 \]
(6)
since $R$ is strongly $\alpha$-skew reversible and strongly $\alpha$-IFP. If we multiply (6) by $w_2^{-1}$ on the left hand side and use (2), then we have
\[ 0 = w_2^{-1} \left( \sum_{j=0}^{n} q_j \alpha^j (w_1) x^j \right) \left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \]
\[ = \left( \sum_{j=0}^{n} q_j x^j \right) \left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \]
(7)
\[ = \left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right). \]
By using (7) and similar arguments as above, we get $p(x)r(x)q(x) = 0$. Hence, $Q(R)$ is strongly $\bar{\alpha}$-symmetric. 

Let $R$ be a ring and $\alpha$ be an endomorphism of $R$. The set of all central regular elements of $R$ is denoted by $\Delta$. It can be seen that $\Delta$ is a multiplicatively closed subset of $R$. Then the map
\[ \bar{\alpha} : \Delta^{-1} R \to \Delta^{-1} R \]
defined by $\bar{\alpha}(u^{-1}r) = \alpha(u)^{-1}\alpha(r)$ is also an automorphism, where $r \in R$ and $u \in \Delta$.

In the following proposition, we are able to remove the condition in [14, Lemma 3.2] which states that “$\alpha(u) = u$ for any central regular element $u$".
And hence, we obtain a generalization of [14, Lemma 3.2] without any condition on the automorphism \( \alpha \).

**Proposition 3.4.** Let \( R \) be a ring and \( \alpha \) be an endomorphism of \( R \). Then \( R \) is strongly \( \alpha \)-symmetric if and only if \( \Delta^{-1} R \) is strongly \( \bar{\alpha} \)-symmetric.

**Proof.** Let \( p(x)q(x)r(x) = 0 \), where \( p(x) = \sum_{i=0}^{m} u^{-1} p_i x^i \), \( q(x) = \sum_{j=0}^{n} v^{-1} q_j x^j \) and \( r(x) = \sum_{k=0}^{l} w^{-1} r_k x^k \in \Delta^{-1} R[\alpha] \) with \( p_i, q_j, r_k \in R \) and \( u, v, w \) are central regular elements in \( R \) for all \( 0 \leq i \leq m, 0 \leq j \leq n \) and \( 0 \leq k \leq l \). Then we have

\[
0 = p(x)q(x)r(x) \\
= \left( \sum_{i=0}^{m} u^{-1} p_i x^i \right) \left( \sum_{j=0}^{n} v^{-1} q_j x^j \right) \left( \sum_{k=0}^{l} w^{-1} r_k x^k \right) \\
= u^{-1} \left( \sum_{i=0}^{m} p_i \alpha^i(v)^{-1} x^i \right) \left( \sum_{j=0}^{n} q_j \alpha^j(w)^{-1} x^j \right) \left( \sum_{k=0}^{l} r_k x^k \right) \\
= u^{-1} \left( \sum_{i=0}^{m} \alpha^i(v)^{-1} p_i x^i \right) \left( \sum_{j=0}^{n} \alpha^j(w)^{-1} q_j x^j \right) \left( \sum_{k=0}^{l} r_k x^k \right).
\]

There exist \( p_i', p_j' \in R \) and central regular elements \( v_1, w_1 \in R \) such that

\[
\alpha^i(v)^{-1} p_i = v_1^{-1} p_i', \\
\alpha^j(w)^{-1} q_j = w_1^{-1} q_j'.
\]

for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Then we have

\[
0 = u^{-1} v_1^{-1} \left( \sum_{i=0}^{m} p_i' \alpha^i(w_1)^{-1} x^i \right) \left( \sum_{j=0}^{n} q_j' x^j \right) \left( \sum_{k=0}^{l} r_k x^k \right) \\
= u^{-1} v_1^{-1} \left( \sum_{i=0}^{m} \alpha^i(w_1)^{-1} p_i' x^i \right) \left( \sum_{j=0}^{n} q_j' x^j \right) \left( \sum_{k=0}^{l} r_k x^k \right).
\]

There exist \( p_i'' \in R \) and central regular element \( w_2 \in R \) such that

\[
\alpha^i(w_1)^{-1} p_i' = w_2^{-1} p_i''
\]

for all \( 0 \leq i \leq m \). Then we obtain

\[
0 = u^{-1} v_1^{-1} w_2^{-1} \left( \sum_{i=0}^{m} p_i'' x^i \right) \left( \sum_{j=0}^{n} q_j' x^j \right) \left( \sum_{k=0}^{l} r_k x^k \right).
\]
Hence, we get
\[
0 = \left( \sum_{i=0}^{m} p_i'' x^i \right) \left( \sum_{j=0}^{l} r_j x^j \right) \left( \sum_{k=0}^{n} q_j x^j \right).
\]
By using the fact that \( R \) is strongly \( \alpha \)-IFP, we have
\[
0 = \left( \sum_{i=0}^{m} p_i'' x^i \right) w_1 \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right).
\]
If we multiply (11) by \((w_2 v_1)^{-1}\) on the left hand side and if we use (8) and (10), we get
\[
0 = (w_2 v_1)^{-1} \left( \sum_{i=0}^{m} p_i'' x^i \right) \alpha \left( \sum_{j=0}^{n} q_j x^j \right).
\]
Since \( R \) is strongly \( \alpha \)-skew reversible and strongly \( \alpha \)-IFP, we obtain that
\[
0 = \left( \sum_{j=0}^{n} q_j x^j \right) \alpha \left( \sum_{i=0}^{m} p_i x^i \right).
\]
By multiplying (12) with \( w_1^{-1} \) on the left hand side and using (9), we get
\[
0 = w_1^{-1} \left( \sum_{j=0}^{n} q_j x^j \right) \alpha \left( \sum_{i=0}^{m} p_i x^i \right).
\]
Then we have
\[
\left( \sum_{i=0}^{m} p_i x^i \right) \left( \sum_{k=0}^{l} r_k x^k \right) \left( \sum_{j=0}^{n} q_j x^j \right) = 0.
\]
Therefore, \( \Delta^{-1} R \) is strongly \( \bar{\alpha} \)-symmetric.

Recall that the ring of Laurent polynomials over a ring \( R \) consists of all formal sums \( \sum_{i=k}^{n} p_i x^i \) with usual addition and multiplication, where \( p_i \in R \) and \( k, n \) are (possibly negative) integers and denoted by \( R[x, x^{-1}] \). Note that an endomorphism \( \alpha \) of \( R \) can be extend an endomorphism \( \bar{\alpha} \) of \( R[x, x^{-1}] \) with \( \bar{\alpha}((\sum_{i=k}^{n} p_i x^i)) = \sum_{i=k}^{n} \alpha(p_i)x^i \).
Corollary 3.5. Let $R$ be a ring with an endomorphism $\alpha$. Then $R[x]$ is strongly $\alpha$-symmetric if and only if so is $R[x, x^{-1}]$.

Proof. Let $\Delta = \{1, x, x^2, \ldots\}$. It can be easily seen that $\Delta$ is a multiplicatively closed subset of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$. By Proposition 3.4, it follows that $R[x, x^{-1}]$ is strongly $\alpha$-symmetric. □

Therefore, we obtain a generalization of the following results.

Corollary 3.6 ([14, Lemma 3.2]). (1) Let $R$ be a ring and $\Delta$ be a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ is symmetric if and only if so is $\Delta^{-1}R$.

(2) For a ring $R$, $R[x]$ is symmetric if and only if so is $R[x; x^{-1}]$.

Let $R$ be a ring with a monomorphism $\alpha$. Recall that the Jordan’s construction of $R$ by $\alpha$ is the minimal extension of $R$ to which $\alpha$ extends as an automorphism. Let

$$A(R, \alpha) = \{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$$

be a subset of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. The multiplication is defined by $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Also, note that $x^{-ir}x^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$ for each $j \geq 0$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following operations:

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$$

and

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^i(r)\alpha^j(s)x^{i+j}$$

for $r, s \in R$ and $i, j \geq 0$. Furthermore, $A(R, \alpha)$ is an extension of $R$ by $\alpha$ and the map $\alpha$ can be extended to an automorphism $\bar{\alpha}$ of $A(R, \alpha)$ with $\bar{\alpha}(x^{-ir}x^i) = x^{-i}\alpha(r)x^i$. In [17], Jordan proved that for any pair of $(R, \alpha)$ such an extension always exists and $A(R, \alpha)$ is called Jordan extension of $R$ by $\alpha$.

Proposition 3.7. Let $R$ be a ring with a monomorphism $\alpha$. Then $R$ is strongly $\alpha$-symmetric if and only if $A(R, \alpha)$ is strongly $\alpha$-symmetric.

Proof. Suppose that $R$ is strongly $\alpha$-symmetric and let $p(y)q(y)r(y) = 0$, where $p(y) = \sum_{i=0}^{m}p_i y^i$, $q(y) = \sum_{j=0}^{n}q_j y^j$ and $r(y) = \sum_{k=0}^{l}r_k y^k$ in $A(R, \alpha)[y; \bar{\alpha}]$ such that $p_i = x^{-i}p_i'x^i$, $q_j = x^{-j}q_j'x^j$, $r_k = x^{-w_k}r_k'x^{w_k}$ for $p_i', q_j', r_k' \in R$ and $u_i, v_j, w_k \geq 0$ for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. Then we have

$$p_i = x^{-\mu}p_i'x^\mu, \quad q_j = x^{-\nu}q_j'x^\nu \quad \text{and} \quad r_k = x^{-\nu}r_k'x^\nu$$

for some $\mu \geq 0$, where $p_i', q_j', r_k' \in R$ for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. Since $p(y)q(y)r(y) = 0$, we have

$$\sum_{t=0}^{m+n+l} \left( \sum_{t=s+k} \left( \sum_{s=i+j} p_i \alpha^i(q_j)\bar{\alpha}(r_k) \right) \right) y^t = 0.$$
Thus

\[(13) \sum_{t=s+k} \sum_{s=i+j} p_i \alpha^i(q_j) \alpha^s(r_k) = \sum_{t=s+k} \sum_{s=i+j} x^{-\mu} p_i' \alpha^i(q_j') \alpha^s(r_k') x^\mu = 0.\]

Let \( p'(y) = \sum_{j=0}^n p_j' y^j \), \( q'(y) = \sum_{j=0}^n q_j' y^j \) and \( r'(y) = \sum_{k=0}^l r_k' y^k \in R[x; \alpha] \). Then by (13), we get \( p'(y)q'(y)r'(y) = 0 \) and \( p'(y)r'(y)q'(y) = 0 \) since \( R \) is strongly \( \alpha \)-symmetric. Therefore, \( p(y)r(y)q(y) = 0 \) and hence, \( A(R, \alpha) \) is strongly \( \alpha \)-symmetric.

Let \( R \) be a ring with an endomorphism \( \alpha \) and \( I \) be an ideal of \( R \). The map \( \bar{\alpha} \) defined by \( \bar{\alpha}(r + I) = \alpha(r) + I \) is an endomorphism of \( R/I \), where \( r \in R \). Note that although \( R \) is a strongly \( \alpha \)-symmetric ring, \( R/I \) need not be a strongly \( \bar{\alpha} \)-symmetric ring. Indeed, when we consider the ring \( R \) is the ring of quaternions with integer coefficients and \( \alpha \) is a monomorphism of \( R \), then \( R \) is a domain and so strongly \( \alpha \)-symmetric; while for any odd prime integer \( q \), we have \( R/qR \cong Mat_2(Z_q) \) by \([7, Exercise 3A]\). Notice that \( Mat_2(Z_q) \) is not strongly \( \alpha \)-symmetric since it is not abelian and thus, the factor ring \( R/qR \) is not strongly \( \bar{\alpha} \)-symmetric.

Let \( S \) be a subset of the ring \( R \) and then the set \( r_R(S) = \{ c \in R \mid Sc = 0 \} \) is called the right annihilator of \( S \) in \( R \). The left annihilator in \( R \), \( l_R(S) \), is defined similarly.

**Proposition 3.8.** Let \( R \) be a ring with an endomorphism \( \alpha \). Suppose that \( R \) is a strongly \( \alpha \)-symmetric ring. If \( I \) is a one-sided annihilator in \( R \) and \( \alpha(I) \subseteq I \), then \( R/I \) is strongly \( \bar{\alpha} \)-symmetric.

**Proof.** Let \( I = r_R(S) \) for some \( S \subseteq R \). We write \( \bar{R} = R/I \) and \( \bar{r} = r + I \), where \( r \in R \). We have \( R \) is symmetric and so, has IFP since \( R \) is strongly \( \alpha \)-symmetric. Thus, by [30, Lemma 1.2], \( I \) is an ideal of \( R \). Let \( \bar{p}(x) \bar{q}(x) \bar{r}(x) = 0 \), where \( \bar{p}(x) = \sum_{i=0}^n p_i x^i \), \( \bar{q}(x) = \sum_{j=0}^n q_j x^j \) and \( \bar{r}(x) = \sum_{k=0}^l r_k x^k \) in \( R[x; \bar{\alpha}] \). Then \( p(x)q(x)r(x) \in I[x; \alpha] \) and so \( Sp(x)q(x)r(x) = 0 \). Since \( R \) is strongly \( \bar{\alpha} \)-symmetric, we obtain \( Sp(x)r(x)q(x) = 0 \) and this implies that \( p(x)r(x)q(x) = 0 \). Therefore, \( R/I \) is strongly \( \bar{\alpha} \)-symmetric. The left annihilator case can be shown similarly. \( \square \)

As a converse of Proposition 3.8, we give the following result.

**Proposition 3.9.** Let \( R \) be a ring with an endomorphism \( \alpha \) and \( I \) be an \( \alpha \)-ideal of \( R \). If \( I \subseteq R/I \) is a strongly \( \bar{\alpha} \)-symmetric ring and \( I \) is an \( \alpha \)-rigid ring without identity, then \( R \) is strongly \( \alpha \)-symmetric.

**Proof.** Let \( \bar{p}(x) \bar{q}(x) \bar{r}(x) \in R[x; \bar{\alpha}] \) such that \( p(x)q(x)r(x) = 0 \). Then \( \bar{p}(x) \bar{q}(x) \bar{r}(x) = 0 \) and we get \( \bar{p}(x) \bar{r}(x) \bar{q}(x) = 0 \) since \( R/I \) is strongly \( \bar{\alpha} \)-symmetric. Thus, \( p(x)r(x)q(x) \in I[x, \alpha] \). By [10, Proposition 3], we have that \( I[x; \alpha] \) is reduced and hence, symmetric. Also, by [23, Proposition 1], we
get that all the possible products of \( p(x), q(x) \) and \( r(x) \) is in \( I[x; \alpha] \). Then we obtain
\[
(q(x)r(x)p(x))^2 = q(x)r(x)[p(x)q(x)r(x)]p(x) = 0
\]
and so \( q(x)r(x)p(x) = 0 \) since \( I[x; \alpha] \) is reduced. Thus, we have
\[
0 = p(x)r(x)[q(x)r(x)p(x)]r(x)q(x)
= [p(x)r(x)q(x)][r(x)p(x)r(x)q(x)]
= [r(x)p(x)r(x)q(x)][p(x)r(x)q(x)]
\]
(14)
by using the fact that \( I[x; \alpha] \) is symmetric and hence, reversible. If we multiply (14) on the right hand side by \( p(x) \), then we obtain
\[
0 = r(x)[p(x)r(x)q(x)p(x)r(x)q(x)p(x)]
= [p(x)r(x)q(x)p(x)r(x)q(x)p(x)]r(x).
\]
(15)
If we multiply (15) on the right hand side by \( q(x) \), then we get \( (p(x)r(x)q(x))^2 = 0 \) and hence, \( p(x)r(x)q(x) = 0 \). Therefore, \( R \) is strongly \( \alpha \)-symmetric.

As a consequence of Proposition 3.8 and Proposition 3.9, we give the following corollary.

**Corollary 3.10.** (1) [14, Proposition 3.5] Let \( R \) be a symmetric ring and \( I \) be an ideal of \( R \). If \( I \) is an annihilator in \( R \), then \( R/I \) is symmetric.

(2) [14, Proposition 3.6(1)] Let \( R \) be a ring and \( I \) be a proper ideal of \( R \). If \( R/I \) is symmetric and \( I \) is reduced (as a ring without identity), then \( R \) is symmetric.

The following example shows that the condition on \( I \) in Proposition 3.9 is necessary.

**Example 3.11.** Let \( R \) be the ring \( U_2(F) \), where \( F \) is a division ring and \( \alpha \) be an automorphism of \( R \) is defined by \( \alpha \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) = \left( \begin{array}{cc} a & -b \\ 0 & c \end{array} \right) \). Then \( R \) is not strongly \( \alpha \)-symmetric because \( R \) is not abelian. Also \( A\alpha(A) = 0 \), but for \( 0 \not= A = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \in I \). Thus, the \( \alpha \)-ideal \( I = \left( \begin{array}{cc} F & F \\ 0 & 0 \end{array} \right) \) of \( R \) is not \( \alpha \)-rigid.

In addition, the factor ring \( R/I \cong F \) is reduced and \( \alpha \) is the identity map on \( R/I \) and hence, \( R/I \) is \( \bar{\alpha} \)-rigid. Therefore, \( R/I \) is strongly \( \bar{\alpha} \)-symmetric.

Let \( S \) be a commutative ring and \( R \) be an algebra over \( S \). Following [5], the Dorroh extension of \( R \) by \( S \) is defined by \( D = R \oplus S \) with multiplication \( (r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2) \), where \( r_i \in R \) and \( s_i \in S \).

For an \( S \)-algebra homomorphism \( \alpha \) of \( R \), \( \alpha \) can be extended to an \( S \)-algebra homomorphism \( \bar{\alpha} : D \rightarrow D \) with \( \bar{\alpha}(r, s) = (\alpha(r), s) \).

**Theorem 3.12.** Let \( S \) be a commutative domain, \( R \) be an algebra over \( S \) and \( \alpha \) be an \( S \)-algebra homomorphism of \( R \). Then \( R \) is strongly \( \alpha \)-symmetric if and only if the Dorroh extension \( D \) of \( R \) by \( S \) is strongly \( \bar{\alpha} \)-symmetric.
Proof. We assume that $R$ is strongly $\alpha$-symmetric. Suppose that $p(x)q(x)r(x) = 0$, where $p(x) = \sum_{i=0}^{m} (a_i, b_i)x^i$, $q(x) = \sum_{j=0}^{n} (c_j, d_j)x^j$ and $r(x) = \sum_{k=0}^{l} (e_k, f_k)x^k \in D[x; \alpha]$. Then we obtain that

$$
0 = \left( \sum_{s=0}^{m+n} \left( \sum_{s=i+j} (a_i, b_i)\alpha^i(c_j, d_j)x^i \right) \right) \left( \sum_{k=0}^{l} (e_k, f_k)x^k \right) = \sum_{t=0}^{m+n+l} \left( \sum_{s=i+j} (a_i, b_i)\alpha^i(c_j, d_j)\alpha^s(e_k, f_k) \right)x^t
$$

(16)

Let $p'(x) = \sum_{i=0}^{m} a_i x^i$, $q'(x) = \sum_{j=0}^{n} d_j x^j$ and $r'(x) = \sum_{k=0}^{l} f_k x^k \in S[x]$. Then by (16), we get $p'(x)q'(x)r'(x) = 0$. Since $S[x]$ is a domain, either $p'(x) = 0$ or $q'(x) = 0$ or $r'(x) = 0$. Let $p'(x) = 0$. If we use the facts that $R$ is an $S$-algebra and $\alpha^i(s) = s$ for each $s \in S$ and $i \in \mathbb{N}$, then the equation (16) becomes

\begin{align*}
0 &= \sum_{t=s+k}^{m+n+l} \left( \sum_{s=i+j} (a_i, 0)(\alpha^i(c_j, d_j)(\alpha^s(e_k, f_k)) \right)x^t \\
&= \sum_{t=s+k}^{m+n+l} \left( \sum_{s=i+j} (a_i, \alpha^i(c_j) + d_j a_i, 0)(\alpha^s(e_k, f_k)) \right)x^t \\
&= \sum_{t=s+k}^{m+n+l} \left( \sum_{s=i+j} (a_i, \alpha^i(c_j)\alpha^s(e_k) + d_j a_i, \alpha^s(e_k) + f_k d_j a_i, 0) \right)x^t \\
&= \sum_{t=s+k}^{m+n+l} \left( \sum_{s=i+j} (a_i, \alpha^i(c_j)\alpha^s(e_k) + a_i \alpha^i(d_j)\alpha^s(e_k) + a_i \alpha^i(c_j)\alpha^s(f_k) + a_i \alpha^i(d_j)\alpha^s(f_k), 0) \right)x^t.
\end{align*}

Let $p''(x) = \sum_{i=0}^{m} a_i x^i$, $q''(x) = \sum_{j=0}^{n} c_j x^j$ and $r''(x) = \sum_{k=0}^{l} e_k x^k \in R[x; \alpha]$. Then $p''(x)q''(x)r''(x) = 0$. Since $R$ is strongly $\alpha$-symmetric, we have $p''(x)q''(x)r''(x) = 0$ and this implies that $p(x)r(x)q(x) = 0$. For the cases $q'(x) = 0$ and $r'(x) = 0$, the proof can be seen by using similar arguments.

\begin{flushright}
\Box
\end{flushright}

**Corollary 3.13** ([14, Proposition 4.2(2)]). Let $S$ be a commutative ring and $R$ be an algebra over $S$. If $S$ is domain and $R$ is symmetric, then the Dorroh extension $D$ is symmetric.
In the following result, we give a criteria to transfer strongly $\alpha$-symmetric property from one ring to another.

**Proposition 3.14.** Let $\phi : R \to R'$ be a ring isomorphism. Then $R$ is a strongly $\alpha$-symmetric ring if and only if $R'$ is a strongly $\phi \alpha \phi^{-1}$-symmetric ring.

**Proof.** The proof is obvious since an isomorphism from $R[x; \alpha]$ to $S[x; \phi \alpha \phi^{-1}]$ can be defined via the isomorphism $\phi$. \qed

We denote the ring of full matrices (resp., upper triangular matrices) of $n \times n$ type on $R$ by $\text{Mat}_n(R)$ (resp., $U_n(R)$) for $n \geq 2$. The following rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R, i = 1, \ldots, n, j = 2, \ldots, n \right\}$$

and

$$V_n(R) = \{ (a_{ij}) \in D_n(R) \mid a_{ij} = a_{i(i+1)(j+1)} \text{ for } i = 1, \ldots, n-2, j = 2, \ldots, n-1 \}$$

are subrings of $\text{Mat}_n(R)$. It is easy to see that $V_n(R) \cong R[x]/(x^n)$, where $(x^n)$ denotes the ideal of $R[x]$ generated by $x^n$. An endomorphism $\alpha$ of a ring $R$ can be extended to an endomorphism $\bar{\alpha}$ of $D_n(R)$ (resp., $V_n(R)$) with $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. We use the same notation $\bar{\alpha}$ for the extension endomorphism of $D_n(R)$ and $V_n(R)$. It is known that $\text{Mat}_n(R)$ and $U_n(R)$ do not have IFP for $n \geq 2$ since $U_2(R)$ is not abelian. Also, note that by [18, Example 1.3], $D_n(R)$ does not have IFP for $n \geq 4$. Thus, $\text{Mat}_n(R)$ and $U_n(R)$ are not strongly $\bar{\alpha}$-IFP and so, are not strongly $\bar{\alpha}$-symmetric for $n \geq 2$. Similarly, $D_n(R)$ is not strongly $\bar{\alpha}$-symmetric for $n \geq 4$. In [2, Proposition 3.7], it is proved that $D_2(R)$ and $D_3(R)$ are strongly $\bar{\alpha}$-IFP whenever $R$ is a $\alpha$-rigid ring. Naturally, one might ask whether $D_2(R)$ and $D_3(R)$ are strongly $\bar{\alpha}$-symmetric when $R$ is an $\alpha$-rigid ring. In the following example, we eliminate the case for $D_3(R)$.

**Example 3.15.** Recall that $D_3(\mathbb{R})[x; \alpha] \cong D_3(\mathbb{R}[x; \alpha])$. Consider the matrices

$$p = \begin{pmatrix} 1 & x^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 3x^4 & x^5 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 2x^2 & 3x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $D_3(\mathbb{R})[x; \alpha]$. Then we have $pqr = 0$, but $prq \neq 0$. Thus, $D_3(\mathbb{R})$ is not strongly $\bar{\alpha}$-symmetric.

In the following theorem, it is proved that $n \times n$ upper triangular matrix ring over a strongly $\alpha$-symmetric ring has a subring which is strongly $\bar{\alpha}$-symmetric.

**Theorem 3.16.** Let $R$ be a ring with an endomorphism $\alpha$. If $R$ is $\alpha$-rigid, then $V_n(R)$ is strongly $\bar{\alpha}$-symmetric.
Proof. Firstly, note that \( V_n(R)[x; \alpha] \cong V_n(R[x; \alpha]) \) and hence, \( p \in V_n(R)[x; \alpha] \) can be written as \( p = (p_1, p_2, \ldots , p_n) \) for some \( p_i \in R[x; \alpha] \). We assume that \( p(x)q(x)r(x) = 0 \), where \( p(x) = \sum_{i=0}^{m} A_i x^i = (p_1, p_2, \ldots , p_n), \quad q(x) = \sum_{j=0}^{n} B_j x^j = (q_1, q_2, \ldots , q_n) \) and \( r(x) = \sum_{k=0}^{l} C_k x^k = (r_1, r_2, \ldots , r_n) \) \( \in \quad V_n(R)[x; \alpha] \), for \( A_i = (a_i^{(j)}) \), \( B_j = (b_j^{(k)}) \), \( C_k = (c_k^{(l)}) \) \( \in \quad V_n(R) \) for \( 1 \leq u, v, s, l, u, v, z, w \leq n \). In this case we obtain that the following system of equations in \( R[x; \alpha] \):

\[
\begin{align*}
(17) \quad & p_1 q_1 r_1 = 0, \\
(18) \quad & p_1 q_1 r_2 + p_1 q_2 r_1 + p_2 q_1 r_1 = 0, \\
(19) \quad & p_1 q_1 r_3 + p_1 q_2 r_2 + p_1 q_3 r_1 + p_2 q_1 r_2 + p_2 q_2 r_1 + p_3 q_1 r_1 = 0, \\
& \vdots \\
(20) \quad & p_1 q_1 r_{n-1} + p_1 q_2 r_{n-2} + \cdots + p_1 q_{n-1} r_1 + \cdots + p_{n-1} q_1 r_1 = 0, \\
(21) \quad & p_1 q_1 r_n + p_1 q_2 r_{n-1} + \cdots + p_1 q_n r_1 + \cdots + p_n q_1 r_1 = 0.
\end{align*}
\]

By the assumption, \( R[x; \alpha] \) is reduced and therefore, \( p(x)q(x) = 0 \) requires \( p(x)R[x; \alpha] q(x) = 0 \) and \( q(x)p(x) = 0 \). Also, \( p(x)q(x)^2 = 0 \) implies \( p(x)q(x) = 0 \) for any \( p(x), q(x) \in R[x; \alpha] \). Considering these facts and (17), we have \( p_1 q_1 r_1 = 0 \) and \( r_1 p_1 q_1 = 0 \). If the equality (18) is multiplied by \( q_1 r_1 \) from the right hand side, then we have \( p_2 q_2 r_1 = 0 \) and \( p_2 r_1 q_1 = 0 \). Now, (18) becomes

\[
(22) \quad p_1 q_1 r_2 + p_1 q_2 r_1 = 0.
\]

If we multiply (22) on the right hand side by \( q_2 r_1 \), we have \( p_1 q_2 r_1 = 0 \) and thus, \( p_1 q_1 r_2 = 0 \). Hence, we obtain \( p_1 r_1 q_2 = 0 \) and \( p_1 r_2 q_1 = 0 \). Similarly, If equality (19) is multiplied by \( q_1 r_1 \) from the right hand side, then we get \( p_3 q_1 r_1 = 0 \) and \( p_3 r_1 q_1 = 0 \). So, (19) becomes

\[
(23) \quad p_1 q_1 r_3 + p_1 q_2 r_2 + p_1 q_3 r_1 + p_2 q_1 r_2 + p_2 q_2 r_1 = 0.
\]

If equality (23) is multiplied by \( q_2 r_1 \) from the right hand side, then we have \( p_2 q_2 r_1 = 0 \) and \( p_2 r_1 q_2 = 0 \). Hence, (23) becomes

\[
(24) \quad p_1 q_1 r_3 + p_1 q_2 r_2 + p_1 q_3 r_1 + p_2 q_1 r_2 + p_2 q_2 r_1 = 0.
\]

If we multiply (24) on the right hand side by \( q_1 r_2, q_3 r_1, q_2 r_2 \), respectively, we obtain \( p_2 q_1 r_2 = 0, \quad p_1 q_2 r_1 = 0, \quad p_1 q_1 r_2 = 0 \) and \( p_1 q_1 r_3 = 0 \). Inductively, assume that \( p_i q_j r_k = 0 \), where \( 1 \leq i, j, k \leq n-1 \) and \( i + j + k = n + 1 \) for \( n \geq 2 \). If we multiply (21) on the right hand side by \( q_1 r_1 \), we get \( p_n q_1 r_1 = 0 \) and \( p_n r_1 q_1 = 0 \). Then (21) becomes

\[
(25) \quad p_1 q_1 r_n + p_1 q_2 r_{n-1} + \cdots + p_1 q_n r_1 + \cdots + p_{n-1} q_2 r_1 = 0.
\]

If we multiply (25) on the right hand side by \( q_2 r_1 \), we have \( p_n -1 q_2 r_1 = 0 \) and \( p_{n-1} q_2 r_2 = 0 \). Continuing this procedure, by multiplying the equation by the appropriate \( q_j r_k \) on the right side, we get \( p_1 q_j r_k = 0 \) and hence, \( p_i r_k q_j = 0 \).
where $1 \leq i, j, k \leq n$ such that $i + j + k = n + 2$. Consequently, we get $p(x)r(x)q(x) = 0$ and therefore, $V_n(R)$ is strongly $\alpha$-symmetric.

By Theorem 3.16, it may be asked whether $V_n(R)$ is strongly $\bar{\alpha}$-symmetric for a strongly $\alpha$-symmetric ring $R$. But, the following example shows that this statement is not correct.

Example 3.17. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, b \in \mathbb{Z}_4 \right\}$$

and let $\alpha$ be an endomorphism of $R$ defined by $\alpha \left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$.

By Example 3.2, we know that $R$ is strongly $\alpha$-symmetric. For

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

in $V_2(R)[x; \bar{\alpha}]$ we have $ABC = 0$, but

$$ACB = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Therefore, $V_2(R)$ is not strongly $\bar{\alpha}$-symmetric.

Corollary 3.18. Let $R$ be a ring and $\alpha$ be an endomorphism of $R$. If $R$ is $\alpha$-rigid, then $R[x]/(x^n)$ is strongly $\alpha$-symmetric, where $(x^n)$ denotes the ideal of $R[x]$ generated by $x^n$.

Proof. It is clear since $R[x]/(x^n) \cong V_n(R)$.

Let $R$ be a ring and let $M$ be an $(R, R)$-bimodule. The ring $T(R, M) = R \oplus M$ is called the trivial extension of $R$ by $M$ with the componentwise addition and the multiplication defined as following:

$$(r, m)(r', m') = (rr', rm' + mr').$$

It is easy to see that this ring is isomorphic to the ring $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ with the usual matrix operations. An endomorphism $\alpha$ of the
ring $R$ can be extended to an endomorphism $\bar{\alpha}$ of $T(R, R)$ by $\bar{\alpha}\left(\begin{array}{c} r \\ r' \end{array}\right) = \alpha(r)\alpha(r')$, where $r, r' \in R$. Also, note that $T(R, 0) \cong R$.

**Corollary 3.19.** Let $R$ be a ring with an endomorphism $\alpha$. If $R$ is an $\alpha$-rigid ring, then $T(R, R)$ is strongly $\bar{\alpha}$-symmetric.

**Proof.** It is clear by Theorem 3.16, since $T(R, R) \cong V_2(R)$. □

If we consider Example 2.5(2), it may be seen that the converse of the Corollary 3.19 is not true. On the other hand, we obtain a generalization of the following results by using Theorem 3.16.

**Corollary 3.20 ([14, Theorem 2.3]).** Let $R$ be a reduced ring. Then $R[x]/(x^n)$ is symmetric, where $(x^n)$ denotes the ideal of $R[x]$ generated by $x^n$.

**Corollary 3.21 ([14, Corollary 2.4]).** If $R$ is a reduced ring, then $T(R, R)$ is symmetric.

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**References**


Fatma Kaynarca
Department of Mathematics
Afyon Kocatepe University
Afyonkarahisar 03200, Turkey
Email address: fkaynarca@aku.edu.tr

Halise Melis Tekin Akcın
CB4 1QB, Cambridge, UK
Email address: meliss65@gmail.com