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THE STRUCTURE OF IDEALS IN THE SEMIRING OF $n\times n$ MATRICES OVER A EUCLIDEAN SEMIRING

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1. Introduction

The structure of ideals in a Euclidean semiring E was given in [1]. It is natural to wonder whether or not this structure in E will carry over into the semiring of $n\times n$ matrices over E. Since a Euclidean semiring is commutative all of its ideals are two sided. However, one sided ideals do exist in the matrix semiring. Consequently, our attention will be directed toward the two sided ideals. The purpose of this paper is to investigate the structure of two sided ideals in the semiring of $n\times n$ matrices over a Euclidean semiring. It is shown that the structure of ideals in the matrix semiring is almost identical to the structure of ideals in the Euclidean semiring. This is surprising since the matrix semiring is neither Euclidean nor commutative.

2. Fundamentals

For this paper, a semiring will be a a set S together with two binary operation called *addition* $(+)$ and *multiplication* $(\cdot)$ such that $(S,+)$ is an abelian semigroup with a zero, $(S,\cdot)$ is a semigroup and multiplication distributes over addition from both the left and the right. A semiring S is said to be *commutative* if $(S,\cdot)$ is commutive and S is said to have an *identity* if there is an element $e\in S$ such that $ae=ea=a$ for all $a\in S$.

DEFINITION. A subset I of a semiring S will be called an *ideal* if I is a subsemigroup of $(S,+)$, $SI\subset I$ and $IS\subset I$.

In this case I is called a *two sided ideal*. If $SI\subset I$, then I is called a *left ideal* while I is called a *right ideal* if $IS\subset I$.

DEFINITION. Let S be a commutative semiring with an identity e. Then $S_p=\{x\in S \mid \text{there exists } y\in S \text{ such that } x=y+e\}\cup\{0\}$ is called *principal part of* S. If $S=S_p$, then S is called a *principal semiring.*

DEFINITION. A Euclidean semiring E is a principal semiring together with a function ϕ: $E \to Z^+$ satisfying the following properties for all a, $b \in E$.

(i) $\phi(a)=0$ if and only if $a=0$.

(ii) If $a+b \neq 0$, then $\phi(a+b) \geq \phi(a)$.

(iii) $\phi(ab)=\phi(a)\phi(b)$.

(iv) If $b \neq 0$, then there exists $p, r \in E$ such that $a=pb+r$ where $r=0$ or $\phi(r) < \phi(b)$.

If S is a semiring, then the set of all $n \times n$ matrices over S will be denoted by $\mathrm{Mat}_n S$. If S is a semiring with an identity, then it is clear that $\mathrm{Mat}_n S$ is a semiring with an identity. The $n \times n$ identity matrix will be denoted by I and the scalar matrix kI will be denoted by $\delta(k)$. It is well known that the center of $\mathrm{Mat}_n S$ is $\{\delta(k) \in \mathrm{Mat}_n S \mid k \in S\}$

3. The structure of ideals in $\mathrm{Mat}_n E$

In a Euclidean semiring E the basic ideals are ideals of the form dT_p where $d, p \in E$ and $T_p = \{x \in E \mid \phi(x) \geq \phi(p)\} \cup \{0\}$. The following structure theorem was proved in [1].

THEOREM 1. *Let A be an ideal in a Euclidean semiring E. Then $A = L \cup dT_p$, where dT_p is maximal in A, $L = \{t \in A \mid \phi(t) < \phi(dp)\}$ and $L \cap dT_p = \{0\}$. Moreover, $L \cup S[p, 2p)$ is a basis for A whose images are bounded by $\phi(2p)$.*

Our purpose here is to investigate the structure of two sided ideals in $\mathrm{Mat}_n E$ to see if their structure is related to the structure of ideals in E. To do this, we use the well known fact that if R is a ring J^* is an ideal in $\mathrm{Mat}_n R$ if and only if J^* is the ring of all $n \times n$ matrices over J for some ideal J in R. It is easy to prove that this fact remains valid if R is a semiring. This establishes somewhat of a correspondence between two sided ideals in $\mathrm{Mat}_n E$ and ideals in E that is very useful.

Let $a \in E$ and $T_a^* = \{(a_{ij}) \in \mathrm{Mat}_n E \mid a_{ij}=0 \text{ or } \phi(a_{ij}) \geq \phi(a)\}$. Clearly the zero matrix, 0, is in T_a^*.

THEOREM 2. *If $S = \mathrm{Mat}_n E$ and $a \in E$, then T_a^* is an ideal in S.*

PROOF. If $A=(a_{ij})$, $B=(b_{ij}) \in T_a^*$, then $a_{ij}=0$ or $\phi(a_{ij}) \geq \phi(a)$ and $b_{ij}=0$ or $\phi(b_{ij}) \geq \phi(a)$. Now if $a_{ij}+b_{ij} \neq 0$, then either $a_{ij} \neq 0$ or $b_{ij} \neq 0$. If $a_{ij} \neq 0$ then $\phi(a_{ij}) \neq 0$ and it follows that $\phi(a_{ij}+b_{ij}) \geq \phi(a_{ij}) \geq \phi(a)$. Similarly, if $b_{ij} \neq 0$, then $\phi(a_{ij}+b_{ij}) \geq \phi(b_{ij}) \geq \phi(a)$. In either case $\phi(a_{ij}+b_{ij}) \geq \phi(a)$. Consequently, $a_{ij}+b_{ij}$

$=0$ or $\phi(a_{ij}+b_{ij})\geq\phi(a)$ and it follows that $A+B\in T_a{}^*$. Now if $C=(c_{ij})\in S$, then $CA=(d_{ij})$ where $d_{ij}=\Sigma c_{ik}a_{kj}$. Clearly, if $d_{ij}\neq 0$, then $c_{it}a_{tj}\neq 0$ for some t and it follows that

$$\phi(d_{ij})=\phi(\Sigma c_{ik}a_{kj})\geq\phi(c_{it}a_{tj})=\phi(c_{it})\phi(a_{tj})\geq\phi(c_{ij})\phi(a)\geq\phi(a).$$

Consequently, $d_{ij}=0$ or $\phi(d_{ij})\geq\phi(a)$ and it follows that $CA\in T_a{}^*$. Similarly, $AC\in T_a{}^*$ and it follows that $T_a{}^*$ is an ideal in S.

The ideal T_a in E and the ideal $T_a{}^*$ in S have a natural relationship. To see this, let $T=\mathrm{Mat}_n T_a$. Clearly T is an ideal in S. Since $T_a=\{x\in E\,|\,\phi(x)\geq\phi(a)\}\cup\{0\}$, it follows that if $A=(a_{ij})\in T$, then $a_{ij}=0$ or $\phi(a_{ij})\geq\phi(a)$. Consequently, $A\in T_a{}^*$ and it follows that $T\subset T_a{}^*$. On the other hand if $B=(b_{ij})\in T_a{}^*$, then $b_{ij}=0$ or $\phi(b_{ij})\geq\phi(a)$ and it follows that $b_{ij}\in T_a$. Hence $B\in T$ and it follows that $T_a{}^*\subset T$. Thus $T_a{}^*=T=\mathrm{Mat}_n T_a$. This proves the following theorem.

THEOREM 3. *If E is a Euclideam semiring, and $a\in E$, then* $T_a{}^*=\mathrm{Mat}_n T_a$.

This theorem allows us to extend the properties of the ideal T_a to the ideal $T_a{}^*$.

THEOREM 4. *Let E be a Euclidean semiring, $S=\mathrm{Mat}_n E$ and $a,b\in E$. Then*

(i) $T_a{}^*\subset T_b{}^*$ *if and only if* $\phi(a)\geq\phi(b)$.

(ii) $T_a{}^*\cup T_b{}^*=T_c{}^*$, *where* $\phi(c)=\min\{\phi(a),\phi(b)\}$.

(iii) $T_a{}^*\cap T_b{}^*=T_c{}^*$, *where* $\phi(c)=\max\{\phi(a),\phi(b)\}$.

(iv) *If $\{a_i\}$ is a sequence of elements in E such that $\phi(a_i)<\phi(a_{i+1})$ then* $\cap T_{a_i}{}^*=0$, *the zero matrix.*

PROOF. (i) From theorem 3 we have $T_a{}^*=\mathrm{Mat}_n T_a$ and $T_b{}^*=\mathrm{Mat}_n T_b$. Consequently $T_a{}^*\subset T_b{}^*$ if and only if $T_a\subset T_b$, and $T_a\subset T_b$ if and only if $\phi(a)\geq\phi(b)$.

(ii) and (iii). Now a, $b\in E$ implies that $\phi(a)\geq\phi(b)$ or $\phi(b)\geq\phi(a)$. But (i) assures that $T_a{}^*\subset T_b{}^*$ or $T_b{}^*\subset T_a{}^*$ and it follows that $T_a{}^*\cup T_b{}^*=T_a{}^*$ or $T_a{}^*\cup T_b{}^*=T_b{}^*$. If $\phi(c)=\min\{\phi(a),\ \phi(b)\}$, then $T_a{}^*\cup T_b{}^*=T_c{}^*$. (iii) is proved in a similar manner.

To prove (iv) we need the following lemma.

LEMMA 5. *If $\{A_k\,|\,k\in J\}$ is a family of ideals in $S=\mathrm{Mat}_n E$, then* $\cap\mathrm{Mat}_n A_k=\mathrm{Mat}_n(\cap A_k)$.

PROOF. If $M=(m_{ij})\in\bigcap \mathrm{Mat}_n A_k$, then $M\in\mathrm{Mat}_n A_k$ for each $k\in J$ and it follows that $m_{ij}\in A_k$ for each i, j. Thus $m_{ij}\in\bigcap A_k$ and it follows that $(m_{ij})=M\in\mathrm{Mat}_n(\bigcap A_k)$. Consequently, $\bigcap\mathrm{Mat}_n A_k\subset\mathrm{Mat}_n(\bigcap A_k)$. Reversing the steps in the above argument will show that $\mathrm{Mat}_n(\bigcap A_k)\subset\bigcap\mathrm{Mat}_n A_k$. Consequently, $\mathrm{Mat}_n(\bigcap A_k)=\bigcap\mathrm{Mat}_n A_k$.

Proof of (iv). If $\{a_i\}$ is a sequence in E such that $\phi(a_i)<\phi(a_{i+1})$ for each i, then $T^*_{a_i}\supset T^*_{a_{i+1}}$ follows from (i). Hence $T_{a_i}\supset T_{a_{i+1}}$ for each i. But $\bigcap T_{a_i}=\{0\}$. Consequently, applying lemma 5 we get

$$\bigcap T^*_{a_i}=\bigcap\mathrm{Mat}_n T_{a_i}=\mathrm{Mat}_n(\bigcap T_{a_i})=\mathrm{Mat}_n\{0\}=0.$$

It was shown in [1] that if E is a Euclidean semiring and A is an ideal in E such that $T_a\subset A$ for some $a\in E$, then $A=K\cup T_m$ where T_m is maximal in A and $K=\{x\in A\mid 0<\phi(x)<\phi(m)\}$. We now establish the matrix semiring form of this theorem.

THEOREM 6. *Let E be a Euclidean semiring, $S=\mathrm{Mat}_n E$ and V^* an ideal in S such that $T_a^*\subset V^*$ for some $a\in E$. Then there exists an $m\in E$ such that T_m^* is maximal in V^* with respect to ideals of the form T_a^* and $V^*=K^*\cup T_m^*$ where*

$$K^*=\{(a_{ij})\in V^*\mid 0<\phi(a_{ij})<\phi(m) \textit{ for some } a_{ij}\}.$$

PROOF. Let $V^*=\mathrm{Mat}_n V$, where V is an ideal in E. Since $T_a^*\in V^*$, it follows that $T_a\subset V$. Consequently, $V=K\cup T_m$, where T_m is maximal in V with respect to ideals of the form T_a. Since $T_m\subset V$, it follows that $T_m^*\subset V^*$. To show that T_m^* is maximal in V^*, suppose $r\in E$ such that $T_m^*\subset T_r^*\subset V^*$. Then $T_m\subset T_r\subset V$ and it follows that $T_r=T_m$ or $T_r=V$. Consequently, $T_r^*=T_m^*$ or $T_r^*=V^*$ and it follows that T_m^* is maximal in V^*.

If $K^*=\{(a_{ij})\in V^*\mid 0<\phi(a_{ij})<\phi(m)$ for some $a_{ij}\}$, then it is clear that $V^*=K^*\cup T_m^*$.

Observe that K^* can be decomposed into two disjoint sets. If we let

$$K_1^*=\{(a_{ij})\in K^*\mid 0\leq\phi(a_{ij})<\phi(m) \text{ for all } a_{ij}\} \text{ and}$$
$$K_2^*=\{(a_{ij})\in K^*\mid \text{there exists } k,t,r \text{ and } s \text{ such that } 0<\phi(a_{kt})<\phi(m) \text{ and } \phi(a_{rs})\geq\phi(m)\},$$

then $K_1^*\cap K_2^*=\phi$, $K^*=K_1^*\cup K_2^*$ and $V^*=K_1^*\cup K_2^*\cup T_m^*$ where $K_1^*\cap K_2^*\cap T_m^*=\phi$.

If d, $a\in E$, then dT_a is an ideal in E. The scalar matrix $\delta(d)=dI$ has the same properties in $S=\mathrm{Mat}_n E$ as d has in E. Since $\delta(d)$ is in the center of S, it is clear that for $a\in E$, $\delta(d)T_a^*$ is an ideal in S. Observe that $\delta(d)T_a^*=\{(da_{ij})\mid a_{ij}=0 \text{ or } \phi(a_{ij})\geq\phi(a)\}$.

We not establish a relation between dT_a and $\delta(d)T_a^*$.

THEOREM 7. *If E is a Euclidean semiring and d, $a\in E$, then* $\delta(d)T_a^*=\mathrm{Mat}_n dT_a$.

PROOF. If $A=(a_{ij})\in\mathrm{Mat}_n dT_a$, then $a_{ij}\in dT_a$ and it follows that $a_{ij}=db_{ij}$ where $b_{ij}=0$ or $\phi(b_{ij})\geq\phi(a)$. Thus $A=(db_{ij})=\delta(d)(b_{ij})=\delta(d)B$, where $B\in T_a^*$, and it follows that $A\in\delta(d)T_a^*$. Consequently, $\mathrm{Mat}_n dT_a\subset\delta(d)T_a^*$. On the other hand, let $C=(c_{ij})\in\delta(d)T_a^*$. Then $C=\delta(d)P$ for some $P=(p_{ij})\in T_a^*$ Now $p_{ij}\in T_a$, $C=\delta(d)(p_{ij})=(dp_{ij})$ and it follows that $C\in\mathrm{Mat}_n d\, T_a$. Consequently, $\delta(d)T_a^*\subset\mathrm{Mat}_n dT_a$ and it follows that $\mathrm{Mat}_n dT_a=\delta(d)T_a^*$.

In order to prove our main structure theorem for ideals in S we need to consider the basis for an ideal in S. Let $P_{r,s}$ be the matrix with 1 as the row r-column s entry and 0 elsewhere. Then it is clear that $\delta(a)P_{r,s}$ is a matrix with a as the row r-column s entry and 0 elsewhere. If $B=(b_{ij})$, then $P_{r,s}BP_{t,q}$ is a matrix with b_{st} as the row r-column q entry and 0 elsewhere. Also $B=(b_{ij})=\Sigma\delta(b_{ij})P_{i,j}$ is a decomposition of B as a linear combination of the matrices $\{P_{i,j}\}$.

LEMMA 8. *If $\{v_\alpha\mid\alpha\in X\}$ is a basis for an ideal V in E, then $\{\delta(v_\alpha)P_{i,j}\mid\alpha\in X\}$ is a basis for V^* in* $\mathrm{Mat}_n E$.

PROOF. If V^* is an ideal in $S=\mathrm{Mat}_n E$, then $V^*=\mathrm{Mat}_n V$ for some ideal V in E. Let $M=(m_{ij})\in V^*$. Then $m_{ij}\in V$ and since $\{V_\alpha\mid\alpha\in X\}$ is a basis for V, it follows that $m_{ij}=\sum_\alpha c_{\alpha_{ij}}v_{\alpha_{ij}}$ where $c_{\alpha_{ij}}\in E$. Now $M=(m_{ij})=\Sigma\delta(m_{ij})P_{i,j}$. But

$$\delta(m_{ij})P_{i,j}=\delta(\sum_\alpha c_{\alpha_{ij}}v_{\alpha_{ij}})P_{i,j}=\sum_\alpha\delta(c_{\alpha_{ij}}v_{\alpha_{ij}})P_{i,j}$$
$$=\sum_\alpha\delta(c_{\alpha_{ij}})\delta(v_{\alpha_{ij}})P_{i,j}=\sum_\alpha\delta(c_{\alpha_{ij}})\{\delta(v_{\alpha_{ij}})P_{i,j}\}$$

since $\delta(m_{ij})$ is a scalar matrix. Consequently M is a linear combination of $\{\delta(v_\alpha)P_{i,j}\}$ and it follows that $\{\delta(v_\alpha)P_{i,j},\ \alpha\in X\}$ is a basis for V^*.

THEOREM 9. *Let E be a Euclidean semiring and V^* an ideal in $S=\mathrm{Mat}_n E$. Then there exist $d, p\in E$ such that $V^*=L^*\cup\delta(d)T_p^*$ where*

$$L^*=\{(a_{ij})\in V^*\,|\,0<\phi(a_{ij})<\phi(d_p)\ \text{for some}\ a_{ij}\},$$

$\delta(d)T_p^*$ is maximal in V^* with respect to ideals of the form $\delta(x)T_y^*$, $L^*\cap\delta(d)T_p^*=\phi$, and ϕ restricted to the entries of the matrices in a basis for V^* is bounded.

PROOF. We know that $V^*=\mathrm{Mat}_n V$ for some ideal V in E. From theorem 1, it follows that $V=L\cup dT_p$ where dT_p is maximal in V and the set of all $x\in V$ such that $\phi(x)<\phi(2dp)$ is a basis for V. Thus $\mathrm{Mat}_n dT_p\subset V^*$. By theorem 7, $\mathrm{Mat}_n dT_p=\delta(d)T_p^*$ and it follows that $\delta(d)T_p^*\subset V^*$. If $b, q\in E$ such that $\delta(d)T_p^*\subset\delta(b)T_q^*\subset V^*$ then $dT_p\subset bT_q<V$ and it follows that $bT_q=dT_p$ or $bT_q=V$. Consequently, $\delta(b)T_q^*=\delta(d)T_p^*$ or $\delta(b)T_q^*=V^*$ and it follows that $\delta(d)T_p^*$ is maximal in V^*. If $L^*=\{(a_{ij})\in V^*\,|\,0<\phi(a_{ij})<\phi(dp)$ for some $a_{ij}\}$ then it is clear that $V^*=L^*\cup\delta(d)T_p^*$ and $L^*\cap\delta(d)T_p^*=\phi$. Now $W=\{v_\alpha\in V\,|\,\phi(v_\alpha)<\phi(2dp)\}$ is a basis for V and it follows from lemma 8 that $\{\delta(v_{\alpha_{ij}})P_{i,j}\,|\,v_{\alpha_{ij}}\in W\}$ is a basis for V^*. Clearly ϕ restricted to the entries of the matrices in this basis is bounded by ϕ $(2dp)$.

The preceding theorem shows that the structure of ideals in the noncommutative semiring $\mathrm{Mat}_n E$ is almost identical to the structure of ideals in the Euclidean semiring E. The difference in the structures is the fact that in E, $V=L\cup dT_p$ with $L\cap dT_p=\{0\}$ while in $\mathrm{Mat}_n E$, $V^*=L^*\cup\delta(d)T_p^*$ with $L^*\cap\delta(d)T_p^*=\phi$.

4. Applications

The applications below will give instances when ideals in $\mathrm{Mat}_n E$ are finitely generated and an example in which ideals in $\mathrm{Mat}_n E$ are not finitely generated.

a. The set of nonnegative integers Z^+ is a Euclidean semiring with $\phi: Z^+\to Z^+$ given by $\phi(n)=n$ for all $n\in Z^+$. If I^* is an ideal in $\mathrm{Mat}_n Z^+$, *then $I^*=\mathrm{Mat}_n I$ for some ideal I in Z^+. It follows from theorem* 1 *that ϕ is bounded on a* *basis for I. Consequently, I has a finite basis. Now from theorem* 9, *it follows that the entries in the matrices in the basis for I^* come from the basis for I, which is finite. Since we can form only a finite number of matrices from a finite set of entries, it follows that there are only a finite number of matrices in the basis for I^*. Consequently, every ideal in* $\mathrm{Mat}_n Z^+$ *is finitely generated.*

b. *A direct application of lemma* 8 *can be used to show that if R is a semiring such that every ideal in R is finitely generated, then every ideal in* $\mathrm{Mat}_n R$ *is finitely generated.*

c. *If E is a Euclidean semiring such that* $\phi(a+b)>\phi(a)$ *for all* $a,b\in E$ *with* $a+b\neq 0$, *then every ideal in* $\mathrm{Mat}_n E$ *is finitely generated.*

PROOF. Suppose I is an ideal in E and $\{a_i\}$ is an infinite irredundant basis for I. Since $a_i\neq 0$ for any i, it follows that $\phi(a_i)>0$ for each i. If $i\neq j$, by property (iv) of a Euclidean semiring, we can write $a_i=qa_j+r$ where $r=0$ or $\phi(r)<\phi(a_j)$. Now $r\neq 0$ since this would give $a_i=qa_j$, a contradiction that the basis is irredundant. Thus $\phi(r)<\phi(a_j)$. Now if $q\neq 0$, then by the assumption, $\phi(a_i)=\phi(qa_j+r)>\phi(qa_j)=\phi(q)\phi(a_j)>\phi(a_j)$. If $q=0$, then $a_i=r$ and $\phi(a_i)=\phi(r)<\phi(a_j)$. In either case, we have $\phi(a_i)>\phi(a_j)$ or $\phi(a_i)<\phi(a_j)$ so that $\phi(a_i)\neq\phi(a_j)$ if $i\neq j$. Thus we can reindex the basis $\{a_i\}$ to give $\{a_{i_k}\}$ where $\phi(a_{i_k})$ is an increasing sequence of integers. Consequently, given an $N>0$, there is a_{i_k} such that $\phi(a_{i_k})>N$. But this contradicts theorem 1 which states that ϕ is bounded on a basis for I. Therefore $\{a_i\}$ is finite and it follows that every ideal in E is finitely generated. Consequently, part (b) above assures that every ideal in $\mathrm{Mat}_n E$ is finitely generated.

We want to give an example of a Euclidean semiring E where every ideal in $\mathrm{Mat}_n E$ is not finitely generated. Let $E=Q_p^+=\{x\in Q^+ \mid x\geq 1\}\cup\{0\}$. It was shown in [1] that Q_p^+ is a Euclidean semiring with ϕ defined as follows: $\phi(0)=0$ and $\phi(r)=1$ if $r\geq 1$. It is clear that the set $N=\{x\in Q_p^+ \mid x>2\}\cup\{0\}$ is an ideal in E. Suppose β is a basis for N and β is finite, say $\beta=\{r_1,r_2,\cdots,r_n\}$. Let r_1 be the least element of β. Then $r_1>2$ and there exists $s\in N$ such that $2<s<r_1$. Since P is a basis for N, $s=pr_k$ for some $r_k\in\beta$ and $p\in E$. Now $p=\frac{s}{r_k}$ and since $s<r_1\leq r_R$, it follows that $p<1$. Consequently, $p\notin E$ and it follows that β cannot generate N. Therefore any basis for N must be infinite. Thus any basis for the ideal $N^*=\mathrm{Mat}_n N$ in $\mathrm{Mat}_n E$ must also be infinite.

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REFERENCES

[1] Dale, L., and Hanson, D.L., *The structure of ideals in a Euclidean semiring*, Kyungpook Math. J. 17(1977) 21—29.

[2] Dale, L., and Pitts, J.D., *Euclidean and Gaussian semirings*, Kyungpook Math. J. 18(1978).