SOME PROPERTIES OF COMPLETELY POSITIVE MAP

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1. Introduction

In [1], Arveson stated that the correspondence between commutant of $\bar{\pi}(a)$ and the set of completely positive maps is an affine order isomorphism. This note states that the extension of $B$-valued inner product can not be carried out for even the simplest sort of pre-Hilbert $B$-module unless $B$ is at least an $AW^*$-algebra [Theorem 2.9].

In §3, in addition to Arveson's statements, it is also given that the correspondence preserves convex combinations [Theorem 3.5] and an equivalence condition for completely positive map [Theorem 3.6].

2. Preliminaries

Definition 2.1. Let $B$ be a $C^*$-algebra. A pre-Hilbert $B$-module is a right $B$-module $X$ equipped with a conjugate bilinear map $\langle \ , \ \rangle: X \times X \rightarrow B$ satisfying:

(i) $\langle x, x \rangle \geq 0 \ \forall x \in X$ ;
(ii) $\langle x, x \rangle = 0$ only if $x = 0$ ;
(iii) $\langle x, y \rangle = \langle y, x \rangle^* \text{ for } x, y \in X$ ;
(iv) $\langle x \cdot b, y \rangle = \langle x, y \rangle b \text{ for } x, y \in X, b \in B.$
The map $\langle \cdot , \cdot \rangle$ will be called a \textit{B-valued inner product} on $X$.

\textbf{Example 2.2.} If $J$ is a right ideal of $B$, then $J$ becomes a pre-Hilbert $B$-module when we define $\langle \cdot , \cdot \rangle$ by $\langle x, y \rangle = y^*x$ for $x, y \in J$.

For a pre-Hilbert $B$-module $X$, define $||\cdot||_x$ on $X$ by $||x||_x = ||\langle x, x \rangle||^{\frac{1}{2}}$.

\textbf{Proposition 2.3.} $||\cdot||_x$ is a norm on $X$ and satisfies:

(i) $||x \cdot b||_x \leq ||x||_x ||b||$ for $x \in X$, $b \in B$;
(ii) $\langle x, y \rangle^* \langle x, y \rangle \leq ||x||_x^2 \langle x, x \rangle$ for $x, y \in X$;
(iii) $||\langle x, y \rangle|| \leq ||x||_x ||y||_x$ for $x, y \in X$.

\textbf{Proof.} [5], [8].

\textbf{Definition 2.4.} A pre-Hilbert $B$-module $X$ which is complete with respect to $||\cdot||_x$ will be called a \textit{Hilbert $B$-module}.

\textbf{Remark 2.5.} For a pre-Hilbert $B$-module $X$, we let $\mathcal{A}(X)$ denote the set of operators $T \in B(X)$ for which there is an operator $T^* \in B(X)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $x, y \in X$. That is $\mathcal{A}(X)$ is the set of bounded operators on $X$ which possess bounded adjoint with respect to the $B$-valued inner product. It is easy to see that for $T \in \mathcal{A}(X)$, the adjoint $T^*$ is unique and belongs to $\mathcal{A}(X)$, so $\mathcal{A}(X)$ is a $\ast$-algebra with involution $T \mapsto T^*$.

\textbf{Lemma 2.6.} $\mathcal{A}(X)$ consists of entirely module maps.

i.e. if $T \in \mathcal{A}(X)$, then $T(x \cdot b) = (Tx) \cdot b$ for $x \in X$, $b \in B$. 

Some Properties of Completely Proof. Take \( y \in X \). Then by properties of \( B \)-valued inner product,
\[
\langle T(x \cdot b), y \rangle = \langle x \cdot b, T^*y \rangle = \langle x, T^*y \rangle b = \langle Tx, y \rangle b = \langle(Tx) \cdot b, y \rangle.
\]

For the balance of this section, \( A \) will be a \( C^* \)-algebra, \( B \) a closed *-subalgebra of \( A \), \( X \) a pre-Hilbert \( B \)-module, and \( Y \) a pre-Hilbert \( A \)-module.

**Lemma 2.7.** For a linear map \( T : X \rightarrow Y \) the followings are equivalent:

(i) \( T \) is a bounded module map of \( B \).
(ii) There is a real \( K \geq 0 \) such that \( \langle Tx, Tx \rangle_a \leq K \langle x, x \rangle_b \) for \( x \in X \).

**Proof.** [1], [5].

We let \( X' \) denote the set of bounded \( B \)-module maps of \( X \) into \( B \). By 2.7 (with \( A = B = Y \)), \( X' \) is precisely the set of linear maps \( \tau : X \rightarrow B \) for which there is a real \( K \geq 0 \) such that \( \tau(x)^* \tau(x) \leq K \langle x, x \rangle \) for \( x \in X \). Each \( x \in X \) gives rise to a map \( \hat{x} \in X' \) defined by \( \hat{x}(y) = \langle y, x \rangle \) for \( y \in X \). We will call \( X \) self-dual if \( X = X' \). According to [5, p.451], \( X' \) is a pre-Hilbert \( B \)-module, that is, \( \langle , \rangle \) can be extended to a \( B \)-valued inner product on \( X' \) and the extension satisfies \( \langle \hat{x}, \tau \rangle = \tau(x) \) for \( x \in X \) and \( \tau \in X' \).

**Theorem 2.8.** Let \( X \) and \( Y \) be pre-Hilbert \( A \)-modules and \( T : X \rightarrow Y \) a bounded module map. Then (i) There exists a bounded module map \( \hat{T} : X' \rightarrow Y' \) (ii) \( (\hat{T} \hat{x})(y) = (Tx) \hat{(y)} \) for \( x \in X \) and \( y \in Y \).

**Proof.** (i) Define \( T^* : Y \rightarrow X' \) by \( (T^*y)(x) = \langle Tx, y \rangle \)
for \( y \in X, \ x \in X \). By Schwarz's inequality \( \|(T^*y)(x)\| \leq \|T\|\|x\|\|y\| \), so \( T^* \) is bounded. Also since

\[
(T^*(y \cdot b))(x) = \langle Tx, y \cdot b \rangle = \langle y \cdot b, Tx \rangle^* \\
= (\langle y, Tx \rangle b)^* = b^* \langle y, Tx \rangle^* \\
= b^* \langle Tx, y \rangle = (\langle T^*y \rangle \cdot b)(x),
\]

\( T^* \) is a bounded module map.

Define \( \tilde{T} : X' \longrightarrow Y' \) by \( (\tilde{T}\tau)(y) = \langle T^*y, \tau \rangle \) for \( y \in Y, \ \tau \in X' \). Since \( \tilde{T} \) is just \( (T^*)^* \), \( \tilde{T} \) is a bounded module map also.

(ii) From the following observation, (ii) is immediate. That is, for \( x \in X, \ y \in Y \),

\[
(\tilde{T}\hat{x})(y) = \langle T^*y, \hat{x} \rangle = \langle \hat{x}, T^*y \rangle^* \\
= (\langle T^*y \rangle(x))^* = \langle Tx, y \rangle^* \\
= \langle y, Tx \rangle = (Tx)(y).
\]

**Theorem 2.9.** Let \( B \) be a \( C^* \)-algebra with the property that for every right ideal \( J \) of \( B \), there is a \( B \)-valued inner product \( \langle \ , \ \rangle \) on \( J' \) satisfying \( \langle \hat{x}, \tau \rangle = \tau(x) \) for all \( x \in J, \ \tau \in J' \). Then \( B \) is an \( AW^* \)-algebra.

**Proof.** Let \( J \) be a right ideal of \( B \). For \( a \in B \), define \( \hat{a} \in J' \) by \( \hat{a}(x) = a^*x \ (x \in J) \) and let \( \tau_i \in J' \) denote the inclusion of \( J \) into \( B \). Notice \( \tau_i \cdot a = \hat{a} \) for \( a \in B \) and that \( \hat{x} = \hat{\hat{x}} \) for \( x \in J \).

Put \( q = \langle \tau_i, \tau_i \rangle \). Then \( q = q^* \) and \( x \in J, \ qx = \tilde{q}(x) \). By the way,

\[
\tilde{q}(x) = \langle \tau_i \cdot x, \tau_i \rangle = \langle \hat{x}, \tau_i \rangle \\
= \langle \hat{x}, \tau_i \rangle = \tau_i(x) = x.
\]

So we have
$q^2 = \langle \tau_i, q, \tau_i \rangle = \langle \tilde{q}, \tau_i \rangle$
$= \langle \tau_i, \tau_i \rangle = q,$

i.e., $q$ is a projection. Put $p = 1 - q$, so $p$ is a projection in $L(J)$, where $L(J)$ is the left annihilator of $J$. For self-adjoint element $a$ of $L(J)$, we have $\tilde{a} = 0$, so

$q a = \langle \tau_i \cdot a, \tau_i \rangle$
$= \langle \tilde{a}, \tau_i \rangle = 0,$

which shows that $a p = a$ for all $a \in L(J)$. That is, $L(J)$ is a principal ideal generated by some projection $p$.

### 3. Completely positive maps

**Definition 3.1.** Let $B$ be a $C^*$-algebra, $A$ a *-algebra and $\phi : A \rightarrow B$ a linear map. We call $\phi$ positive if $\phi(a^*a) \geq 0 \ \forall a \in A$.

For $n = 1, 2, \ldots$, $\phi$ induces a map $\phi_n$ from algebra $A_{(n)}$ of $n \times n$ matrices with entries in $A$ (made into a *-algebra by setting $[a_{ij}]^* = [a_{ij}]^*$ for matrices $[a_{ij}] \in A_{(n)}$) into the corresponding $C^*$-algebra $B_{(n)}$ defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$; we say that $\phi$ is completely positive if each of the induced map $\phi_n$ is positive.

**Remark 3.2.** According to [7, p.194], a linear map $\phi : A \rightarrow B$ is completely positive if and only if

$$\sum_{i,j} b_i \cdot \phi(a_i^*a_j) b_j \geq 0 \ \text{for} \ a_1, \ldots, a_n \in A, \ b_1, \ldots, b_n \in B.$$
tensor product $A \otimes B$, which becomes a right $B$-module when we set $(a \otimes b) \cdot \beta = a \otimes b \beta$ for $b, \beta \in B$, $a \in A$.

Define
$$[,] : (A \otimes B) \times (A \otimes B) \to B$$
$$\left( \sum_{j=1}^{n} a_j \otimes b_j, \sum_{i=1}^{m} \alpha_i \otimes \beta_i \right) \mapsto \sum_{i}^{m} \alpha_i \otimes B_i$$
for $a_1, \ldots, a_n$, $\alpha_i, \ldots, \alpha_m \in A$, $b_1 \ldots b_n$, $\beta_1, \ldots, \beta_m \in B$.

$[,]$ is clearly well-defined and conjugate-bilinear. Since $\phi$ is completely positive, for with $x \in A \otimes B$, $[x, x] \geq 0$, since $\phi$ is $*$-map, $[x, y^*] = [y, x]^*$, and $[x \cdot b, y] = [x, y] b$ for $x, y \in A \otimes B$ and $b \in B$.

Put
$$N = \{ x \in A \otimes B : [x, x] = 0 \}.$$ Then $N$ is a submodule of $A \otimes B$ and $X_0 = A \otimes B/N$ is a pre-Hilbert $B$-module with $B$-valued inner product
$$\langle x+N, y+N \rangle = [x, y] \text{ for } x, y \in A \otimes B.$$ 

Following T.W. Palmer [3], we call an element $v$ of the $*$-algebra $A$ quasi-unitary if $vv^* = v^*v = v + v^*$ and say that $A$ is a $U^*$-algebra if it is the linear span by its quasi-unitary elements. All Banach $*$-algebras are $U^*$-algebra and $A$ is a $U^*$-algebra iff it is spanned by its unitaries [3].

**Theorem 3.3.** Let $A$ be a $U^*$-algebra with $1$, $B$ a $C^*$-algebra with $1$, and $\phi : A \to B$ a completely positive map. Then

(i) there is a Hilbert $B$-module $X$, a $*$-representation $\pi$ of $A$ on $X$, and an element $e \in X$ such that $\phi(a) = \langle \pi(a)e, e \rangle$ for $a \in A$. 
(ii) the set \{\pi(a)(\tau \cdot b) : a \in A, b \in B\} spans a dense subspace of \(X\).

\textbf{Proof.} \[5\], \[6\], \[7\]. In particular, note that \(\pi(a)(x+N) = a \cdot x + N \forall x \in A \otimes B\) and \(\pi(a) \in \mathcal{O}(X)\) (i.e., \(\pi(a)\) is a \(B\)-module map), \(X\) a completion of \(X_0\).

Let \(A\) be a \(U^*\)-algebra with 1, and \(B\) a \(C^*\)-algebra. If \(X, \pi\) and \(e(\epsilon = 1 \otimes 1 + N)\) are as in 3.3, we may define a \(*\)-representation \(\pi\) of \(A\) on the self-dual Hilbert \(B\)-module \(X'\) by \(\pi(a) = \pi(a)^* \in \mathcal{O}(X')\) for \(a \in A\). Suppose \(\phi : A \rightarrow B\) is another completely positive map. We write \(\phi \leq \phi'\) if \(\phi - \phi'\) is completely positive and let \([0, \phi]\) denote the set of completely positive maps from \(A\) into \(B\) which are \(\leq \phi\).

For \(T \in \mathcal{O}(X')\), define \(\phi_T : A \rightarrow B\) by \(\phi_T(a) = \langle T\pi(a)\epsilon, \epsilon \rangle\). Notice that \(\phi_T = \phi\) and that the map \(T \mapsto \phi_T\) is a linear map of \(\mathcal{O}(X')\) into the space of linear transformations of \(A\) into \(B\), also that \(X'\) becomes a right \(B\)-module if we set \((\tau \cdot b)(x) = b* \pi(x)\) for \(\tau \in X', b \in B, x \in X\).

\textbf{Lemma 3.4.} Let \(A\) be a \(C^*\)-algebra with 1 and let \(a \in A, a \geq 0\). Then there exists a unique element \(b \in A\) such that \(b \geq 0\) and \(b^2 = a\).

\textbf{Proof.} \[2\], \[7\].

\textbf{Theorem 3.5.} Under the above circumstances,

(i) for each \(T \in \pi(A)'\) with \(0 \leq T \leq I_{X'}\), the formula \(\phi_T(a) = \langle T\pi(a)\epsilon, \epsilon \rangle\) defines a completely positive map such that \(\phi_T \leq \phi\);

(ii) the correspondence \(T \mapsto \phi_T\) described in (i) is a bijection of \(\{T \in \pi(A)' : 0 \leq T \leq I_{X'}\}\) onto \([0, \phi]\);

(iii) the correspondence preserves convex combinations,
where $\hat{\pi}(A)'$ denotes the commutant of $\pi(A)$ in $\mathcal{A}(X')$.

**Proof.** (i) For $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, set

$$x = \sum_{j=1}^n \pi(a_j)(e \cdot b_j) \in X \quad \text{(by 3.3, this is possible).}$$

Then

$$\sum_{j, l} b_l \phi_r(a_l^* a_j) b_j = \sum_{j, l} b_l \langle T\hat{\pi}(a_l^* a_j) e, e \rangle b_j$$

$$= \sum \langle T\hat{\pi}(a_l^* a_j) e \cdot b_j, e \cdot b_j \rangle$$

$$= \sum \langle T\hat{\pi}(a_l^* a_j) e \cdot b_j, \hat{\pi}(a_j^* e) \rangle$$

(since $T \in \hat{\pi}(A)'$)

$$= \langle T(\sum \hat{\pi}(a_j^* e) \cdot b_j), \sum \hat{\pi}(a_j^* e) \cdot b_j \rangle$$

$$= \langle T(\sum \hat{\pi}(a_j^* e) \cdot b_j)^\wedge, \sum \hat{\pi}(a_j^* e) \cdot b_j \rangle$$

(by the above notice)

$$= \langle T(\sum \hat{\pi}(a_j^* e) \cdot b_j)^\wedge, (\sum \pi(a_i)(e \cdot b_i))^\wedge \rangle$$

(by 2.8)

$$= \langle T\hat{x}, \hat{x} \rangle$$

$$= \langle T^{\frac{1}{2}}\hat{x}, T^{\frac{1}{2}}\hat{x} \rangle \geq 0 \quad \text{(by 3.4).}$$

Thus $\phi_r$ is completely positive. But $\phi_r \leq \phi$ is to be shown in (ii).

(ii): If $T \in \hat{\pi}(A)'$ and $\phi_r = 0$, then

$$\langle T(\pi(a_1)(e \cdot b))^\wedge, (\pi(a_2)(e \cdot b))^\wedge \rangle$$

$$= \langle T\hat{\pi}(a_1^* a_1)(e \cdot b_1)^\wedge, (e \cdot b_2)^\wedge \rangle$$

$$= \langle T\hat{\pi}(a_2^* a_1)(e \cdot b_1)^\wedge, (e \cdot b_2)^\wedge \rangle$$

$$= b_2^* \langle T\hat{\pi}(a_2^* a_1) e, e \rangle b_1$$

$$= b_2^* \phi_r(a_2^* a_1) b_1 = 0$$

for $a_1, a_2 \in A$, $b_1, b_2 \in B$.

So $\langle T(\hat{X}_0), X_0 \rangle = 0$, or $\langle T(X), X \rangle = 0$; hence $T = 0$ by 2.8.

Thus the correspondence is one-one.

To show that the correspondence is onto, take $\phi \in [0, \phi]$.

By 3.3 there exists a $*$-representation $\rho$ of $A$ on a Hilbert
Some properties of completely $B$-module $Y$ and a $d \in Y$ such that $\phi(a) = \langle \rho(a) d, d \rangle$ for $a \in A$ and the set $\{ \rho(a) (d \cdot b) : a \in A, \ b \in B \}$ spans a dense subspace $Y_0$ of $Y$. Since $\phi \leq \phi$, there is a well-defined bounded module map $W : X_0 \to Y_0$ such that $W(\pi(a) (e \cdot b)) = \rho(a) (d \cdot b)$ for $a \in A, \ b \in B$ and $\langle w_\xi, w_\zeta \rangle \leq \langle x, x \rangle$ for $x \in X_0$. $W$ extends to a bounded module map $\bar{W} : X \to Y$. Also

$$W\pi(a)(\pi(a)(e \cdot b)) = w\pi(a)(a \otimes b + N)$$

$$= w(a^2 \otimes b + N)$$

$$= \rho(a^2)(d \cdot b)$$

$$= \rho(a) \rho(a)(d \cdot b) = \rho(a) W(\pi(a)(e \cdot b)),$$

i.e., $W\pi(a)$ and $\rho(a)w$ agree on $X_0$ for $a \in A$. Hence $W\pi(a) = \rho(a) W$ for $a \in A$. By 2.8, we get a bounded module map $\bar{W} : X \to Y$ extending $W$. It is clear from the proof of 2.8 that

$$\langle \bar{W} \tau, \bar{W} \tau \rangle \leq \langle \tau, \tau \rangle \text{ for } \tau \in X'.$$

Let $\bar{W}^* : Y' \to X'$ be the adjoint of $\bar{W}$ and put $T = \bar{W}^* W$, so $T \in \Theta(X')$ and $T = T^*$. For $\tau \in X'$, we have $\langle T \tau, \tau \rangle = \langle \bar{W} \tau, \bar{W} \tau \rangle \leq \langle \tau, \tau \rangle$, so $0 \leq \langle T \tau, \tau \rangle \leq \langle \tau, \tau \rangle$, hence $0 \leq T \leq I$. Since $\bar{W} \pi(a) = \tilde{\rho}(a) \bar{W}$, $\bar{W}(a) \bar{W}^* = \bar{W}^* \tilde{\rho}(a)$ for $a \in A$. Hence for any $a \in A$, we have

$$T \pi(a) = \bar{W}^* \bar{W} \pi(a)$$

$$= \bar{W}^* \tilde{\rho}(a) \bar{W}$$

$$= \pi(a) \bar{W}^* \bar{W} = \pi(a) T, \ \text{i.e., } T \in \pi(A)' .$$

Finally, for $a \in A$,

$$\phi_\tau(a) = \langle T \pi(a) \varepsilon, \varepsilon \rangle = \langle \bar{W} \pi(a) \varepsilon, \bar{W} \varepsilon \rangle$$

$$= \langle \bar{W} \pi(a) \varepsilon, \bar{W} \varepsilon \rangle$$

$$= \langle \rho(a) d, d \rangle = \phi(a).$$
These complete the proof of (i) and (ii).

(iii) The set \( K = \{ T \in \hat{\pi}(A)' : 0 \leq T \leq I_{X'} \} \) is obviously convex; so is \([0, \phi]\). If \( S, T \in K \) and \( 0 \leq \lambda \leq 1 \), and \( R = \lambda S + (1 - \lambda)T \), then for \( a \in A \),

\[
\phi_R(a) = \phi(a)\phi_{\lambda S + (1 - \lambda)T}(a) \\
= \langle (\lambda S + (1 - \lambda)T')\hat{\pi}(a)\check{\varepsilon}, \check{\varepsilon} \rangle \\
= \lambda \phi_S(a) + (1 - \lambda)\phi_T(a),
\]

i.e., \( \phi_{\lambda S + (1 - \lambda)T} = \lambda \phi_S + (1 - \lambda)\phi_T \).

Thus the correspondence preserves convex combinations.

Theorem 3.6. A completely positive map \( \phi \) on \( A \) satisfies \( \phi \leq \phi \) if and only if there exists an operator \( T \in \mathcal{A}(X') \) such that \( 0 \leq T \leq I_{X'} \), \( T\hat{\pi}(a) = \hat{\pi}(a)T \) for all \( a \in A \), and \( \phi(a) = \langle T\hat{\pi}(a)\check{\varepsilon}, \check{\varepsilon} \rangle \) for all \( a \in A \).

Proof. Suppose \( \phi \in [0, \phi] \). Then by Theorem 3.5, \( \phi(a) = \phi_T(a) = \langle T\hat{\pi}(a)\check{\varepsilon}, \check{\varepsilon} \rangle \) and also by Theorem 3.5, it is clear.

Conversely, since \( \phi_T = \phi \) and \( \sum_{i,j} b_i^* (\phi - \phi_T)(a_i^* a_j) b_j = \langle (I - T)^{\check{\varepsilon}} \check{\varepsilon}, \check{\varepsilon} \rangle \geq 0 \), for \( a_1, \ldots, a_n \in A, \ b_1, \ldots, b_n \in B \) and \( x = \sum x_j a_j (e^* b_j) \in X \), the proof is immediate.

References

4. W.L. Paschke, Completely positive maps on U*-algebras,
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