ON FINITE DIMENSIONAL $C^*$-SUBALGEbras
OF AF $C^*$-ALGEBRA

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1. Introduction

The set of traces on a $C^*$-algebra is a very useful invariant of the algebra and there have been some significant recent advances concerning the relationship between traces, finiteness and comparability of elements. For example a simple $C^*$-algebra with a finite trace is a finite algebra [3]. An approximately infinite dimensional algebra, that is AF $C^*$-algebra, is a $C^*$-algebra which is an inductive limit of a sequence of finite $C^*$-algebras. The study of AF $C^*$-algebra was begun by Bratteli [2] following earlier more specialized studies by Glimm [6] and Diximier [4]. Elliott showed that if $A$ is an AF $C^*$-algebra, then $A$ is classified up to isomorphism by $K_0(A)$, considered as a partially ordered abelian group, [5]. The relation between trace and $K_0(A)$ has been studied by J. Cuntz and G.K Pedersen. In this paper we study the finite dimensional $C^*$-subalgebras of AF $C^*$-algebra by using the trace and the partially ordered abelian group $K_0$. 
2. Preliminaries

Let $A$ be a $C$-algebra. A trace on $A$ is a function $\phi: A_+ \to [0, \infty]$ such that

i) $\phi(\alpha x) = \phi(x)$ if $x \in A_+$ and $\alpha \in \mathbb{R}$,

ii) $\phi(x + y) = \phi(x) + \phi(y)$ if $x$ and $y$ belong to $A_+$,

iii) $\phi(u^* xu) = \phi(x)$ for all $X$ in $A_+$ and all unitaries $u$ in $A$.

In here $A$ is a $C^*$-algebra with unit containing $A$ as a closed ideal and $A_+$ is the set of all positive elements in $A$. We say that $\phi$ is finite if $\phi(x) < \infty$ for $x \in A_+$ and $\phi$ is semi-finite if for each $x \in A_+$, $\phi(x')$ is the supremum of the numbers $\phi(y)$ for those $y \in A_+$ such that $y \leq x$ and $\phi(y) < +\infty$. Clearly $\phi$ may be unbounded functional on $A$. $y \leq x$ means that $x - y \in A_+$ for $x, y \in A$. If a trace $\phi$ is finite, then $\phi$ can be extended to $A$ as a positive linear functional on $A$. $\phi$ is lower semi-continuous if for each $\alpha \in \mathbb{R}_+$, the set $\{x \in A_+ | \phi(x) \leq \alpha\}$ is closed. The trace has deep relation with the type of von Neumann algebras. A cone $M$ in the positive part of a $C^*$-algebra $A$ is called hereditary if $0 \leq x \leq y$ and $y \in M$ implies $x \in M$ for each $x$ in $A$. A $*$-subalgebra $B$ of $A$ is hereditary if $B^*$ is hereditary in $A$.

**Lemma 2 ([3])**. Let $B$ a hereditary $C^*$-subalgebra of $A$. Each finite trace $\rho$ on $B$ has an extension to a semi-finite lower semi-continuous trace $\hat{\rho}$ on $A$. 
3. Abelian group $K_0$

Let $A$ be a $*$-algebra. We present the construction of $K_0(A)$, which in genera yields a pre-ordered abelian group, built from the family of self adjoint projections in all matrix algebras over $A$. Let $e, f$ be projections in $A$. $e$ and $f$ are $*$-equivalent, written $e \equiv f$; if there is an element $w \in A$ such that $w^* = e$ $ww^* = e$ $w^*w = f$. We define

$$P(A) = \bigcup_{n=1}^{\infty} \text{projections in } M_n(A).$$

In here $M_n(A) = \{ [a_{ij}]_{m \times n} | a_{ij} \in A \}$. Given $e, f \in P(A)$ $e \equiv f$ mean that

$$\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

for some suitable sized zero matrices. And define $e, f \in P(A)$ to be stably $*$-equivalent, written $e \preceq f$ provided

$$e \oplus g \equiv f \oplus g$$

for some $g \in P(A)$, i.e.,

$$\begin{bmatrix} e & 0 \\ 0 & g \end{bmatrix} \equiv \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}.$$

For $e \in P(A)$, we use $[e]$ to denote the equivalence class of $e$ with respect to $\equiv$. If $e_1, e_2, f_1, f_2 \in P(A)$ with $e_1 \equiv e_2$ and $f_1 \equiv f_2$, then $e_1 \oplus f_1 \equiv e_2 \oplus f_2$. Hence we see that $\oplus$ induces a well-defined binary operation $+$ on the set of equivalence classes $P(A)/\equiv$, where $[e] + [f] = [e \oplus f]$ for all $e, f \in P(A)$. Then the operation is commutative and associative. Moreover the semi-group $(P(A)/\equiv, +)$ satisfies cancellation law: so $(P(A)/\equiv, +)$ is an abelian group.

Denote
\[ P(A)/\approx_0, +) = K_0(A). \]

For any \(*\)-algebra \(A\), we set
\[ K_0(A)_+ = \{ [e]| e \in P(A) \}. \]

For any \(x, y \in K_0(A)\) we define
\[ x \leq y \text{ on } K_0(A) \text{ if and only if } y - x \in K_0(A)_+. \]

The relation \(\leq\) \(K_0(A)\) is a pre-order. A \(C^*\)-algebra \(A\) is an \(AF\) \(C^*\)-algebra if \(A\) is the norm-closure of the union of finite dimensional \(C^*\)-algebras \(A_n\).

**Theorem 3.1 ([1]).** If \(A\) is an \(AF\) \(C^*\)-algebra, then \(K_0(A)\) is a partially ordered abelian group.

**Proposition 3.2.** Let \(A\) be a \(AF\) \(C^*\)-algebra and \(p, q\) be projections in \(A\). If \(\phi(p) \leq \phi(q)\) for all nonzero traces \(\phi\) on \(A\), then \([p] \leq [q]\) in \(K_0(A)\).

**Proof.** We may assume that \(p, q\) lie in a finite dimensional subalgebra \(A_1\) by replacing \(p\) and \(q\) by equivalent projections. By [2. Theorem 2.2], we can find an increasing sequence \((A_n)_{n=1}^\infty\) of finite dimensional subalgebras containing \(A_1\) and \(A\) is the norm closure of \(\bigcup_{n=1}^\infty A_n\). If \(\phi(p) \leq \phi(q)\) for all trace \(\phi\) on \(A\), then \(\phi(p) \leq \phi(q)\) for all trace \(\phi\) on \(A_n\) for all \(n\). If not; let \(e\) be the unit of the finite dimensional \(C^*\) subalgebra \(A_1\). There exists an integer \(n_0\) and a trace \(\phi_{n_0}\), on \(A_{n_0}\) such that \(\phi_{n_0}(p) > \phi_{n_0}(q)\) and \(\phi_{n_0}(e) = \alpha\), for some \(\alpha > 0\). Let \(\phi'_{n_0} = \frac{1}{\alpha} \phi_{n_0} |_{A_0} e\). Since \(eA_1e = A_1 \subset eA_{n_0}e\), \(\phi'_{n_0}\) is a trace on \(eA_{n_0}e\) such that \(\phi'_{n_0}(p) > \phi'_{n_0}(q)\) and \(\phi'_{n_0}(e) = 1\). Then for \(n > n_0\) there...
exists a trace $\phi_n$ on $e\Lambda_n e$ such that $\phi_n|_{\Lambda_n} = \phi_n$. Hence there exists a trace $\phi_n$ on $e\Lambda_n e$ such that $\phi_n(p) > \phi_n(q)$ and $\phi_n(e) = 1$ for $n > n_0$. Let $e\Lambda e = B$ and $\tilde{\phi}_n$ be an extension of $\phi_n$ to a state on $B$. Since $B$ has a unit $e$, $\{\tilde{\phi}_n\}$ has a weak*-limit $\tilde{\phi}$. Then $\tilde{\phi}$ is a tracial state and $\tilde{\phi}(e) = 1$.

Since $B$ is a hereditary subalgebra of $A$ and $\tilde{\phi}|_{\Lambda_n}$ is a finite trace on $B$, by Lemma 2.1 $\tilde{\phi}$ extended to a trace on $A$. Furthermore $\tilde{\phi}(p) > \tilde{\phi}(q)$ and this contradicts to the hypothesis. Hence $[p] \leq [q]$ in $K_0(A_n)$. Since $[p] \leq [q]$ in $K_0(A)$ if and only if $[p] \leq [q]$ in $K_0(A_n)$ for some $n$, $[p] = [q]$ in $K_0(A)$.

Since $K_0(A)$ is a partially ordered group for an AF C*-algebra $A$, if $p, q$ are projections and $\phi(p) = \phi(q)$ for all traces $\phi$ on AF C*-algebra $A$, then $[p] = [q]$ in $K_0(A)$.

**Proposition 3.3.** Let $A$ be an AF C*-algebra and $p, q$ be projections in $A$. Then $[p] = [q]$ in $K_0(A)$ if and only if $p \geq q$ in $A$.

**Proof.** Clearly $p \geq q$ implies $[p] = [q]$. In AF C*-algebra by [1. Lemma 20] if $[p] = [q]$ in $K_0(A)$, then $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \ast \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$ for some suitable sized zero matrix. Hence there exists a $w \in M_n(A)$ such that $w^*w = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ and $ww^* = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ for some $n$. Since $e, f$ are in $A$ and the zero matrices in the above is of the same sized, there exists a partial isometry $w' \in A$ such that $w = \begin{bmatrix} w' & 0 \\ 0 & 0 \end{bmatrix}$. 


4. Main results.

Let $A$ be a $*$-algebra. A set $n \times n$ $*$-matrix units in $A$ is a set of $n \times n$ matrix units $\{e_{ij} | i, j = 1, \ldots, n\}$ of elements of $A$ such that $e_{ij} e_{km} = \delta_{jm} e_{im}$ and $e_{ij}^* = e_{ij}$ for all $i, j$.

In this case $e_{11}, \ldots, e_{nn}$ are orthogonal projections. A $*$-matricial subbasis in $A$ is a set $\{e_{ij}^i | i = 1, \ldots, k, p, q = 1, \ldots, n(i)\}$ of elements of $A$ such that

1) $\{e_{ij}^i | p, q = 1, \ldots, n(i)\}$ is a set of $n(i) \times n(i)$ $*$-matrix units for each $i = 1, \ldots, k$;

2) $e_{ij}^i e_{ij}^{i'} = 0$ for all $i, j, p, q, r, s$ with $i \neq j$.

Then $e^{(i)} = \sum_{p=1}^{n(i)} e_{ij}^p$ are mutually orthogonal projections in $A$. If a $*$-algebra $A$ has a $*$-matricial subbasis $\{e_{ij}^i\}$ that spans $A$, then $\{e_{ij}^i\}$ is a $*$-matricial basis for $A$. In this case $\sum_{i=1}^k \sum_{j=1}^{n(i)} e_{ij}^i$ is a unit of $A$. Thus a $*$-algebra is matricial if and only if it has a $*$-matricial basis.

**Theorem 4.** Let $A$ be an AF $C^*$-algebra with unit acting on a separable Hilbert space $H$. Suppose that $M \subset A$ and $N \subset A$ are $*$-isomorphic finite dimensional $C^*$-subalgebras of $A$. Then there exists a unitary element $u$ in $A$ such that $uM u^* = N$.

**Proof.** Suppose that $\{E_{ij}^k | i, j = 1, \ldots, n_k, k = 1, \ldots, n\}$ and $\{F_{ij}^k | i, j = 1, \ldots, n_k, k = 1, \ldots, n\}$ are $*$-matricial basis of $M$ and $N$ respectively. We may assume that $M, N$ have the same unit with $A$. We show that there exists a partial
isometry \( V^k \subset A \) with initial projection \( E_{i1}^k \) and terminal projection \( F_{i1}^k \) for \( k = 1, \ldots, n \). Let \( U = \sum_{k=1}^{n} \sum_{i=1}^{n_k} F_{i}^k V^k E_{i1}^k \).

\[
u E_{ij}^* = \left( \sum_{k} \sum_{i=1}^{n_k} F_{i}^k V^k E_{i1}^k \right) (E_{ij}^*) \left( \sum_{p} \sum_{i=1}^{n_p} F_{i}^p V^* E_{i1}^p \right)
\]

\[
= \left( \sum_{k} \sum_{q} F_{qi}^k V^k E_{i1}^k \right) (E_{ij}^*) \left( \sum_{p} \sum_{i=1}^{n_p} E_{ip}^k (V^*) F_{ip}^p \right)
\]

\[
= F_{i1}^r V^r E_{ij}^r E_{ij}^r (V^*) F_{ij}^r
\]

\[
= F_{i1}^r V^r E_{i1}^r (V^*) F_{ij}^r = F_{ij}^r.
\]

Therefore

\[
u M \nu^* = N \quad \text{and} \quad \nu \nu^* = \sum_{k, q, \psi, \sigma} (F_{qi}^k V^k E_{i1}^k) (F_{ip}^q V^* E_{ij}^q)^*
\]

\[
= \sum_{k, q, \psi, \sigma} \delta_{k, q} \delta_{\psi, \sigma} F_{qi}^k V^k E_{i1}^k E_{ip}^q (V^*) F_{ij}^q
\]

\[
= \sum_{k, q} F_{qi}^k V^k E_{i1}^k E_{ij}^q (V^k)^* F_{ij}^q
\]

\[
= \sum_{k, q} F_{qi}^k = I
\]

Similarly \( \nu^* \nu = I \) Hence \( \nu \) is the unitary that we want.

Let \( \phi \) be a trace on \( A \). Let \( p_i^k = EE_{i1}^k, Q_i^k = EF_{i1}^k \) for central projection \( E \subset A \). Then

\[
\sum_{k=1}^{n} \sum_{i=1}^{n_k} p_i^k = \sum_{k=1}^{n} \sum_{i=1}^{n_k} Q_i^k = E.
\]

We put

\[
E_i^k = \sum_{i=1}^{n_k} E_{i1}^k
\]
Let
\[ S(i) = \sum_{k=1}^{n} E^k - E^k_{ii} + E^k_{li} + E^k_{il} \]
and
\[ V(i) = \sum_{k=1}^{n} F^k - F^k_{ii} + F^k_{li} + F^k_{il} \]

Then
\[ S(i) P^k_S(i)^* = \left( \sum_{i=1}^{n} E^i - E^i_{ii} + E^i_{li} + E^i_{il} \right) \left( EE^k_{ll} \right) \]
\[ \cdot \left( \sum_{r=1}^{n} E^r - E^r_{ii} + E^r_{li} + E^r_{il} \right)^* \]
\[ = \left( \sum_{i=1}^{n} \delta_{ik} EE^i_{ll} E^k_{li} - EE^k_{ll} E^k_{li} - EE^k_{li} E^k_{il} \right) \]
\[ + EE^k_{li} E^k_{il} + EE^k_{ii} E^k_{il} \]
\[ \left( \sum_{r=1}^{n} E^r - E^r_{ii} + E^r_{li} + E^r_{il} \right)^* \]
\[ = \sum_{r} \delta_{rk} EE^k_{ll} \left( E^r - E^r_{ii} E^r_{li} + E^r_{il} \right) \]
\[ = EE^k_{li} E^k_{il} - EE^k_{ii} E^k_{il} E^k_{li} - EE^k_{il} E^k_{li} + EE^k_{ll} E^k_{il} \]
\[ + EE^k_{li} E^k_{il} = EE^k_{ll} = P^k_{ll}. \]

Moreover \( S(i) S(i)^* = S(i)^* S(i) = \sum_{k=1}^{n} E^k = I. \) By similar computation \( V(\alpha) Q^k_{l} V(\alpha)^* = Q^k_{l}. \)
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Since $S_{(\ell)}$, $V_{(\ell)}$ are unitary and trace is invariant under inner automorphisms, $\phi(EE^k_1) = \phi(EE^k_{ij}) = \phi(EE^k_{jj}) = \phi(EE^k_{jj})$ for all $k=1, \ldots, n$.

Since $E$ is a central projection, $\phi(E^k_1) = \phi(F^k_1)$ for all trace $\phi$ on $A$. By Proposition 3.2 $[E^k_1] = [F^k_1]$ in $K_0(A)$ and by Proposition 3.3 there exists a partial isometry $V^k \subseteq A$ with initial projection $E^k_1$ and terminal projection $F^k_1$.

References


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