Kyungpook Mathematical Journal
Volume 32, Number 1, June, 1992

ON MULTIPLICATION MODULES

Eun Sup Kim and Chang Woo Choi

1. Introduction

In this note all rings are commutative rings with an identity and all modules are unital. Let R be a ring and M an R-module. Then M is called a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$. If N is a submodule of M then $(N : M) = \{r \in R : rM \subseteq N\}$. It is clear that every cyclic R-module is a multiplication module. Let P be a maximal ideal of a ring R. An R-module M is called *P-torsion* provided for each $m \in M$ there exists $p \in P$ such that $(1-p)m = 0$. On the other hand M is called *P-cyclic* provided there exist $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$. For given an R-module M, we consider the associated ideal $\theta(M) = \sum_{x \in M}(Rx : M)$.

In Section 2 we investigate multiplication modules. We show that an R-module M is a multiplication module if and only if $Rm = \theta(M)m$ for all $m \in M$.

In Section 3 some properties of multiplication modules are studied.

2. Multiplication modules

Let R be a commutative ring with identity and M an R-module. Then M is called a *multiplication module* if for each submodule N of M there exists an ideal I of R such that $N = IM$. Let N be a submodule of a multiplication module M. It is well known that M is a multiplication module if and only if $N = (N : M)M$ for all submodules N of M. An R-module M is called a *locally cyclic* if M_p is a cyclic R_p-module for all maximal ideals P of R.

Received November 5, 1991.
This research was partially supported by TGRC-KOSEF.

Theorem 1. *Let R be a ring and let M be an R-module. Then the following statements are equivalent.*

(i) *M is a multiplication module*

(ii) *$Rm = \theta(M)m$ for all $m \in M$.*

Proof. Suppose M is a multiplication R-module. Let $m \in M$. Then $Rm = (Rm : M)M$. Thus $M = \sum_{m\in M} Rm = \sum_{m\in M}(Rm : M)M = \theta(M)M$. where $\theta(M) = \sum_{m\in M}(Rm : M)$. Now let $x \in M$. Then

$$\begin{aligned} Rx &= (Rx : M)M = (Rx : M)\theta(M)M \\ &= \theta(M)(Rx : M)M = \theta(M)Rx. \end{aligned}$$

Therefore $Rx = \theta(M)x$ for all $x \in M$.

Conversely, suppose (ii) holds. Let P be a maximal ideal of R. If $\theta(M) \subseteq P$ then for any $m \in M$, $Rm = Pm$ by hypothesis and hence M is P-torsion for all maximal ideal P of R. Otherwise $\theta(M) \not\subseteq P$, and hence $(Rx : M) \not\subseteq P$ for some $x \in M$. Then $(1-q)M \subseteq Rx$ for some $q \in P$. By [4, Theorem 1.2], M is a multiplication R-module.

Theorem 1 has two corollaries which we wish to mention. The first is an immediate consequence of the theorem and the second is an alternative proof of the well known result [4, Corollary 1.4.] following by our technique.

Corollary 2. *Let R be a domain and let M be a faithful multiplication R-module. Then M is finitely generated and locally cyclic.*

Proof. By Theorem 1, $Rm = \theta(M)m$ for all $m \in M$ and hence $R(Rm) = R(\theta(M)m) = \theta(M)(Rm)$ for all $m \in M$. But Rm is a faithful R-module by [4, Lemma 4.1] and so Rm is a finitely generated faithful multiplication R-module. By [4, Theorem 3.1], $\theta(M) = R$. Thus M is finitely generated and locally cyclic by [1, Theorem 1].

Corollary 3. *Let I be a multiplication ideal of a ring R and M a multiplication R-module. Then IM is a multiplication R-module.*

Proof. By the theorem $Ri = \theta(I)i$, $Rm = \theta(M)m$ for all $i \in I$, $m \in M$. Thus $Rim = \theta(I)\theta(M)im$. Clearly $\theta(I)\theta(M) \subseteq \theta(IM)$ and so $Rim = \theta(IM)im$. Therefore $Rx = \theta(IM)x$ for all $x \in IM$. By Theorem 1, IM is a multiplication module.

For an R-module homomorphism $f : M \longrightarrow N$, our next result shows a criterion that it makes onto.

Theorem 4. *Let $f : M \longrightarrow N$ be a homomorphism of R-modules. Then the following statements are equivalent.*

(i) *For each maximal ideal P of R, the induced map $f_{[P]} : M/PM \longrightarrow N/PN$ given by $m + PM \mapsto f(m) + PN$ is onto and $N/f(M)$ is a multiplication R-module.*

(ii) *f is onto.*

Proof. (ii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (ii). Note that $f(M) + PN = N$ for all maximal ideals P of R. This implies $P(N/f(M)) = N/f(M)$. Since $N/f(M)$ is a multiplication R-module, $f(M) = N$. For, suppose M is a multiplication R-module and $M = PM$ for all maximal ideal P of R. If M is nonzero, then there exists a maximal ideal Q of R such that M is Q-cyclic by [4, Theorem 2.5] and hence $M \neq QM$ by [4, Theorem 1.2] and [8, Lemma 6]. This is a contradiction and so our theorem is proved.

Compare the next result with [4, Corollary 2.4].

Proposition 5. *Let M be an R-module which is P-cyclic for only finitely many maximal ideals P of R. Then M is a multiplication module if and only if M is cyclic.*

Proof. As we remarked above, cyclic modules are multiplication modules. Conversely, suppose M is a multiplication module. Let $P_1, P_2, \cdots, P_n$ be the maximal ideals of R such that M is P-cyclic. Then $M \neq P_iM$ for all $1 \leq i \leq n$. Put $P_iM = N_i$. Then N_i is a maximal submodules of M for each $1 \leq i \leq n$ by [4, Theorem 2.5]. These N_i are the only maximal submodules of M. Indeed, suppose that there exists a maximal submodule N of M such that $N \neq N_i$ for all $1 \leq i \leq n$. Then again by [4, Theorem 2.5], there exists a maximal ideal P of R such that $N = PM \neq M$. By [8, Lemma 6], M is P-cyclic. By hypothesis $P = P_i$ for some $1 \leq i \leq n$. This implies $N = N_i$ for some $1 \leq i \leq n$, a contradiction. Thus M has only finitely many maximal submodules. Hence M is cyclic by [4, Theorem 2.8].

3. Some properties of multiplication modules

Let R be a commutative ring with identity and M an R-module. In this section we investigate some properties of multiplication modules. In particular, we prove Fitting's Lemma in terms of multiplication module.

Theorem 6. *Let M be a multiplication R-module satisfying descending* *chain conditions on multiplication submodules and let $f \in End_R(M)$. Then f is a one–to–one function if and only if f maps onto M.*

Proof. Suppose f maps onto M. $\mathrm{Ker}(f) = IM$ for some ideal I of R. Thus $0 = f(\mathrm{Ker}f) = f(IM) = If(M) = IM = \mathrm{Ker}f$ and hence f is a one–to–one function.

Conversely, suppose that f is one–to–one and consider the chain of R-submodules $M \supseteq f(M) \supseteq f^2(M) \supseteq \cdots$. Since M is a multiplication R–module, so is every homomorphic images of M. By hypothesis, this chain will terminate after a finite number of steps, say n steps; then $f^n(M) = f^{n+1}(M)$. Given an arbitrary $x \in M, f^n(x) = f^{n+1}(y)$ for some $y \in M$. As f is assumed to be a one–to–one function, f^n also enjoys this property, whence $x = f(y)$. This implication is that $M = f(M)$ and so f maps onto M.

Proposition 7. *Let M be a multiplication R-module. Then*

(i) *Every submodule of M is fully invariant for all $f \in End_R(M)$.*

(ii) *$f \in End_R(M)$ is an epimorphism if and only if $(f|N) : N \to N$ is an epimorphism for all submodule N of M.*

Proof. (i) Let N be a submodule of M. By hypothesis, $N = IM$ for some ideal I of R. Let $f \in \mathrm{End}_R(M)$. Then $f(N) = f(IM) = If(M) \subseteq IM = N$. i.e., $f(N) \subseteq N$ for all submodule N of M. Hence every submodule of M is fully invariant for all $f \in \mathrm{End}_R(M)$.

(ii) The sufficiency is obvious. Conversely, let N be any submodule of M. Then $N = IM$ for some ideal I of R. This implies $f(N) = f(IM) = If(M) = IM = N$. This completes the proof.

Note that Proposition 7 (ii) gives at once that every epimorphism of a multiplication R-module is an automorphism.

Next we note a further property of multiplication modules.

Proposition 8. *Let M be an R-module and let $R_0 \subseteq R$ be a subring of R. If M is a multiplication R_0-module, then M is a multiplication R-module.*

Proof. Let N be a R-submodule of M. Then N is a R_0-submodule of M. Since M is a multiplication R_0-module, there exist an ideal I_0 of R_0 such that $N = I_0M$. Thus $N = I_0M = I_0(RM) = (I_0R)M$. Since I_0R is an ideal of R, M is a multiplication R-module.

Theorem 9. *Let M be a multiplication R-module satisfying descending chain conditions on multiplication submodules and let $f \in End_R(M)$.*

Then, for some n, $M = f^n M \oplus f^{-n}0$.

Proof. Consider the sequence $M \supset f(M) \supset f^2(M) \supset \cdots$. Since every homomorphic images of multiplication modules are multiplication ones and M satisfies descending chain conditions on multiplication submodules by hypothesis, the sequence becomes stationary after n steps, say. Thus $f^{(n)}(M) = f^{n+1}(M) = \cdots = f^{2n}(M) = \cdots$. Therefore f^n induces an endomorphism on multiplication module $f^{(n)}(M)$ which is an epimorphism, hence an automorphism by Proposition 7. Thus $f^n(M) \cap f^{-n}0 = 0$. Now take any $m \in M$, then $f^n(m) = f^{2n}(n)$ for some $n \in M$, hence $m - f^n(n) \in Ker(f^n)$. Since $m = f^n(n) + (m - f^n(n))$, $M = f^n M \oplus f^{-n}0$. This completes the proof.

Corollary 10. *If a free R-module M is a multiplication module, then M is isomorphic to a single factor of R i.e.* $M \cong R$.

Proof. Suppose M is isomorphic to a direct sum of R more than two. Define $f : M \to \oplus R$ by $(m_1, m_2, m_3, \cdots) \to (m_2, m_1, m_3, \cdots)$. Then f is an R-automorphism of M. Let $N = R \oplus \{0\} \oplus R \oplus \cdots \oplus R \oplus \cdots$ be a submodule of M. Then $f(N) = \{0\} \oplus R \oplus R \oplus \cdots \oplus R \oplus \cdots \not\subseteq N$, a contradiction.

Remark. Corollary 10 shows that if M is a multiplication module as a vector space, then the dimension of M is always 1.

We close this section with additional simple properties of multiplication modules.

Proposition 11. *Let M be an R-algebra and a multiplication R-module. If* $f \in End_R(M)$, *then f is a monomorphism.*

Proof. $Ker f = IM$ for some ideal I of R. Let $x \in Ker f$. Then $x = \alpha_1 m_1 + \cdots + \alpha_n m_n$ for some $\alpha_i \in I$, $m_i \in M$ $(1 \leq i \leq n)$. Since $y = y1 \in IM = Ker f$ for all $y \in I$, $0 = f(\alpha_i) = f(\alpha_i 1) = \alpha_i f(1) = \alpha_i$. This implies $x = 0$. This completes the proof.

ACKNOWLEDGEMENT. The authors wish to thank Patrick F. Smith for his helpful comments and suggestions.

References

[1] D.D. Anderson, *Some remarks on multiplication ideals*, Math. Japonica 4 (1980),

463-469.

[2] F.W. Anderson and K.R. Fuller, *Rings and categories of modules*(Springer-Verlag, 1974).

[3] A. Barnard, *Multiplication module*, J. Algebra 71(1981), 174-178.

[4] Z.A. EL-BAST and P.F. Smith, *Multiplication modules*, Communication in Algebra, Vol.16,(1988), 755-779.

[5] Thomas W. Hungerford, Algebra, New York, Springer Verlag, 1974.

[6] J. Lambek, Lectures on rings and modules, London 1966.

[7] G.M. Low and P.F. Smith, *Multiplication modules and ideals*, To appear in Communication in Algebra.

[8] Patrick, F. Smith, *Some remarks on multiplication modules*, Arch. Math. Vol.50, (1988), 223-235.

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.