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# FIXED POINT THEORY OF MULTIFUNCTIONS IN TOPOLOGICAL VECTOR SPACES, II

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## 1. Introduction

This is a continuation of our previous work [**P5**] which will be called Part I. In Part I, we applied an existence theorem of maximizable quasiconcave functions on convex spaces to obtain coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points of convex-valued multifunctions defined on convex subsets of a topological vector space. Consequently, we could generalize and unify many of historically well-known Brouwer or Kakutani type fixed point theorems to a class of multifunctions more general than weakly inward (outward) upper hemicontinuous ones defined on convex subsets of topological vector spaces having sufficiently many linear functionals.

After we completed Part I, some new results on Kakutani maps were obtained by the author and Bae [**PB**] and Idzik [**I**]. Moreover, in [**P6-9**], fixed point theorems on acyclic maps defined on convex subsets of topological vector spaces were studied. Further, certain coincidence theorems on acyclic maps or more general class of multifunctions were applied to the KKM theory in the author's previous works [**P6,10**]. Especially, [**P10**] was motivated by the works of Ben-El-Mechaiekh and Deguire [**BD1,2**] on a very large class of "admissible" upper semicontinuous multifunctions with non-convex values.

The purpose in this paper is, first, to give common generalizations of some results in [**P5**], [**PB**], and [**I**]. This will give more adequate understanding on the nature of the results on convex-valued multifunctions in Part I. Our second purpose is to obtain new fixed point or related results on compact composites of non-convex valued "admissible" upper semicontinuous multifunctions defined on convex subsets of topological vector spaces having sufficiently many linear functionals.

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Many results in **[P6-9]** are substantially generalized. Consequently, well-known results of Fan, Halpern and Bergman, Browder, Fitzpatrick and Petryshyn, Reich, Ha, Granas and Liu, Idzik, and many others are improved, extended, and unified.

## 2. Preliminaries

We mainly follow our previous works **[P5,8]**.

A *multifunction* (or *map*) $F : X \to 2^Y$ is a function from a set $X$ into the set $2^Y$ of nonempty subsets of a set $Y$. As usual, $F$ also denotes its graph; that is, $(x,y) \in F$ if and only if $y \in Fx$.

For topological spaces $X$ and $Y$, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed subset $B$ of $Y$, $\{x \in X : Fx \cap B \neq \emptyset\}$ is closed; *closed* if $F$ is closed in $X \times Y$; and *compact* if the range $F(X)$ is contained in a compact subset of $Y$.

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* $X$ is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hull is called a *polytope.* A nonempty subset $L$ of a convex space $X$ is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$ **[L1]**. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in $X$, where $x \in X$.

A Hausdorff topological vector space is abbreviated as a t.v.s. In Part I, we assumed that every t.v.s. is real, but not in this paper. Let $E$ be a t.v.s. and $E^*$ its topological dual. A multifunction $F : X \to 2^E$ defined on a topological space $X$ is said to be *upper hemicontinuous* (u.h.c.) if for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$ and for any real $\alpha$, the set $\{x \in X : \sup p(Fx) < \alpha\}$ is open in $X$. Note that an upper semicontinuous (u.s.c.) function $F : X \to 2^E$ is upper demicontinuous (u.d.c.) and that an u.d.c. function is u.h.c.

Let $c(E)$ denote the set of nonempty closed subsets of $E$, $cc(E)$ the set of nonempty closed convex subsets of $E$, and $kc(E)$ the set of nonempty compact convex subsets of $E$. Bd, Int, and — will denote the boundary, interior, and closure, resp., with respect to $E$.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of $X$ at $x$,

$I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) := x + \bigcup_{r>0} r(X - x), \quad O_X(x) := x + \bigcup_{r<0} r(X - x).$$

A function $F : X \to 2^E$ is said to be *weakly inward* [resp. *outward*] if $Fx \cap \overline{I}_X(x) \neq \emptyset$ [resp. $Fx \cap \overline{O}_X(x) \neq \emptyset$] for each $x \in \operatorname{Bd} X \backslash Fx$.

For $p \in \{\operatorname{Re} h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) := \inf\{|p(u - v)| : u \in U, \ v \in V\}.$$

Recall that a real-valued function $f : X \to \mathbb{R}$ on a topological space $X$ is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is open for each $r \in \mathbb{R}$. If $X$ is a convex set in a vector space, then $f$ is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : fx > r\}$ [resp. $\{x \in X : fx < r\}$] is convex for each $r \in \mathbb{R}$.

Given a class $\mathbb{L}$ of multifunctions, $\mathbb{L}(X, Y)$ denotes the set of multifunctions $T : X \to 2^Y$ belonging to $\mathbb{L}$, and $\mathbb{L}_c$ the set of finite composites of multifunctions in $\mathbb{L}$.

For topological spaces $X$ and $Y$, we define

$T \in \mathbb{K}(X, Y) \iff T$ is a Kakutani map; that is, $Y$ is a convex space and $T$ is u.s.c. with $Tx \in kc(Y)$ for $x \in X$.

$T \in \mathbb{V}(X, Y) \iff T$ is an acyclic map; that is, $T$ is u.s.c. with compact acyclic values.

We now introduce an abstract class $\mathfrak{A}$ of multifunctions as in [**P10**]:

A class $\mathfrak{A}$ of multifunctions is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope $P$, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Note that $\mathbb{C}$, $\mathbb{K}$, and $\mathbb{V}$ are examples of $\mathfrak{A}$. See Park [**P8**]. Moreover, the class of approachable maps in a t.v.s. [**BD1**] also belongs to $\mathfrak{A}$. For other examples of related classes, see [**BD1,2**], [**L4**], [**P8**].

$T \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset $K$ of $X$, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma x \subset Tx$ for each $x \in K$.

A class $\mathfrak{A}_c^\kappa$ will be called *admissible*. The class $\mathbb{K}_c^+$ due to Lassonde [**L2**] and $\mathbb{V}_c^+$ due to Park, Singh, and Watson [**PSW**] are examples of $\mathfrak{A}_c^\kappa$. Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\kappa$. For other examples, see Park [**P10**].

## 3. Convex-valued multifunctions in a t.v.s.

In this section, we give common generalizations of some results in [**P4**], [**PB**], and [**I**]. In fact, fixed point or related results on convex-valued multifunctions defined on convex subsets of a t.v.s. are generalized.

For a convex space $X$, let $\hat{X}$ denote the set of all u.s.c. quasiconcave real functions defined on $X$.

The following can be obtained by the same proof for [**P5**, Theorem 3]:

THEOREM A. *Let $X$ be a convex space, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $E$ a t.v.s. with topological dual $E^*$, $B : E^* \to 2^{\hat{X}}$ a multifunction with convex graph, and $F, G : X \to 2^E$. Suppose that for each $h \in E^*$,*

(1) $X_h := \{x \in X : \sup \operatorname{Re} h(Fx) \geq \inf \operatorname{Re} h(Gx)\}$ *is compactly closed;*
(2) *for each $x \in K$ and each $g \in Bh$, $gx = \max g(X)$ implies $x \in X_h$; and*
(3) *for each $x \in X \backslash K$ and each $g \in Bh$, $gx = \max g[x, L]$ implies $x \in X_h$.*

*Then there exists an $x \in \bigcap \{X_h : h \in E^*\}$.*

For a subset $S$ of $E$, let $\operatorname{coc} S$ denote the intersection of all closed halfspaces containing $S$ [**I**]. Here, a closed halfspace means a set $\{x \in E : \operatorname{Re} hx \geq r\}$ for some $h \in E^*$ and $r \in \mathbb{R}$.

The following existence theorem of zero is the main result in this section:

THEOREM 1. *Let $X, L, K, E$, and $B$ be the same as in Theorem A. Let $F : X \to 2^E$ satisfy the following for each $h \in E^*$:*

(1.1) $X_h := \{x \in X : \inf \operatorname{Re} h(Fx) \leq 0\}$ *is compactly closed in $X$;*
(1.2) *for each $x \in K$ and $g \in Bh$, $gx = \max g(X)$ implies $x \in X_h$; and*
(1.3) *for each $x \in X \backslash K$ and $g \in Bh$, $gx = \max g[x, L]$ implies $x \in X_h$.*

*Then there exists an $x \in X$ such that $0 \in \operatorname{coc} Fx$.*

*Proof.* Suppose that for each $x \in X$ we have $0 \notin \operatorname{coc} Fx$. Then 0 does not belong to a closed halfspace containing $Fx$; that is, there exist an $h \in E^*$ and a $t \in \mathbb{R}$ such that $\operatorname{Re} h(0) = 0 < t \leq \operatorname{Re} hy$ for all $y \in Fx$. So $\inf \operatorname{Re} h(Fx) > 0$ and hence $x \notin X_h$. From (1.1)-(1.3), by considering the ordered pair $(0, F)$ instead of $(F, G)$, all of the requirements of Theorem A are satisfied. Therefore, we should have an $x \in \bigcap\{X_h : h \in E^*\}$, a contradiction.

We need the following:

LEMMA. *For a subset $S$ of a t.v.s. $E$,*
(i) *if $E$ is locally convex and $S \in cc(E)$, then $\operatorname{coc} S = S$; and*
(ii) *if $E^*$ separates points of $E$ and $S \in kc(E)$, then $\operatorname{coc} S = S$.*

*Proof.* It suffices to show that $\operatorname{coc} S \subset S$ under the hypothesis of (i) or (ii). Suppose that there exists a $y \in (\operatorname{coc} S)\backslash S$. Then, by the standard separation theorems on a t.v.s., there exist an $h \in E^*$ and a $t \in \mathbb{R}$ such that $\operatorname{Re} hy < t < \operatorname{Re} hz$ for all $z \in S$. This implies $y \notin \operatorname{coc} S$, a contradiction.

From Theorem 1 and Lemma, we have the following in [**PB**, Theorem 3]:

COROLLARY 1.1. *In Theorem 1, further, suppose that either*

(A) *$E^*$ separates points of $E$ and $\overline{\operatorname{co}}\, Fx$ is compact for each $x \in X$; or*

(B) *$E$ is locally convex.*

*Then there exists an $x \in X$ such that $0 \in \overline{\operatorname{co}}\, Fx$.*

As we showed in [**P5**], [**PB**], Corollary 1.1 has many equivalent or particular known results. Moreover, Corollary 1.1 is equivalent to [**P5**, Theorem 5], which was stated for a closed-convex-valued multifunction.

From Theorem 1, by putting $X = K$ and $Bh = (\operatorname{Re} h)|_X$ for each $h \in E^*$, we have the following:

COROLLARY 1.2. *Let $E$ be a t.v.s., $K \in kc(E)$, and $F : K \to 2^E$ satisfy the following for each $p \in \{\operatorname{Re} h : h \in E^*\}$:*

(1) *$\{x \in K : \inf p(Fx) \leq 0\}$ is closed; and*

(2) *for each $x \in \operatorname{Bd} K$, $px = \max p(K)$ implies $\inf p(Fx) \leq 0$.*

*Then there exists an $x \in K$ such that $0 \in \operatorname{coc} Fx$.*

*Proof.* It suffices to show that $x \in \operatorname{Bd} K$ in (2) can be replaced by $x \in K$. In fact, if $x \in \operatorname{Int} K$, then there exists an open neighborhood

$U$ of the origin of $E$ such that $x+U \subset K$; and if $px = \max p(K)$, then $px + pu \le px$ or $pu \le 0$ for all $u \in U$. Therefore, we must have $p = 0$ and hence $\inf p(Fx) \le 0$. This completes our proof.

REMARK. If $F$ is u.h.c., (1) clearly holds. In this case, Corollary 1.2 reduces Idzik [**I**, Theorem 1]. Moreover, note that (2) can be replaced by

(2)′ for each $x \in \operatorname{Bd} K$, there exists a $\lambda > 0$ and a $u \in Fx$ such that $x + \lambda u \in K$.

In fact, if $px = \max p(K)$, then $p(x + \lambda u) = px + \lambda pu \le px$, whence $pu \le 0$. Since $u \in Fx$, we have $\inf p(Fx) \le 0$. Idzik [**I**, Theorem 10] obtained a particular form of Corollary 1.2 under the assumption (2)′. Further, note that [**I**, Corollary 11] is a particular form of [**PB**, Theorem 2].

From Theorem 1, we have the following fixed point theorem:

THEOREM 2. *Let $X$ be a convex space, $L$ a c-compact subset of $X$, $K$ a nonempty compact subset of $X$, $E$ a t.v.s. containing $X$ as a subset, and $F : X \to 2^E$. Suppose that for each $p \in \{\operatorname{Re} h : h \in E^*\}$ the following holds:*

(2.0) *$p|_X$ is continuous on $X$;*
(2.1) *$X_p := \{x \in X : \inf p(Fx) \le px\}$ is compactly closed in $X$;*
(2.2) *$d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \operatorname{Bd} X$; and*
(2.3) *$d_p(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$.*

*Then there exists an $x \in X$ such that $x \in \operatorname{coc} Fx$.*

*Proof.* We use Theorem 1 with $Bh = \{\operatorname{Re} h|_X\}$ for each $h \in E^*$. Considering $Fx - x$ instead of $Fx$ in Theorem 1, (2.1) implies (1.1). Since $I_X(x) = E$ for $x \in \operatorname{Int} X$, (2.2) is actually equivalent to the following:

(2.2)′ $d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K$.

We show that (2.2)′ implies (1.2) for $Fx - x$. Let $x \in K$ such that $px = \max p(X)$. Suppose that $\inf p(Fx) > px$. Then, for any $v \in Fx$, $z = x + r(u - x) \in I_X(x)$, $u \in X$, and $r > 0$, we have

$$|p(v - z)| \ge p(v - x) + rp(x - u) \ge p(v - x)$$

and hence

$$d_p(Fx, \overline{I}_X(x)) = d_p(Fx, I_X(x)) \ge pv - px \ge \inf p(Fx) - px > 0.$$

This contradicts (2.2)$'$. Similarly, (2.3) implies (1.3). Therefore, by Theorem 1, we have an $x \in X$ such that $0 \in \operatorname{coc}(Fx - x)$; that is, $x \in \operatorname{coc} Fx$.

The following is a dual form of Theorem 2 and a surjectivity result:

THEOREM 2$'$. *Let $X, L, K, E$, and $F$ be the same as in Theorem 2. Suppose that for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$ the following holds:*

(2.0) *$p|_X$ is continuous on $X$;*
(2.1)$'$ *$X_p = \{x \in X : \sup p(Fx) \geq px\}$ is compactly closed in $X$;*
(2.2)$'$ *$d_p(Fx, \overline{O}_X(x)) = 0$ for every $x \in K \cap \mathrm{Bd}\, X$; and*
(2.3)$'$ *$d_p(Fx, \overline{O}_L(x)) = 0$ for every $x \in X \backslash K$.*

*Then there exists an $x \in X$ such that $x \in \operatorname{coc} Fx$. Moreover, if $F$ is u.h.c., then $(\operatorname{coc} F)(X) \supset X$.*

*Proof.* Considering $2x - Fx$ instead of $Fx$ in Theorem 1, as in the proof of Theorem 2, we conclude that $\operatorname{coc} F$ has a fixed point. For the surjectivity result, let $y \in X$. Consider $Fx - y$ instead of $Fx$ and $[y, L]$ instead of $L$ in Theorem 1. Then there exists an $x \in X$ such that $0 \in \operatorname{coc}(Fx - y)$; that is, $y \in \operatorname{coc} Fx$. This completes our proof.

REMARKS. 1. In Theorems 2 and 2$'$, we do not require any concrete connection between topologies of $X$ and $E$ except

(2.0) $p|_X \in \hat{X}$ (that is, $p|_X$ is continuous) for all $p \in \{\mathrm{Re}\, h : h \in E^*\}$.

In order to assure the continuity of $p|_X$ for all $p \in \{\mathrm{Re}\, h : h \in E^*\}$, it is sufficient to assume that

(i) as a convex space, $X$ has any topology finer than the relative weak topology with respect to $E$, and

(ii) $E$ has a topology finer than its weak topology.

2. Therefore, in some results in Part I, the expression "$E$ a t.v.s. containing $X$ such that the topology of $X$ is finer than its relative topology w.r.t. $E$" can be replaced by "$E$ a t.v.s. containing $X$ as a subset, and assume (2.0)". Such results in Part I are Corollaries 3.1, 4.1, 4.2, 5.1, and 6.1, and Theorems 6,7, and 8.

In view of Lemma, Theorems 2 and 2$'$ impliy the following in [**PB**, Theorem 4], [**P5**, Theorem 6]:

COROLLARY 2.1. *In Theorems 2 and 2′, further, suppose that either*

(A) *$E^*$ separates points of $E$ and $\overline{\mathrm{co}}\, Fx$ is compact for each $x \in X$; or*

(B) *$E$ is locally convex.*

*Then there exists an $x \in X$ such that $x \in \overline{\mathrm{co}}\, Fx$. Moreover, for the outward case, if $F$ is u.h.c., then $(\overline{\mathrm{co}}\, F)(X) \supset X$.*

As we showed in [**PB**], [**P5**], Corollary 2.1 has many particular known results. One of the simplest case of Corollary 2.1 is given in [**P4**] as a generalization of the Brouwer fixed point theorem.

From Theorem 2 or 2′ with $X = K$, we have the following:

COROLLARY 2.2. *Let $E$ be a t.v.s., $K \in kc(E)$, and $F : K \to c(E)$ an u.h.c. multifunction such that for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$ and $x \in \mathrm{Bd}\, K$, we have*

$$d_p(Fx, \overline{I}_K(x)) = 0 \quad [d_p(Fx, \overline{O}_K(x)) = 0, \text{ resp.}].$$

*Then there exists an $x \in K$ such that $x \in \mathrm{coc}\, Fx$. Further, for the outward case, we have $(\mathrm{coc}\, F)(K) \supset K$.*

Note that Corollary 2.2 contains Idzik [**I**, Corollary 2] and many well-known fixed point theorems. See Part I.

## 4. Compact admissible maps in a t.v.s.

In this section, we obtain mainly sufficient conditions for the existence of fixed points of compact admissible maps defined on a convex subset of a t.v.s. $E$ on which its topological dual $E^*$ separates points. Our arguments are based on a geometric property of a convex set and a variational inequality related admissible maps developed in our previous work [**P10**]. Our new results extend earlier works on acyclic maps in [**P7-9**] to admissible maps.

We begin with the following particular form of minimax inequality with respect to an admissible map given in [**P10**, Theorem 11].

THEOREM B. *Let $X$ be a convex space, $Y$ a Hausdorff space, $T \in \mathfrak{A}_c^\kappa(X, Y)$ a compact multifunction, and $f, g : X \times Y \to \overline{\mathbb{R}}$ two extended real-valued functions such that*

(1) *$g(x, y) \le f(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each* $x \in X$, $y \mapsto g(x,y)$ *is lower semicontinuous on* $Y$*; and*
(3) *for each* $y \in Y$, $x \mapsto f(x,y)$ *is quasiconcave on* $X$.

*Then there exists a* $\overline{y} \in \overline{T(X)}$ *such that*

$$\sup_{x\in X} g(x,\overline{y}) \le \sup_{(x,y)\in T} f(x,y).$$

REMARK. For $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, Theorem B is essentially due to Granas and Liu [**GL**, Theorem 7.1] as a generalization of the celebrated 1972 minimax inequality of Ky Fan [**F5**]. When $f = g$, $T \in \mathbb{K}(X,Y)$, and $Y$ is a compact convex space, Theorem B reduces to Ha [**H**, Theorem 1], where the Hausdorffness of $X$ is superfluous. A far-reaching generalization of Theorem B is given in a recent work of the author [**P10**].

The following is a variant of Theorem B with a lopsided saddle point.

THEOREM 3. *Let* $X$ *be a compact convex space,* $Y$ *a Hausdorff space, and* $T \in \mathfrak{A}_c^\kappa(X,Y)$. *Let* $\phi : X \times Y \to \mathbb{R}$ *be a continuous function such that for each* $y \in Y$, $x \mapsto \phi(x,y)$ *is quasiconvex on* $X$. *Then there exists an* $(x_0, y_0) \in T$ *such that*

$$\phi(x_0,y_0) \le \phi(x,y_0) \quad \text{for all} \quad x \in X.$$

*Proof.* Since $X$ is compact, we may assume that $T \in \mathfrak{A}_c(X,Y)$. Define $f : X \times Y \to \mathbb{R}$ by

$$f(x,y) = \min_{z\in X} \phi(z,y) - \phi(x,y)$$

for $(x,y) \in X \times Y$. Then it is easy to see that $f$ is continuous on $X \times Y$ [**A**, p.70] and satisfies (1), (2), and (3) of Theorem B with $f = g$. Moreover, $T$ is compact since it is u.s.c. and compact-valued. Therefore, by Theorem B, there exists a $\overline{y} \in Y$ such that

$$\sup_{x\in X} f(x,\overline{y}) \le \sup_{(x,y)\in T} f(x,y).$$

Since $x \mapsto \phi(x,\overline{y})$ is continuous on the compact set $X$, there exists an $\overline{x} \in X$ such that $\phi(\overline{x},\overline{y}) = \min_{z\in X} \phi(z,\overline{y})$ or $f(\overline{x},\overline{y}) = 0$. Hence, we have

$$0 \le \sup_{(x,y)\in T} f(x,y).$$

Since the graph of $T$ is closed and hence compact in $X \times Y$, the supremum in the above inequality is attained. This completes our proof.

REMARK. For $T \in \mathbb{K}(X, Y)$, Theorem 3 reduces to Ha [**H**, Theorem 2]. For $X = Y$ and $T = 1_X$, the identity function of $X$, Theorem 3 reduces to Fan [**F5**, Corollary 1]. Moreover, for a normed vector space $E = Y$, a single-valued $T$, and $\phi(x, y) = \|x - y\|$, Theorem 3 reduces to Fan [**F4**, Theorem 2].

A direct consequence of Theorem 3 is as follows:

COROLLARY 3.1. *Let $E$ be a metric t.v.s. where the metric $d$ on $E$ has been chosen so that balls are convex, $X$ a compact convex subset of $E$, and $T \in \mathfrak{A}_c^\kappa(X, E)$. Then there exists an $(x_0, y_0) \in T$ such that*

$$d(x_0, y_0) \le d(x, y_0) \quad \text{for all} \quad x \in X.$$

*Further, if $T \in \mathfrak{A}_c^\kappa(X, X)$, then $T$ has a fixed point.*

*Proof.* In view of Theorem 3 with $\phi = d$, it suffices to show that, for each $y \in E$, $x \mapsto d(x, y)$ is quasiconvex on $X$. In fact, for each real $\lambda$,

$$\{x \in X : d(x, y) < \lambda\} = X \cap \{x \in E : d(x, y) < \lambda\}$$

is convex. This shows the first part. The second part is trivial.

REMARK. Note that certain axiom of the metric $d$ is not necessary. If $T$ is single-valued, Corollary 3.1 reduces to Cellina [**C**], Fan [**F4**], Rassias [**Ra**], and Park [**P3**], which in turn generalize the well-known fixed point theorems of Brouwer [**B**] and Schauder [**S**]. Moreover, for $T \in \mathbb{K}(X, X)$, Corollary 3.1 generalizes other well-known theorems of Kakutani [**Kk**] and Bohnenblust and Karlin [**BK**]. See [**P1-5**].

Let $\mathcal{P}$ denote the family of all weakly continuous seminorms on a t.v.s. $E$.

As another application of Theorem 3, we obtain the following Ky Fan type fixed point theorem:

THEOREM 4. *Let $X$ be a compact convex space, $E$ a t.v.s. on which $E^*$ separates points such that $E$ contains $X$ as a subset and, for each $p \in \mathcal{P}$, $(x, y) \mapsto p(x - y)$ is continuous for $(x, y) \in X \times E$, and $T \in \mathfrak{A}_c^\kappa(X, E)$. Then either $T$ has a fixed point or there exist an $(x_0, y_0) \in T$ and a $p \in \mathcal{P}$ such that*

$$0 < p(x_0 - y_0) \le p(x - y_0) \quad \text{for all} \quad x \in \overline{I}_X(x_0).$$

*Proof.* It suffices to prove for $T \in \mathfrak{A}_c(X, E)$. Suppose that $T$ has no fixed point. Then, for each $x \in X$, the origin 0 of $E$ does not belong to the compact set $K := x - Tx$. For each $z \in K$ there exists a linear functional $\ell_z \in E^*$ such that $\ell_z(z) \neq 0$. Since $\ell_z$ is continuous, there exists an open neighborhood $U_z$ of $z$ such that $\ell_z(y) \neq 0$ for every $y \in U_z$. Let $\{U_{z_1}, \ldots, U_{z_n}\}$ be a finite subcover of the cover $\{U_z\}_{z \in K}$ of $K$ and

$$p_x(y) := \sum_{i=1}^{n} |\ell_{z_i}(y)| \quad \text{for each} \quad y \in E.$$

Then $p_x \in \mathcal{P}$ such that $p_x(z) > 2\delta_x$ for all $z \in K$ for some $\delta_x > 0$.

Since $T$ is u.s.c., there exists an open neighborhood $V_x$ of $x$ in $X$ such that $p_x(u-v) > \delta_x$ for all $u \in V_x$ and $v \in Tu$. Since $\{V_x : x \in X\}$ covers $X$ and $X$ is compact, there exists a finite subcover $\{V_{x_1}, \ldots, V_{x_k}\}$ of $X$. Let $p := \max\{p_{x_i} : 1 \leq i \leq k\}$ and $\delta := \min\{\delta_{x_i} : 1 \leq i \leq k\} > 0$. Then $p \in \mathcal{P}$ and $p(x - y) > \delta$ for all $(x, y) \in T$.

We define a function $\phi : X \times Y \to \mathbb{R}$ by $\phi(x, y) = p(x - y)$ for $(x, y) \in X \times Y$, where $Y := T(X)$ is compact. Then clearly $\phi$ and $T$ satisfy all of the requirements of Theorem 3. Therefore, there exists an $(x_0, y_0) \in T$ such that

$$0 < p(x_0 - y_0) \leq p(x - y_0) \quad \text{for all} \quad x \in X.$$

Now we show that the above inequality holds for all $x \in \overline{I}_X(x_0)$. In fact, for $x \in I_X(x_0)\backslash X$, there exist $u \in X$ and $r > 1$ such that $x = x_0 + r(u - x_0)$. Suppose that $p(x - y_0) < p(x_0 - y_0)$. Since

$$\frac{1}{r}x + (1 - \frac{1}{r})x_0 = u \in X,$$

we have

$$p(u - y_0) \leq \frac{1}{r}p(x - y_0) + (1 - \frac{1}{r})p(x_0 - y_0) < p(x_0 - y_0),$$

which is a contradiction. Therefore $p(x_0 - y_0) \leq p(x - y_0)$ holds for all $x \in I_X(x_0)$; that is, for all $x \in \overline{I}_X(x_0)$.

THEOREM 4$'$. *Let $X$ be a compact convex space, $E$ a t.v.s. on which $E^*$ separates points such that $E$ contains $X$ as a subset and, for each $p \in \mathcal{P}$, $(x,y) \mapsto p(x-y)$ is continuous for $(x,y) \in X \times E$, and $T \in \mathfrak{A}_c^\kappa(X,E)$. Let $T' : X \to 2^E$ be given by $T'x = 2x - Tx$ for $x \in X$. Suppose that $T' \in \mathfrak{A}_c^\kappa(X,E)$. Then either $T$ has a fixed point or there exist an $(x_0, y_0) \in T$ and a $p \in \mathcal{P}$ such that*

$$0 < p(x_0 - y_0) \le p(x - y_0) \quad \text{for all } x \in \overline{O}_X(x_0).$$

*Proof.* By the above inward case, there exist an $(x_0, y_1) \in T'$ and a $p \in \mathcal{P}$ such that

$$0 < p(x_0 - y_1) \le p(x' - y_1) \quad \text{for all} \quad x' \in I_X(x_0).$$

For $x \in O_X(x_0)$, let $x' = 2x_0 - x$ and $y_1 = 2x_0 - y_0$ where $y_0 \in Tx_0$. Then we have

$$0 < p(x_0 - y_0) \le p(x - y_0) \quad \text{for all} \quad x \in O_X(x_0),$$

and hence, for all $x \in \overline{O}_X(x_0)$. Therefore the conclusion of Theorem 4 holds for $\overline{O}_X(x_0)$ instead of $\overline{I}_X(x_0)$.

REMARKS. 1. As in Remarks of Theorems 2 and 2$'$, in Theorems 4 and 4$'$, as a convex space, $X$ can have any topology such that, for each $p \in \mathcal{P}$, $(x,y) \mapsto p(x-y)$ is continuous for $(x,y) \in X \times E$.

2. Note that the $x_0$ in the conclusions of Theorems 4 and 4$'$ belong to $\mathrm{Bd}\, X$. In fact, suppose that $x_0 \in \mathrm{Int}\, X$. Then $x_0$ is an internal point and $I_X(x_0) = O_X(x_0) = E$. By putting $x = y_0$, we have $0 < p(x_0 - y_0) \le 0$ in the conclusions, which is a contradiction.

3. For a locally convex t.v.s. and $T \in \mathbb{K}(X,E)$, Theorems 4 and 4$'$ reduce to Park [**P2**, Theorem 2] and Reich [**R2**, Theorem 2], and improve Ha [**H**, Theorem 3] and Fan [**F4**, Theorem 1]. For $T \in \mathbb{V}(X,E)$, Theorems 4 and 4$'$ reduce the results in [**P7,9**].

As a direct consequence of Theorem 4, we have the following:

THEOREM 5. *Let $X$ be a compact convex space, $E$ a t.v.s. on which $E^*$ separates points such that $E$ contains $X$ as a subset and, for each $p \in \mathcal{P}$, $(x,y) \mapsto p(x-y)$ is continuous for $(x,y) \in X \times E$, and*

*$T \in \mathfrak{A}_c^\kappa(X, E)$. If $T$ satisfies one of the following conditions, then $T$ has a fixed point.*

*For each $x \in \operatorname{Bd} X$,*

(0) *for each $y \in Tx$ and each $p \in \mathcal{P}$, $p(y-x) > 0$ implies $p(y-x) > p(y-z)$ for some $z \in \overline{I}_X(x)$.*

(i) *for each $y \in Tx$, there exists a number $\lambda$ (real or complex, depending on whether the vector space $E$ is real or complex) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1-\lambda)y \in \overline{I}_X(x).$$

(ii) $Tx \subset \overline{I}_X(x)$.

(iii) *for each $y \in Tx$, there exists a number $\lambda$ (as in (i)) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1-\lambda)y \in X.$$

(iv) $Tx \subset IF_X(x) := \{x + c(u-x) : u \in X,\ \operatorname{Re} c > 1/2\}$.

(v) $Tx \subset X$.

(vi) $T(X) \subset X$.

*Proof.* (0) Clear from Theorem 4.

(i) For any $p \in \mathcal{P}$ satisfying $p(y-x) > 0$, put $z = \lambda x + (1-\lambda)y$ in (0). Then we have

$$p(y-z) = p(\lambda y - \lambda x) = |\lambda| p(y-x) < p(y-x)$$

since $|\lambda| < 1$.

(ii) If $Tx \subset \overline{I}_X(x)$, then for each $y \in Tx$, we can choose $\lambda = 0$ in (i).

(iii) Since $X \subset I_X(x)$, we clearly have (iii) $\Longrightarrow$ (i).

(iv) Note that (iv) $\Longleftrightarrow$ (iii) [**R1**].

(v) If $Tx \subset X$, then for each $y \in Tx$, we can choose $\lambda = 0$ in (iii).

(vi) Clearly, we have (vi) $\Longrightarrow$ (v).

Remarks. 1. If $T' \in \mathfrak{A}_c^\kappa(X, E)$ as in Theorem 4′, then the inward sets in (0), (i), (ii), and (iv) can be replaced by the corresponding outward sets.

2. Even for **V** and for a locally convex t.v.s., to the best of our knowledge, only Case (vi) of Theorem 5 is known except Park [**P8,9**]. In this case, $X$ is an *lc* space and hence Theorem 5(vi) follows from

Begle [**B**, Theorem 1]. Note that in (0) "each $y \in Tx$" can not be replaced by "there is a $y \in Tx$" as noted by Reich [**R1**, Example 1.2]. For some other conditions equivalent to (0), see [**R3**].

3. Following Halpern [**Ha**], for a subset $D$ of a normed vector space $E$, we define

$$n_D(x) = \{y \in E : y \neq x, \ \|y - x\| \leq \|y - z\| \quad \text{for all} \quad z \in D\}$$

for $x \in D$, and consider a "nowhere normal outward" multifunction $T : D \to 2^X$; that is,

$$(0)' \qquad Tx \cap n_D(x) = \emptyset \quad \text{for} \quad x \in D.$$

Then $(0)'$ clearly implies (0). Particular forms of Theorem 5 for **V** with Condition $(0)'$ for a normed vector space are given by Fitzpatrick and Petryshyn [**FP**, Theorem 3(i)], Reich [**R2**, Theorem 3.3(a)], Halpern [**Ha**, Theorem 20], and Halpern and Bergman [**HB**, Theorem 2.1].

4. Particular forms of Theorem 5(ii) for **V** are given by Fitzpatrick and Petryshyn [**FP**, Corollary 1] and Halpern [**Ha**, Corollaries 21 and 22]. Moreover, for the outward case in Theorem 5(ii), we have the surjectivity $X \subset T(X)$ as in [**P5**] and Halpern [**Ha**, Corollary 23]. Note that Halpern [**Ha**, Theorem 19] is a simple consequence of Theorem 5(vi).

5. For a $T \in \mathbb{K}(X, E)$, single-valued or multi-valued, each case of Theorem 5 generalizes historically well-known results as follows:

(0) Reich [**R5**, Theorem 7], [**R3**, Theorems 1 and 2], and Browder [**Br2**, Corollary to Theorem 9].

(i) Park [**P2**, Theorem 4].

(ii) Browder [**Br1**, Theorems 1 and 2] and Halpern and Bergman [**HB**, Theorems 4.1 and 4.3].

(iii) Fan [**F4**, Theorem 3], Ha [**H**, Theorem 4], and Kaczynski [**Ka**, Théorème 1].

(iv) Reich [**R2**, Theorem 3.1].

(v) Rothe [**Ro**].

(vi) Brouwer [**B**], Schauder [**S**, Satz 1], Tychonoff [**T**, Satz], Kakutani [**Kk**, Theorem 1], Bohnenblust and Karlin [**BK**], Glicksberg [**G**, Theorem], Fan [**F1**, Theorem 1], [**F3**, Corollaire 3], and Granas and Liu [**GL**, Theorem 10.5].

6. As in Reich [**R3**], Condition (0) can be reformulated using the subdifferential $\partial p$ of $p$. Moreover, as Reich [**R3**] noted, Theorem 5 is also valid for lower semicontinuous $T : X \to 2^E$ if $E$ is completely metrizable and if $T$ has closed convex values. This follows from the Michael selection theorem and Theorem 4.

7. In Theorem 5, since $T$ is compact, if $T$ is convex-valued, then upper semicontinuity of $T$ can be replaced by upper demicontinuity or upper hemicontinuity. For the literature, see [**P5**].

8. If $T \in \mathbb{K}(X, E)$, then more general conditions than (0)-(vi) suffice for the existence of fixed points. See [**P5,6**]. In this case, e.g., (ii) can be replaced by $Tx \cap \overline{I}_X(x) \neq \emptyset$. However, this is not true even for $T \in \mathbb{V}(X, E)$.

EXAMPLE. Let $E = \mathbb{R}^2$, $X = [-1, 1] \times \{0\}$, and $T \in \mathbb{V}(X, E)$ such that, for each $x \in X$, $Tx$ is the union of two segments joining $(-2, 0)$ and $(0, 1)$, $(0, 1)$ and $(2, 0)$. Then $T$ is a constant acyclic map and $Tx \cap \overline{I}_X(x) \neq \emptyset$ for $x \in X$. However, $T$ has no fixed point.

As an application of Theorem 5, we have the following:

COROLLARY 5.1. *Let $E$ be a t.v.s. on which $E^*$ separates points and $K \in kc(E)$. Then a continuous affine map $f : K \to E$ satisfying $K \subset fK$ has a fixed point.*

*Proof.* Let $T = f^{-1} : fK \to 2^K$. Then $T \in \mathbb{K}(fK, fK)$. Therefore, by Theorem 5(vi), there exists an $x \in K$ such that $x \in f^{-1}x$; that is, $x = fx$.

For the outward case (ii), we have the following surjectivity result:

THEOREM 6. *Let $X$, $E$, and $T$ be the same as in Theorem 5 such that $T' \in \mathfrak{A}_c^\kappa(X, E)$, where $T'x = 2x - Tx$ for $x \in X$. If $Tx \subset \overline{O}_X(x)$ for each $x \in \operatorname{Bd} X$, then $T$ has a fixed point and $T(X) \supset X$.*

*Proof.* By Theorem 5(ii), $T$ has a fixed point. Suppose $X \not\subset T(X)$. We may assume that the origin 0 is a point of $X \backslash T(X)$. The complement $U$ of $T(X)$ is a neighborhood of 0, so we can choose $c > 1$ such that $cU \supset X$. Then $cT(X)$ is disjoint from $X$, and so the map $cT$ can have no fixed point. However, since the weakly outward set $\overline{O}_X(x)$ is closed under the multiplication by a constant $c > 1$ ([**HB**, Lemma 4.2]), $cTx \subset \overline{O}_X(x)$ for all $x \in X$. This is a contradiction.

REMARK. We followed the proof of [**HB**, Theorem 4.3], which is the single-valued case of Theorem 6.

Untill now in this section, we mainly considered admissible maps defined on a compact convex subset of a t.v.s. having sufficiently many linear functionals. However, for a locally convex t.v.s., we have a result on compact admissible maps in our previous work [**P10**]:

THEOREM C. *Let $X$ be a nonempty convex subset of a locally convex t.v.s. $E$. If $T \in \mathfrak{A}_c(X, X)$ is compact, then $T$ has a fixed point.*

As an application of Theorem C, we obtain the following acyclic version of Reich's theorem [**R4**] on condensing maps with the Leray-Schauder boundary condition. For the definition of condensing multifuntions, see Su and Sehgal [**SS**].

THEOREM 7. *Let $C$ be a nonempty closed subset of a locally convex t.v.s. $E$ and $T : C \to 2^E$ an u.s.c. multifunction with closed acyclic values. Suppose that $T$ has a bounded range, and that there is a point $w \in \operatorname{Int} C$ such that*

(L-S) *for every $x \in \operatorname{Bd} C$ and $y \in Tx$,*

$$y - w \neq m(x - w) \quad \text{for all} \quad m > 1.$$

*Then $T$ has a fixed point if one of the following holds:*

(i) *$T$ is compact.*
(ii) *$T$ is condensing and $E$ is quasi-complete.*
(iii) *$T$ is condensing with compact values and $C$ is quasi-complete.*

*Proof.* Just follow the proof of Reich [**R4**] and use Theorem C instead of Himmelberg [**Hi**, Theorem 2].

REMARK. If $T$ has convex values, then Theorem 7 reduces to Reich [**R4**, Theorem].

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