ON 3-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLDS*

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The various structures on almost contact metric manifolds have been studied by many authors([2],[5],[6],[9],[11]). The purpose of the present paper is to study the structures on 3-dimensional almost contact metric manifolds.

1. Preliminaries

Let $M$ be a $(2n+1)$-dimensional differentiable manifold of class $C^\infty$ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field $\phi_i^h$ of type $(1,1)$, a vector field $\xi^h$ and a 1-form $\eta_i$ satisfying

\begin{equation}
\phi_j^i \phi_i^h = -\delta_j^h + \eta_j^i \xi^h, \phi_i^h \xi^i = 0, \eta_i \phi_j^i = 0, \eta_i \xi^i = 1,
\end{equation}

where the indices $h, i, j \ldots$ run over the range $\{1, 2, \ldots, 2n + 1\}$. Such a set of a tensor field $\phi$ of type $(1,1)$, a vector $\xi$ and a 1-form $\eta$ is called an almost contact structure and a manifold with an almost contact structure an almost contact manifold. If, in an almost contact manifold, there is given a Riemannian metric $g_{ji}$ such that

\begin{equation}
g_{is} \phi_j^i \phi_s^i = g_{ji} - \eta_j^i \eta_i, \eta_i = g_{ii} \xi^i,
\end{equation}

then the almost contact structure is said to be metric and the manifold is called an almost contact metric manifold([1]).

An almost contact structure $(\phi, \xi, \eta)$ on $M$ is said to be normal if

\[ N_{ji}^h + (\partial_j \eta_i - \partial_i \eta_j) \xi^h = 0 \text{ or } [N(x,y) = [\phi, \phi](x,y) + (d\eta)(x,y) = 0], \]

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where
\[ N_{ji}^h = \phi_i^l \partial_i \phi_j^h - \phi_j^l \partial_i \phi_i^h - (\partial_j \phi_i^l - \partial_i \phi_j^l) \phi_i^h \]
is the Nijenhuis tensor formed with \( \phi_i^h \) and \( \partial_j = \partial/\partial x^j \). We denote \( \nabla \) the covariant differentiation with respect to the Riemannian connection of \( g \) and denote \( \phi_{ji} = \phi_j^h g_{hi} \). An almost contact metric structure \((\phi, \xi, \eta, g)\) on \( M \) is said to be

(a) contact ([1]), if \( \phi_{ji} = \frac{1}{2} (\partial_j \eta_i - \partial_i \eta_j) \),
(b) \( K \)-contact ([1]), if \( \nabla_i \eta_j = \phi_{ij} \),
(c) nearly Sasakian ([5]), if \( \nabla_k \phi_j^h + \nabla_j \phi_k^h = -2g_{kj} \xi^h + \delta_k^h \eta_j + \delta_j^h \eta_k \),
(d) quasi Sasakian ([2]), if \( \phi_{ji} \) is closed and \((\phi, \xi, \eta)\) is normal,
(e) Sasakian ([1]), if \( \phi_{ji} = \frac{1}{2} (\partial_j \eta_i - \partial_i \eta_j) \) and \((\phi, \xi, \eta)\) is normal,
(f) nearly cosymplectic ([1],[3]), if \( \phi_j^h \) is Killing,
(g) quasi cosymplectic ([7]), if \( \nabla_k \phi_{ji} + \phi_k^h \phi_j^i \nabla_i \phi_{si} - \eta_j \phi_i^j \nabla_i \eta_i = 0 \),
(h) closely cosymplectic ([4]), if \( \phi_j^h \) is Killing and \( \eta_i \) is closed,
(i) almost cosymplectic ([9]), if \( \phi_{ji} \) and \( \eta_j \) are closed,
(j) cosymplectic ([1]), if \( \phi_{ji} \) and \( \eta_j \) are closed and \((\phi, \xi, \eta)\) is normal.

We note the following schematic array of structures ([8]).

If we put
\[ E_{kjih} = \phi_{kj} \phi_{ih} - (g_{ki} - \eta_k \eta_i)(g_{jh} - \eta_j \eta_h) + (g_{ji} - \eta_j \eta_i)(g_{kh} - \eta_k \eta_h), \]
then we have $E_{kjih}E^{kjih} = 12n(n - 1)$.

In a 3-dimensional almost contact metric manifold, $E_{kjih}$ is a zero tensor.

Thus we have

**Lemma 1.** In a 3-dimensional almost contact metric manifold, $E_{kjih}$ vanishes identically, that is,

$$
(1.3) \phi_{kjih} = (g_{ki} - \eta_{k}\eta_{i})(g_{jh} - \eta_{j}\eta_{h}) - (g_{ji} - \eta_{j}\eta_{i})(g_{kh} - \eta_{k}\eta_{h}).
$$

where $\Gamma_{ji} = g_{ji} - \eta_{j}\eta_{i}$.

On the other hand, it is well known that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold. Therefore the curvature tensor $K_{kjih}$ of a 3-dimensional almost contact metric manifold $M$ is given by

$$
(1.4) K_{kjih} = -k_{ki}\delta_{j}^{h} + K_{ji}\delta_{k}^{h} - g_{ki}K_{j}^{h} + g_{ji}K_{k}^{h} + \frac{K}{2}(g_{ki}\delta_{j}^{h} - g_{ji}\delta_{k}^{h}),
$$

where $K_{ji}$ and $K$ are the Ricci tensor and the scalar curvature of the manifold respectively.

Differentiating (1.3) covariantly, we obtain, in a 3-dimensional almost contact metric manifold,

$$
\phi_{ih}\nabla_{e}\phi_{kj} + \phi_{kjih}\nabla_{e}\phi_{ih} = -\eta_{i}\nabla_{e}\eta_{k} + \eta_{k}\nabla_{e}\eta_{i})\Gamma_{jih} - \Gamma_{ki}(\eta_{h}\nabla_{e}\eta_{j} + \eta_{j}\nabla_{e}\eta_{h}) + (\eta_{i}\nabla_{e}\eta_{j} + \eta_{j}\nabla_{e}\eta_{i})\Gamma_{kh} + \Gamma_{ji}(\eta_{h}\nabla_{e}\eta_{k} + \eta_{k}\nabla_{e}\eta_{h})
$$

Transvecting this equation with $\phi^{kj}$ and using $\phi^{kj}\phi_{kj} = 2$, we have

$$
(1.5) \nabla_{e}\phi_{ih} = (\nabla_{e}\eta_{i})\phi_{h}^{i} - (\nabla_{e}\eta_{i})\phi_{i}^{h}.
$$

### 2. 3-dimensional $K$-contact manifolds

Let $M$ be a 3-dimensional $K$-contact manifold. Then we have

$$
(2.1) \nabla_{i}\xi^{h} = \phi_{i}^{h}.
$$

The equation (2.1) shows that $\xi^{h}$ is a Killing vector field. Hence we have

$$
(2.2) \nabla_{j}\phi_{i}^{h} + K_{ti}^{h}\xi^{t} = 0.
$$
It is well known ([1]) that on a 3-dimensional $K$-contact manifold the Ricci tensor satisfies
\begin{equation}
K_{jt} \xi^t = 2 \eta_j.
\end{equation}

Differentiating $\phi^h_i \eta_h = 0$ covariantly and taking account of (2.1) and (2.2), we obtain
\begin{equation}
K_{tji}^h \xi^t \eta_h = g_{ji} - \eta_j \eta_i.
\end{equation}

Transvecting (1.4) with $\xi^k \eta_h$ and taking account of (2.3) and (2.4), we have
\begin{equation}
K_{ji} = \left( \frac{K}{2} - 1 \right) g_{ji} + \left( 3 - \frac{K}{2} \right) \eta_j \eta_i.
\end{equation}

Substituting (2.5) into (1.4), we obtain
\begin{equation}
K_{kji}^h = (2 - \frac{K}{2})(g_{ki} \delta_j^h - g_{ji} \delta_k^h)
+ \left( \frac{K}{2} - 3 \right)[(\eta_k \delta_j^h - \eta_j \delta_k^h) \eta_i + (g_{ki} \eta_j - g_{ji} \eta_k) \xi^h].
\end{equation}

Transvecting (2.6) with $\xi^k$ and taking account of (2.2), we obtain
\begin{equation}
\nabla_j \phi^h_i = \delta_j^h \eta_i - g_{ji} \xi^h,
\end{equation}
which shows that $M$ is a Sasakian manifold.

Thus we have the following

**Theorem 2.** A 3-dimensional $K$-contact manifold is a Sasakian manifold.

**Remark.** S.Tanno also showed the same result in [12,13].

### 3. 3-dimensional nearly Sasakian manifolds

Let $M$ be a 3-dimensional nearly Sasakian manifold. Then we have
\begin{equation}
\nabla_k \phi_{ji} + \nabla_j \phi_{ki} = -2g_{kj} \eta_i + g_{ki} \eta_j + g_{ji} \eta_k.
\end{equation}

For a nearly Sasakian manifold the vector field $\xi^h$ is Killing([5]), that is,
\begin{equation}
\nabla_j \eta_i + \nabla_i \eta_j = 0.
\end{equation}

We define the tensor field $H_{ji}$ by putting
\begin{equation}
\nabla_j \eta_i = \phi_{ji} + H_{ji}.
\end{equation}
From the skew-symmetry of $\phi_{ji}$ and (3.2), it follows that $H_{ji}$ is skew-symmetric. Set $H_j^i = H_{ja}^a_i$. Then we have the following equations [10].

(3.4) \[ H_{ji}^i + H_{it}^i = 0, \]

(3.5) \[ H_{ji}\xi^i = 0, \]

(3.6) \[ K_{tji\delta}^i = -\nabla_j \phi_{i\delta} - \nabla_j H_{i\delta} = (g_{ji} + H_{ji} H_{i}^i)\eta^j - (g_{jh} + H_{ji} H_{i}^i)\eta_i. \]

On the other hand, transvecting (1.3) with $\phi^k_m$, we obtain

(3.7) \[ (g_{ji} - \eta_j \eta_i)\phi_{m\delta}^i - (g_{jh} - \eta_j \eta_h)\phi_{m\delta}^i = (g_{mj} - \eta_m \eta_j)\phi_{i\delta}. \]

Transvecting (3.7) with $H_{i}^h$ and taking account of (3.5), we have

(3.8) \[ (g_{ji} - \eta_j \eta_i)H_{it}^i \phi_{m\delta}^i - H_{ij} \phi_{m\delta}^i = (g_{mj} - \eta_m \eta_j)H_{it}^i \phi_{i\delta}. \]

Taking the symmetric part of (3.8) with respect to $l$ and $m$ and using (3.4), we can find

(3.9) \[ -H_{lj} \phi_{mi} - H_{mj} \phi_{li} = (g_{mj} - \eta_m \eta_j)H_{lt}^l \phi_{i}^l + (g_{lj} - \eta_l \eta_j)H_{mt}^t \phi_{i}^t. \]

Transvecting (3.9) with $\phi_{p}^i$ and taking account of (3.5), we obtain

(3.10) \[ H_{lj}(g_{pm} - \eta_p \eta_m) + H_{mj}(g_{pl} - \eta_p \eta_l) \]
\[ = H_{lp}(g_{jm} - \eta_j \eta_m) + H_{mp}(g_{lj} - \eta_l \eta_j). \]

Transvecting (3.10) with $g_{mj}$ and taking account of (3.5) and $H_{i}^i = 0$, we have $H_{lp} = 0$. Hence (3.3) and (3.6) show that

$\nabla_j \eta_i = \phi_{ji}, \nabla_k \phi_{ji} = -g_{kj} \eta_i + g_{ki} \eta_j.$

Therefore $M$ is a Sasakian manifold. Thus we have

**Theorem 3.** A 3-dimensional nearly Sasakian manifold is a Sasakian manifold.

**4. 3-dimensional nearly cosymplectic manifolds**

Suppose that $M$ is a 3-dimensional nearly cosymplectic manifold. Then $\phi^k_j$ is Killing by the definition and it is known ([3]) that the vector field $\xi$ is Killing. Hence we have

(4.1) \[ \nabla_i \phi_{jk} + \nabla_j \phi_{ik} = 0, \]
Transvecting (4.2) with $\eta^t$, we have

\[(4.3) \quad (\nabla_t \xi_j)\eta^t = 0.\]

Transvecting (1.5) with $\eta^c$ and using (4.3), we have

$$\eta^t \nabla_t \phi_{ih} = 0,$$

which and (4.1) imply

\[(4.4) \quad \eta^t \nabla_i \phi_{ht} = 0.\]

Since $\phi_{ht} \eta^t = 0$, we find

$$\eta^t (\nabla_i \phi_{ht}) + \phi_{ht} \nabla_i \eta^t = 0,$$

which and (4.4) imply

\[(4.5) \quad (\nabla_i \eta_t) \phi^t_h = 0.\]

Substituting (4.5) into (1.5), we have

\[(4.6) \quad \nabla_c \phi_{ih} = 0.\]

Transvecting (4.5) with $\phi^t_j$, we have

$$\nabla_i \eta_j = 0,$$

which and (4.6) show that $M$ is a cosymplectic manifold.

Thus we have

**Theorem 4.** Every 3-dimensional nearly cosymplectic manifold is a cosymplectic manifold.

### 5. 3-dimensional quasi cosymplectic manifolds

Let $M$ be a 3-dimensional quasi cosymplectic manifold. Then we have

\[(5.1) \quad \nabla_k \phi_{ji} + \phi^t_k \phi^t_j \nabla_t \phi_{si} - \eta_j \phi^t_k \nabla_t \eta_i = 0.\]

Transvecting (5.1) with $\eta^j$, we obtain

\[(5.2) \quad \phi^t_k \nabla_t \eta_i = \phi^t_i \nabla_k \eta_t.\]
Transvecting (5.2) with $\eta^k \phi^i_j$, we find
\begin{equation}
\eta^s \nabla_s \eta_j = 0.
\tag{5.3}
\end{equation}

Transvecting (5.2) with $\phi^k_j$ and using (5.3), we have
\begin{equation}
\phi^s_j \phi^i_k \nabla_s \eta_t = -\nabla_j \eta_i.
\tag{5.4}
\end{equation}

Transvecting (5.1) with $\eta^k$, we have
\begin{equation}
\eta^k \nabla_k \phi_{ji} = 0.
\tag{5.5}
\end{equation}

From (1.3) and (5.4), we have
\[(\Upsilon_{ji} \Upsilon_{st} - \Upsilon_{si} \Upsilon_{jt}) \nabla^s \eta^t = -\nabla_j \eta_i,
\]
which and (5.3) imply
\begin{equation}
\Upsilon_{ji} \nabla_t \eta^t = \nabla_i \eta_j - \nabla_j \eta_i.
\tag{5.6}
\end{equation}

Transvecting (5.6) with $\phi^i_j$, we obtain
\begin{equation}
\nabla_t \eta^t = 0.
\tag{5.7}
\end{equation}

Substituting (5.7) into (5.6), we find
\begin{equation}
\nabla_i \eta_j - \nabla_j \eta_i = 0,
\tag{5.8}
\end{equation}
which shows that $\eta$ is closed.

From (1.5), we have
\begin{equation}
\phi^t_k \phi^s_j \nabla_t \phi_{si} = \phi^t_k (\nabla_t \eta_j) \eta_i.
\tag{5.9}
\end{equation}

From (5.1), (5.2), (5.8) and (5.9), we find
\begin{equation}
\nabla_k \phi_{ji} + \phi^t_j (\nabla_k \eta_i) \eta_i - \phi^t_k (\nabla_i \eta_i) \eta_j = 0.
\tag{5.10}
\end{equation}

By the cyclic sum of (5.10) with respect to the indices $k$, $j$ and $i$, we find
\[\nabla_k \phi_{ji} + \nabla_j \phi_{ik} + \nabla_i \phi_{kj} = 0,
\]
which means that the 2-form $\phi_{ji}$ is closed.

Thus we have the following
Theorem 5. Every 3-dimensional quasi cosymplectic manifold is an almost cosymplectic manifold.

Remark 1. In [8], Z. Olszak constructed almost cosymplectic structures with non-parallel vector field $\xi$ on certain Lie groups in every odd dimension. Hence a 3-dimensional almost cosymplectic manifold is not cosymplectic in general.

Remark 2. Let $M$ be a 2-dimensional manifold (surface) which does not have constant curvature 1 and let $TM$ its tangent bundle with the fibre coordinates $v^1, v^2$. Then the tangent sphere bundle $\pi : T_1M \to M$ is a hypersurface of $TM$ given by $(v^1)^2 + (v^2)^2 = 1$ and we can find a contact metric structure on $T_1M$ which is not Sasakian. (See. Blair [1])

Remark 3. By theorems 2,3,4 and 5, the array of structures in 3-dimensional almost contact metric manifolds is reduced to the following diagram.

![Diagram]

References


On 3-dimensional almost contact metric manifolds


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