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CONTROLLABILITY OF INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, we will study controllability of some case as an initial condition φ included in some approximated phase space. To this prove we used to the Schauder fixed point theorem.

1. Introduction

In this paper, we will study controllability of some case as an initial condition φ included in some approximated phase space. Let X be a Banach space with norm $\|\cdot\|$. If $x : (-r, \alpha+\xi] \to X$, then for any $t \in (-r, \alpha+\xi)$ let $x_t : (-r, 0] \to X$ be defined by $x_t(\theta) = x(t+\theta), -r < \theta \leq 0$. We consider the following integrodifferential equation:

$$
\begin{aligned}
(1) \qquad & \frac{dx(t)}{dt} = Ax(t) + \int_0^t g(t,s,x_s)ds + f(t,x_t) + Bu(t), \quad t \geq 0 \\
& x_0 = \varphi \in \mathcal{B},
\end{aligned}
$$

where $g : I \times I \times \mathcal{B} \to X$, $f : I \times \mathcal{B} \to X, \xi > 0, I = [0, \xi]$ are continuous functions, $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous operator semigroup $T(t)$ on X. Also, the control function $u(\cdot)$ is given in $L^2(I, U)$, a Banach space of admissible control functions, with U a Banach and B is a bounded linear operator from U into X. Henriquez ([3,4]) studied the existence of solutions, periodic solutions and stability of a class of partial functional differential equations with unbounded delay. Several authors ([1,5,6]) investigated the approximate controllability and controllability of nonlinear integrodifferential systems

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in Banach space by method of fixed point. In this paper we will use the Schauder fixed point theorem to obtain the controllability of equation (1). To apply the Schauder fixed theorem we will employ a compactness condition on the compositions of the semigroup $T(t)$ and the functions f, g, B.

2. Preliminaries

We will use the phase space $\mathcal{B}$ introduced by Hale and Kato ([2]). $\mathcal{B}$ will be a linear space of functions mapping $(-r, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(C1) If $x : (-r, \alpha+\xi) \to X, \xi > 0$ is continuous on $[\alpha, \alpha+\xi)$ (α: fixed) and $x_\alpha \in \mathcal{B}$ then for every t in $[\alpha, \alpha+\xi)$ the following conditions holds:
 (i) x_t is in $\mathcal{B}$;
 (ii) $\|x(t)\| \leq K\|x_t\|_{\mathcal{B}}$;
 (iii) $\|x_t\|_{\mathcal{B}} \leq M(t-\alpha)\sup\{\|x(s)\| : \alpha \leq s \leq t\} + N(t-\alpha)\|x_\alpha\|_{\mathcal{B}}$,
where $K \geq 0$ is a constant; $M, N : [0, \infty) \to [0, \infty)$, M is continuous and N is locally bounded and K, M and N are independent of $x(\cdot)$.

(C2) For the function $x(\cdot)$ in (C1), x_t is a $\mathcal{B}$-valued continuous function on $[\alpha, \alpha+\xi)$.

(C3) The space $\mathcal{B}$ is complete.

We list the following hypotheses;

(H1) If A is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$ satisfying $\|T(t)\| \leq L_1$.

(H2) The linear operator W from U into X, defined by

$$Wu = \int_0^\xi T(\xi - s)Bu(s)ds,$$

has an invertible operator W^{-1} defined on $L^2(I; U)/\ker W$, and there exist positive constants L_2, L_3 such that $\|B\| \leq L_2$, $\|W^{-1}\| \leq L_3$ (see [7] for construction of W).

(H3) There is a compact set $V_t \subset X$ such that $T(t)f(s, \psi), T(t)\int_0^s g(s, \tau, \psi)d\tau$, $T(t)Bu(s) \in V_t$, for $\psi \in \mathcal{B}$ and $0 \leq s \leq \xi$.

DEFINITION 2.1. A function $x : (-r, \xi) \to X, 0 < \xi$, is a *mild solution* of the Cauchy problem if $x_0 = \varphi$ and the restriction $x : [0, \xi) \to X$ is continuous and satisfies the integral equation:

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x_s)ds$$
$$+ \int_0^t T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds + \int_0^t T(t-s)Bu(s)ds, \quad t > 0.$$

DEFINITION 2.2. The system (1) is said to be *controllable* on the interval I if, for every initial function $x_0 \in \mathcal{B}, x_1 \in X$, there exists a control $u \in L^2(I; U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(\xi) = x_1$.

3. Main Result

THEOREM 3.1 *If the hypotheses* (H1)$\sim$(H3) *are satisfied, then the system* (1) *is controllable on* I.

Proof. By Henriquez ([3]), we obtained the existence of a mild solution of system (1) together above hypotheses. Using the hypothesis $(H2)$, define the control

$$u(t) = W^{-1}\left[x_1 - T(\xi)\varphi(0) - \int_0^\xi T(\xi - s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi - s)f(s,x_s)ds\right](t).$$

Also, using this control, we shall show that operator G is defined in below

$$(Gx)(t) = T(t)\varphi(0) + \int_0^t T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds + \int_0^t T(t-s)f(s,x_s)ds + \int_0^t T(t-s)Bu(s)ds$$

has a fixed point. This fixed point is then a solution of (1). Clearly $(Gx)(\xi) = x_1$. This means that the control u steers solution of the abstract functional integrodifferential system with the initial function φ to x_1 at ξ if we have a fixed point of nonlinear operator G. First of all, let us define the function $y(t) = T(t)\varphi(0)$ for $t \geq 0$ and $y_0 = \varphi$. Then from the properties of $\mathcal{B}$ we infer that $y_t \in \mathcal{B}$ and let $\|y_t - \varphi\|_{\mathcal{B}} \le \eta$ for $0 \le t \le \xi_1$, where ξ_1 is some constant and $\eta < \delta$. Let ξ be any constant such that $0 < \xi = \min\{\xi_1, \frac{\delta-\eta}{\tilde{M}L_1\zeta}\}$ and

$$\mathcal{Y}_\xi(\varphi) = \{x \in \mathcal{Y}_\xi; \|x_0 - \varphi\|_{\mathcal{B}} = 0, \|x_t - \varphi\|_{\mathcal{B}} \le \delta, 0 \le t \le \xi\},$$

where $\mathcal{Y}_\xi$ is the space of all functions $x : (-r, \xi] \to X$ such that $x_0 \in \mathcal{B}$ and the restriction $x : [0, \xi] \to X$ is continuous and let $\|\cdot\|_{\mathcal{Y}}$ be a seminorm in $\mathcal{Y}_\xi$ defined by

$$\|x\|_{\mathcal{Y}} = \|x_0\|_{\mathcal{B}} + \sup\{\|x(s)\| : 0 \le s \le \xi\}$$

and $\zeta = L_7\xi + L_6 + L_2L_3\{\|x_1\| + L_1\|\varphi(0)\| + L_1L_7\xi^2 + L_1L_6\xi\}$, $\tilde{M} = \max\{M(t) : 0 \le t \le \xi\}$ since M is a continuous function. Then, $\mathcal{Y}_\xi(\varphi)$ is clearly a nonempty, bounded, convex and closed subset of $\mathcal{Y}_\xi$ (see [3]). Now we define the map G on $\mathcal{Y}_\xi(\varphi)$ by

$$\begin{aligned}(Gx)(t) = T(t)\varphi(0) &+ \int_0^t T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds \\ &+ \int_0^t T(t-s)f(s,x_s)ds + \int_0^t T(t-\sigma)BW^{-1}[x_1 - T(\xi)\varphi(0) \\ &- \int_0^\xi T(\xi-s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau](\sigma)d\sigma.\end{aligned}$$

Now, we show that G is map from $\mathcal{Y}_\xi(\varphi)$ into $\mathcal{Y}_\xi(\varphi)$. Since subsequent arguments will demonstrate the continuity of Gx we need only ascertain that $(Gx)(t) \in \mathcal{Y}_\xi(\varphi)$. Let $z = Gx$. Since Gx is continuous on $[0, \xi]$, $z \in \mathcal{Y}_\xi$. Set

$$\begin{aligned}w(t) &= \int_0^t T(t-s)f(s,x_s)ds, \\ v(t) &= \int_0^t T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds, \\ \rho(t) &= \int_0^t T(t-\sigma)BW^{-1}\Big[x_1 - T(\xi)\varphi(0) - \int_0^\xi T(\xi-s) \\ &\qquad \cdot \int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau\Big](\sigma)d\sigma.\end{aligned}$$

Then we can write map Gx as $z = y + v + w + \rho$. Here we estimate the following term;

$$\begin{aligned}\|z_t - \varphi\|_{\mathcal{B}} &\le \|y_t - \varphi\|_{\mathcal{B}} + \|v_t\|_{\mathcal{B}} + \|w_t\|_{\mathcal{B}} + \|\rho_t\|_{\mathcal{B}} \\ &\le \eta + \|v_t\|_{\mathcal{B}} + \|w_t\|_{\mathcal{B}} + \|\rho_t\|_{\mathcal{B}}.\end{aligned}$$

Since f, g are continuous functions, we can choose some constants L_6, L_7 such that $\|f(t,\psi)\| \le L_6, \|g(t,s,\psi)\| \le L_7$ for all $\psi \in B_\delta(\varphi)$. From Axiom (C_1) (iii) we have

$$\begin{aligned}&\|v_t\|_{\mathcal{B}} + \|w_t\|_{\mathcal{B}} + \|\rho_t\|_{\mathcal{B}} \\ &\quad \le M(t)\sup\{\|v(s)\| : 0 \le s \le t\} + M(t)\sup\{\|w(s)\| : 0 \le s \le t\} \\ &\qquad + M(t)\sup\{\|\rho(s)\| : 0 \le s \le t\}\end{aligned}$$

and

$$\begin{aligned}&\|v(t)\| + \|w(t)\| + \|\rho(t)\| \\ &\quad \le L_1\xi\{L_7\xi + L_6 + L_2L_3[\|x_1\| + L_1\|\varphi(0)\| + L_1L_7\xi_7 + L_1L_6\xi]\} \\ &\quad \le \frac{\delta - \eta}{\tilde{M}}.\end{aligned}$$

Thus we may deduce that $\|z_t - \varphi\|_{\mathcal{B}} \le \delta$ and consequently $(Gx)(t) \in \mathcal{Y}_\xi(\varphi)$. Next we will prove that G maps $\mathcal{Y}_\xi(\varphi)$ into a precompact subset of $\mathcal{Y}_\xi(\varphi)$. To prove this, we first show that, for every fixed $t \in J$, the set

$$\mathcal{Y}_\xi(\varphi)(t) = \{(Gx)(t) : x \in \mathcal{Y}_\xi(\varphi)\}$$

is precompact in X. This is clear for $t = 0$, since $\mathcal{Y}_\xi(\varphi)(0) = \{\varphi(0)\}$. Let $t > 0$ be fixed and for $0 < \varepsilon < t$ define

$$\begin{aligned}(G_\varepsilon x)(t) &= T(t)\varphi(0) + \int_0^{t-\varepsilon} T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds \\ &\quad + \int_0^{t-\varepsilon} T(t-s)f(s,x_s)ds + \int_0^{t-\varepsilon} T(t-\sigma)BW^{-1}[x_1 - T(\xi)\varphi(0) \\ &\quad - \int_0^\xi T(\xi - s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau](\sigma)d\sigma.\end{aligned}$$

From hypothesis (H_3) since $T(t)F(s,\psi), T(t)\int_0^s g(s,\tau,\psi)d\tau, T(t)Bu(s) \in V_t$, the set

$$\mathcal{Y}_\varepsilon(\varphi)(t) = \{(G_\varepsilon x)(t) : x \in \mathcal{Y}_\xi(\varphi)\}$$

is precompact in X for every ε, $0 < \varepsilon < t$. Furthermore, for $x \in \mathcal{Y}_\xi(\varphi)$, we have

$$\begin{aligned}
&\|(Gx)(t) - (G_\varepsilon x)(t)\| \\
&\leq \left\| \int_{t-\varepsilon}^{t} T(t-\sigma)BW^{-1}\Big[x_1 - T(\xi)\varphi(0) - \int_0^\xi T(\xi - s) \right. \\
&\qquad \left. \cdot \int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau\Big](\sigma)d\sigma \right\| \\
&\quad + \left\| \int_{t-\varepsilon}^{t} T(t-s)\int_0^s g(s,\tau,x_\tau)d\tau ds \right\| \\
&\quad + \left\| \int_{t-\varepsilon}^{t} T(t-s)f(s,x_s)ds \right\| \\
&\leq \varepsilon L_1L_2L_3[\|x_1\| + L_1\|\varphi(0)\| + L_1L_7\xi^2 + L_1L_6\xi] + \varepsilon L_1[L_7\xi + L_6],
\end{aligned}$$

which implies that $\mathcal{Y}_\varepsilon(\varphi)(t)$ is totally bounded, that is, precompact in X. If we can show that the image of $\mathcal{Y}_\varepsilon(\varphi)$ under G is an equicontinuous family of functions we can apply the Arzela-Ascoli theorem to conclude that $G(\mathcal{Y}_\varepsilon(\varphi))$ is precompact in $\mathcal{Y}_\varepsilon(\varphi)$. For that, let $t_2 > t_1 > 0$. Then, we have

$$\begin{aligned}
&\|(Gx)(t_1) - (Gx)(t_2)\| \\
&\leq \|T(t_1)\varphi(0) - T(t_2)\varphi(0)\| \\
&\quad + \left\| \int_0^{t_1} [T(t_1-\sigma) - T(t_2-\sigma)]BW^{-1}\Big[x_1 - T(\xi)\varphi(0) \right. \\
&\quad - \int_0^\xi T(\xi-s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau\Big](\sigma)d\sigma \\
&\quad - \int_{t_1}^{t_2} T(t_2-\sigma)BW^{-1}\Big[x_1 - T(\xi)\varphi(0) \\
&\quad \left. - \int_0^\xi T(\xi-s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^\xi T(\xi-\tau)f(\tau,x_\tau)d\tau\Big](\sigma)d\sigma \right\| \\
&\quad + \left\| \int_0^{t_1} [T(t_1-s) - T(t_2-s)]\Big[\int_0^s g(s,\tau,x_\tau)d\tau + f(s,x_s)\Big]ds \right. \\
&\quad \left. - \int_{t_1}^{t_2} T(t_2-s)\Big[\int_0^s g(s,\tau,x_\tau)d\tau + f(s,x_s)\Big]ds \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|T(t_1)\varphi(0) - T(t_2)\varphi(0)\| \\
&+\left\| \int_0^{t_1} [T(t_1-\varepsilon-\sigma) - T(t_2-\varepsilon-\sigma)]T(\varepsilon)BW^{-1}\Big[x_1 - T(\xi)\varphi(0) \right. \\
&\left. - \int_0^{\xi} T(\xi-s)\int_0^s g(s,\tau,x_\tau)d\tau ds - \int_0^{\xi} T(\xi-\tau)f(\tau,x_\tau)d\tau\Big](\sigma)d\sigma \right\| \\
&+(t_2-t_1)L_1L_2L_3[\|x_1\| + L_1\|\varphi(0)\| + L_1L_7\xi^2 + L_1L_6\xi] \\
&+\left\| \int_0^{t_1} [T(t_1-\varepsilon-s) - T(t_2-\varepsilon-s)]T(\varepsilon) \right. \\
&\left. \cdot\left[\int_0^s g(s,\tau,x_\tau)d\tau + f(s,x_s)\right]ds \right\| + (t_2-t_1)L_1[L_7\xi + L_6].
\end{aligned}$$

Since $T(\varepsilon)F(s,\psi), T(\varepsilon)\int_0^s g(s,\tau,\psi)d\tau, T(\varepsilon)Bu(s)$ are included in a compact set V_ε, for all $0 \leq \tau \leq s \leq \xi$ and all $x \in \mathcal{Y}_\xi(\varphi)$ and $T(\cdot)x, x \in V_\varepsilon$ is continuous, the right side of above inequality tends to zero as $t_2 - t_1 \to 0$ which is independent of $x \in \mathcal{Y}_\xi(\varphi)$. So, $G(\mathcal{Y}_\xi(\varphi))$ is an equicontinuous family of functions.

Also, $G(\mathcal{Y}_\xi(\varphi))$ is bounded in $\mathcal{Y}_\xi$, and so by the Arzela-Ascoli theorem, $G(\mathcal{Y}_\xi(\varphi))$ is precompact. Hence, from the Schauder fixed point theorem, G has a fixed point in $\mathcal{Y}_\xi(\varphi)$. Any fixed point of G is a mild solution of (1) on I satisfying

$$(Gx)(t) = x(t) \in X.$$

Thus, the system (1) is controllable on I.

4. Example

Consider the abstract functional integrodifferential equation of the form:

$$\begin{aligned}
\frac{\partial z(t,x)}{\partial t} &= (k(x)z_x(t,x))_x + \int_0^t G(t,s,z(s-r))ds \\
&\quad +F(t,z(t-r)) + Bu(t), \quad x \in [0,1], \quad t \in I \qquad (2)\\
z(t,x) &= \varphi(t,x) \in \mathcal{L}.
\end{aligned}$$

Let the space $\mathcal{L} =: L^p([-r,0];X)$ endowed with the seminorm

$$|||\varphi|||_p = \left(\|\varphi(0)\|^p + \int_{-r}^{0} \|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

Then the phase space $\mathcal{L}$ satisfies the axiom (C1)-(C3). Also, let $X = L^2(I,R)$. We define an operator

$$A : X \to X \quad \text{and} \quad (Az)(t)(x) = -(k(x)z_x(t,x))_x$$

with $k \in H^1(0,1)$ and

$$D(A) = \{z \in X : (k(x)z_x(t,x))_x \in X, z(t,0) = z(t,1) = 0\}.$$

Since operator A is compact under above conditions, A generates compact semigroup $T(t)$. And $B : U \to X$, with $U \subset I$, is a linear operator such that there exists an invertible operator W^{-1} on $L^2(I,U)/\ker W$, where W is defined by

$$Wu = \int_0^{\xi} T(\xi - s)Bu(s)ds.$$

The mapping $G : I \times I \times \mathcal{L} \to X$, $F : I \times \mathcal{L} \to X, \xi > 0, I = [0,\xi]$ are continuous functions. Define $g(t,s,\varphi)(x) = G(t,s,\varphi(-r)x), f(t,\varphi)(x) = F(t,\varphi(-r)x)$. Then equation (2) can be formulated abstractly as

$$\frac{dz(t)}{dt} = Az(t) + \int_0^t g(t,s,z_s)ds + f(t,z_t) + Bu(t), \quad t \geq 0$$
$$z_0 = \varphi \in \mathcal{L}.$$

Further, all the conditions in the above theorem are satisfied. Hence, the system (2) is controllable on I.

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