CONVERGENCE THEOREMS OF
MODIFIED ISHIKAWA ITERATIVE
SEQUENCES WITH MIXED ERRORS FOR
ASYMPTOTICALLY QUASI-NONEXPANSIVE
MAPPINGS IN BANACH SPACES

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ABSTRACT In this paper, we will discuss some sufficient and
necessary conditions for modified Ishikawa iterative sequence
with mixed errors to converge to fixed points for asymptoti-
cally quasi-nonexpansive mappings in Banach spaces. The re-
sults presented in this paper extend, generalize and improve
the corresponding results in Liu [4,5] and Ghosh-Debnath [2]

1. Introduction

Let $E$ be a Banach space with $\| \cdot \|$, $C$ be a nonempty subset of $E$, $N$ be the set of all positive integers and $F(T)$ be the set of all fixed
points of $T$.

DEFINITION 1.1 Let $T : C \to C$ be a mapping.

(1) The mapping $T$ is said to be asymptotically quasi-nonexpansive
if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such

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that
\[\|T^n x - p\| \leq (1 + k_n)\|x - p\|,\]
for all \(x \in C\) and \(p \in F(T)\).
If \(k_n = 0\) for \(n = 1, 2, \cdots\), then the mapping \(T\) is said to be a \textit{quasi-nonexpansive}.

(2) The mapping \(T\) is said to be \textit{asymptotically nonexpansive} if there exists a sequence \(\{k_n\}\) in \([0, \infty)\) with \(\lim_{n \to \infty} k_n = 0\) such that
\[\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|\]
for all \(x, y \in C\) and \(n = 1, 2, \cdots\).
If \(k_n = 0\) for \(n = 1, 2, \cdots\), then the mapping \(T\) is said to be a \textit{nonexpansive}.

Remark 1.1. From Definition 1.1, it is known that, if \(T\) is an asymptotically nonexpansive mapping and \(F(T)\) is a nonempty set, then \(T\) is an asymptotically quasi-nonexpansive mapping. But the converse is not true in general.

Definition 1.2. Let \(C\) be a nonempty subset of \(E\), \(T : C \to C\) be a mapping and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) be three sequences in \([0, 1]\).

(1) The sequence \(\{x_n\}\) defined by

\[
\begin{aligned}
x_1 & \in C, \\
x_{n+1} & = (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \\
y_n & = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad \forall \ n \in \mathbb{N}
\end{aligned}
\]

is called the \textit{modified Ishikawa iterative sequence with mixed errors}, where \(\{u_n\}\) and \(\{v_n\}\) are two bounded sequences in \(C\).

(2) In (1.1), if \(\beta_n \equiv 0, v_n \equiv 0\) for all \(n = 1, 2, \cdots\), then the sequence \(\{x_n\}\) defined by

\[
\begin{aligned}
x_1 & \in C, \\
x_{n+1} & = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u_n, \quad \forall \ n \in \mathbb{N}
\end{aligned}
\]
is called the modified Mann iterative sequence with mixed errors.

The concept of an asymptotically nonexpansive mapping was introduced by Gobel-Kirk [3] in 1972. It is well known that, if $E$ is a uniformly convex Banach space, $C$ is a nonempty closed bounded convex subset of $E$, then an asymptotically nonexpansive mapping defined on $C$ has a fixed point in $C$ (see, Browder [1]).


Motivated and inspired by Liu's results [4,5], in this paper, we will obtain some sufficient and necessary conditions for modified Ishikawa iterative sequences with mixed errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in Banach spaces. The results presented in this paper extend, generalize and improve the corresponding results of Ghosh-Debnath [2], Gobel-Kirk [3] and Liu [4,5].

2. Main results

In order to prove the our main results, we will first prove the following lemma 2.1.

**Lemma 2.1.** Let $E$ be a real Banach space, $C$ be a nonempty convex subset of $E$, $T : C \rightarrow C$ be an asymptotically quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1). Then

(a) $\|x_{n+1} - p\| \leq (1 + k_n)^2 \|x_n - p\| + m_n, \quad \forall \ n \in \mathbb{N}, \forall \ p \in F(T),$ where $m_n = \alpha_n (1 + k_n) \|u_n\| + \|u_n\|$.
(b) There exists a constant $M > 0$ such that
\[\|x_{n+m} - p\| \leq M\|x_n - p\| + M \sum_{j=n}^{n+m-1} m_j, \quad \forall n, m \in \mathbb{N}, \forall p \in F(T).\]

Proof. (a) Since $T$ is an asymptotically quasi-nonexpansive mappings, for all $p \in F(T)$, we have
\begin{align*}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_n T^m y_n + u_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^m y_n - p\| + \|u_n\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 + k_n)\|y_n - p\| + \|u_n\|
\end{align*}
and
\begin{align*}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n T^m x_n + v_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|T^m x_n - p\| + \|v_n\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n(1 + k_n)\|x_n - p\| + \|v_n\|.
\end{align*}
Substituting (2.2) into (2.1), it can be obtained that
\begin{align*}
\|x_{n+1} - p\| &\leq (1 - \alpha_n)\|x_n - p\| \\
&\quad + \alpha_n(1 + k_n)(1 - \beta_n)\|x_n - p\| + \alpha_n(1 + k_n)\|x_n - p\| \\
&\quad + \|v_n\| + \|u_n\| \\
&= (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 + k_n)(1 - \beta_n)\|x_n - p\| \\
&\quad + \alpha_n\beta_n(1 + k_n)^2\|x_n - p\| + \alpha_n(1 + k_n)\|v_n\| + \|u_n\| \\
&\leq (1 - \alpha_n)(1 + k_n)^2\|x_n - p\| + \alpha_n(1 - \beta_n)(1 + k_n)^2\|x_n - p\| \\
&\quad + \alpha_n\beta_n(1 + k_n)^2\|x_n - p\| + m_n \\
&= (1 + k_n)^2\|x_n - p\| + m_n,
\end{align*}
where $m_n = \alpha_n(1 + k_n)\|v_n\| + \|u_n\|$. This completes the proof of (a).
(b) If \( a \geq 0 \), then \( 1 + a \leq e^a \) and \( (1 + a)^2 \leq e^{2a} \). Therefore, from (a) we can obtain that

\[
\|x_{n+m} - p\| \leq (1 + k_{n+m-1})^2 \|x_{n+m-1} - p\| + m_{n+m-1}
\leq e^{2k_{n+m-1}} \|x_{n+m-1} - p\| + m_{n+m-1}
\leq e^{2k_{n+m-1}}[(1 + k_{n+m-2})^2 \|x_{n+m-2} - p\| + m_{n+m-2}]
+ m_{n+m-1}
\leq e^{2(k_{n+m-1} + k_{n+m-2})} \|x_{n+m-2} - p\|
+ e^{2k_{n+m-1}}m_{n+m-2} + m_{n+m-1}
\leq e^{2(k_{n+m-1} + k_{n+m-2})} \|x_{n+m-2} - p\|
+ e^{2k_{n+m-1}}(m_{n+m-1} + m_{n+m-2})
\leq \cdots
\leq e^{2\sum_{j=1}^{n+m-1} k_j} \|x_n - p\| + e^{2\sum_{j=1}^{n+m-1} k_j} \sum_{j=n}^{n+m-1} m_j
\leq M \|x_n - p\| + M \sum_{j=n}^{n+m-1} m_j,
\]

where \( M = e^{2\sum_{j=1}^{\infty} k_j} \). This completes the proof of (b).

We also need the following lemma in the proof of our main results.

**Lemma 2.2** [5]. Let \( \{a_n\} \), \( \{b_n\} \) and \( \{\lambda_n\} \) be three nonnegative real sequences such that \( a_{n+1} \leq (1 + \lambda_n)a_n + b_n \), \( \forall n \in \mathbb{N} \), \( \sum_{n=1}^{\infty} b_n < \infty \), \( \sum_{n=1}^{\infty} \lambda_n < \infty \). Then

(a) \( \lim_{n \to \infty} a_n \) exist.

(b) If \( \lim \inf_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

Now, we are in a position to prove the our main theorems. \( D(y, S) \) denotes the distance of \( y \) to set \( S \), that is, \( D(y, S) = \inf\{\|y - s\| : s \in S\} \).
Theorem 2.1. Let $E$ be a real Banach space, $C$ be a nonempty convex subset of $E$, $T : C \to C$ be an asymptotically quasi-non-expansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1), where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in $C$, that is, $\sum_{n=1}^{\infty} ||u_n|| < \infty$, $\sum_{n=1}^{\infty} ||v_n|| < \infty$. Then the iterative sequence $\{x_n\}$ converges to a fixed point $p$ if and only if

$$\lim_{n \to \infty} \inf D(x_n, F(T)) = 0.$$ 

Proof. From Lemma 2.1 (a), we have

$$(2.3) \quad ||x_{n+1} - p|| \leq (1 + k_n)^2 ||x_n - p|| + m_n, \quad \forall \ p \in F(T), \ \forall \ n \in \mathbb{N},$$

where $m_n = \alpha_n (1 + k_n) ||v_n|| + ||u_n||$. Since $0 \leq \alpha_n \leq 1$, $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} ||u_n|| < \infty$ and $\sum_{n=1}^{\infty} ||v_n|| < \infty$, we have $\sum_{n=1}^{\infty} m_n < \infty$.

From (2.3), we have

$$D(x_{n+1}, F(T)) \leq (1 + k_n)^2 D(x_n, F(T)) + m_n.$$ 

Since $\lim \inf_{n \to \infty} D(x_n, F(T)) = 0$, by Lemma 2.2, we can obtain that

$$\lim_{n \to \infty} D(x_n, F(T)) = 0.$$ 

Now, we have to prove that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. from Lemma 2.1, there exists a constant $M > 0$ such that

$$(2.4) \quad ||x_{n+m} - p|| \leq M ||x_n - p|| + M \sum_{j=n}^{n+m-1} m_j, \quad \forall \ p \in F(T), \ \forall \ n, m \in \mathbb{N}.$$ 

Since $\lim_{n \to \infty} D(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} m_n < \infty$, there exists a constant $N_1$ such that for all $n \geq N_1$,

$$D(x_n, F(T)) < \frac{\epsilon}{4M} \quad \text{and} \quad \sum_{j=N_1}^{\infty} m_j < \frac{\epsilon}{6M}.$$
So, \( D(x_{N_1}, F(T)) < \frac{\varepsilon}{4M} \).

We note that there exists \( p_1 \in F(T) \) such that \( \|x_{N_1} - p_1\| < \frac{\varepsilon}{3M} \).

From (2.4), we can obtain that for all \( n \geq N_1 \),

\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
\leq M\|x_{N_1} - p_1\| + M \sum_{j=N_1}^{N_1+m-1} m_j + M\|x_{N_1} - p_1\| \\
+ M \sum_{j=N_1}^{N_1+m-1} m_j \\
< M \frac{\varepsilon}{3M} + M \frac{\varepsilon}{6M} + M \frac{\varepsilon}{3M} + M \frac{\varepsilon}{6M} \\
= \varepsilon.
\]

Since \( \varepsilon \) is an arbitrary positive number, this implies that \( \{x_n\} \) is a Cauchy sequence, therefore, \( \lim_{n \to \infty} x_n \) exists. Let \( \lim_{n \to \infty} x_n = p \). It will be proven that \( p \) is a fixed point, that is, \( p \in F(T) \). Let \( \varepsilon > 0 \).

Since \( \lim_{n \to \infty} x_n = p \), there exists a natural number \( N_2 \) such that for all \( n \geq N_2 \),

\[
(2.5) \quad \|x_n - p\| < \frac{\varepsilon}{2(2 + k_1)}.
\]

\( \lim_{n \to \infty} D(x_n, F(T)) = 0 \) implies that there exists a natural number \( N_3 \geq N_2 \) such that for all \( n \geq N_3 \),

\[
D(x_n, F(T)) < \frac{\varepsilon}{3(4 + 3k_1)}.
\]

Therefore, there exists a \( \bar{p} \in F(T) \) such that

\[
(2.6) \quad \|x_{N_3} - \bar{p}\| < \frac{\varepsilon}{2(4 + 3k_1)}.
\]
From (2.5) and (2.6), we have
\[
\|Tp - p\| \leq \|Tp - \hat{p} + \hat{p} - Tx_N + x_N - \hat{p} + \hat{p} - x_N + x_N - p\|
\]
\[
\leq \|Tp - \hat{p}\| + 2\|Tx_N - \hat{p}\| + \|x_N - \hat{p}\| + \|x_N - p\|
\]
\[
\leq (1 + k_1)\|p - \hat{p}\| + 2(1 + k_1)\|x_N - \hat{p}\| + \|x_N - p\|
\]
\[+ \|x_N - p\|
\]
\[
\leq (1 + k_1)\|x_N - p\| + (1 + k_1)\|x_N - \hat{p}\|
\]
\[+ 2(1 + k_1)\|x_N - \hat{p}\| + \|x_N - \hat{p}\| + \|x_N - p\|
\]
\[
= (2 + k_1)\|x_N - p\| + (4 + 3k_1)\|x_N - \hat{p}\|
\]
\[
< (2 + k_1)\frac{\bar{\epsilon}}{2(2 + k_1)} + (4 + 3k_1)\frac{\bar{\epsilon}}{2(4 + 3k_1)}
\]
\[
= \bar{\epsilon}.
\]
Since $\bar{\epsilon}$ is an arbitrary positive number, we can obtain that $Tp = p$, that is, $p$ is a fixed point of $T$. This completes the proof of Theorem 2.1.

**Theorem 2.2.** Let $E$ be a real Banach space, $C$ be a nonempty convex subset of $E$, $T : C \to C$ be an quasi-nonexpansive mapping satisfying $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T)$ be a nonempty set. Let $\{x_n\}$ be the modified Ishikawa iterative sequence with mixed errors defined in (1.1), where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in $C$. Then the iterative sequence $\{x_n\}$ converges to a fixed point $p$ if and only if
\[
\liminf_{n \to \infty} D(x_n, F(T)) = 0.
\]

**Proof.** From the Definition 1.1(1), a quasi-nonexpansive mapping is asymptotically quasi-nonexpansive mapping. Therefore, Theorem 2.2 can be proven from Theorem 2.1 immediately.

**Remark 2.1.** (1) Theorem 2.1 extends Theorem 1 in Liu [5], in terms of mixed errors as more general errors.

(2) Theorem 2.2 generalizes and improves the corresponding results in Liu [4,5] and Ghosh-Debnath [2].
REFERENCES


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