A STUDY ON FAITHFUL AND MONOGENIC \(R\)-GROUPS

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ABSTRACT. Throughout this paper, we will consider that \(R\) is a near-ring and \(G\) is an \(R\)-group. We initiate the study of monogenic and strongly monogenic \(R\)-groups and their basic properties. Also, we investigate some properties of \(D_G\), faithful \(R\)-groups and monogenic \(R\)-groups and we determine that when near-rings are rings.

1. Introduction

In this paper, \(R\) is a near-ring, that is, \(R\) is an algebraic system \((R, +, \cdot)\) with two binary operations \(+\) and \(\cdot\) such that \((R, +)\) is a group (not necessarily abelian), \((R, \cdot)\) is a semigroup and the left distributive law holds: \(a(b + c) = ab + ac\) for all \(a, b, c\) in \(R\). If \(R\) has a unity 1, then \(R\) is called unitary. If 0 is the neutral element of the group \((R, +)\) then the left distributive law implies the identity \(a0 = 0\) for all \(a \in R\). However, \(0a\) is not equal to 0, in general. An element \(d\) in \(R\) is called distributive if \((a + b)d = ad + bd\) for all \(a\) and \(b\) in \(R\). A near-ring \(R\) with \((R, +)\) is abelian is called an abelian near-ring.

We consider the following notations: Given a near-ring \(R\),

\[ R_0 = \{a \in R \mid 0a = 0\} \]

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which is called the \textit{zero symmetric part} of \( R \),

\[ R_c = \{ a \in R \mid 0a = a \} = \{ a \in R \mid ra = a, \text{ for all } r \in R \} \]

which is called the \textit{constant part} of \( R \), and

\[ R_d = \{ a \in R \mid a \text{ is distributive} \} \]

which is called the \textit{distributive part} of \( R \).

We note that \( R_0 \) and \( R_c \) are subnear-rings of \( R \), but \( R_d \) is not a subnear-ring of \( R \). A near-ring \( R \) with the extra axiom \( 0a = 0 \) for all \( a \in R \), that is, \( R = R_0 \) is said to be \textit{zero symmetric}, also, in case \( R = R_c \), \( R \) is called a \textit{constant} near-ring, and in case \( R = R_d \), \( R \) is called a \textit{distributive} near-ring. From the Pierce decomposition theorem, we get

\[ R = R_0 \oplus R_c \]

as additive groups. So every element \( a \in R \) has a unique representation of the form \( a = b + c \), where \( b \in R_0 \) and \( c \in R_c \).

An \textit{ideal} of \( R \) is a subset \( I \) of \( R \) such that (i) \( (I, +) \) is a normal subgroup of \( (R, +) \), (ii) \( a(I + b) - ab \subseteq I \) for all \( a, b \in R \), (iii) \( (I + a)b - ab \subseteq I \) for all \( a, b \in R \). If \( I \) satisfies (i) and (ii) then it is called a \textit{left ideal} of \( R \). If \( I \) satisfies (i) and (iii) then it is called a \textit{right ideal} of \( R \).

On the other hand, a \textit{(two-sided)} \( R \)-\textit{subgroup} of \( R \) is a subset \( H \) of \( R \) such that (i) \( (H, +) \) is a subgroup of \( (R, +) \), (ii) \( RH \subseteq H \) and (iii) \( HR \subseteq H \). If \( H \) satisfies (i) and (ii) then it is called a \textit{left \( R \)-subgroup} of \( R \). If \( H \) satisfies (i) and (iii) then it is called a \textit{right \( R \)-subgroup} of \( R \).

Note that normal \( R \)-subgroups of \( R \) may not be ideals of \( R \).

Similarly, a subset \( H \) of \( R \) such that \( RH \subseteq H \) is called a left \( \mathit{R-subset} \) of \( R \), a subset \( H \) of \( R \) such that \( HR \subseteq H \) is called a right \( \mathit{R-subset} \) of \( R \), and a left and right \( \mathit{R-subset} \) \( H \) is said to be a \textit{(two-sided)} \( R \)-\textit{subset} of \( R \).

Also, a subset \( H \) of \( R \) is called a \textit{base} (of equality) if for all \( a, b \in R \) and all \( x \in H \) \( xa = xb \) implies \( a = b \).
Let \((G, +)\) be a group (not necessarily abelian). In the set
\[
M(G) := \{ f : G \to G \}
\]
of all the self maps of \(G\), if we define the sum \(f + g\) of any two mappings \(f, g\) in \(M(G)\) by the rule \(x(f + g) = xf + xg\) for all \(x \in G\) (called the pointwise addition of maps) and the product \(f \cdot g\) by the rule \(x(f \cdot g) = (xf)g\) for all \(x \in G\), then \((M(G), +, \cdot)\) becomes a near-ring. It is called the self map near-ring of the group \(G\) or near-ring of self maps on \(G\).

Also, if we define the set
\[
M_0(G) := \{ f \in M(G) \mid of = o \}
\]
for additive group \(G\) with identity \(o\), then \((M_0(G), +, \cdot)\) is a zero symmetric near-ring.

Let \(R\) and \(S\) be two near-rings. Then a mapping \(\theta\) from \(R\) to \(S\) is called a near-ring homomorphism if for all \(a, b \in R\), (i) \((a + b)\theta = a\theta + b\theta\) and (ii) \((ab)\theta = a\theta b\theta\).

We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as for rings ([1]).

Let \(R\) be any near-ring and \(G\) an additive group. Then \(G\) is called an \(R\)-group if there exists a near-ring homomorphism
\[
\theta : (R, +, \cdot) \to (M(G), +, \cdot).
\]
Such a homomorphism \(\theta\) is called a representation of \(R\) on \(G\), we write that \(xr\) (right scalar multiplication in \(R\)) for \(x(\theta r)\) for all \(x \in G\) and \(r \in R\). If \(R\) is unitary, then \(R\)-group \(G\) is called unitary. Thus an \(R\)-group is an additive group \(G\) satisfying (i) \(x(a + b) = xa + xb\), (ii) \(x(ab) = (xa)b\) and (iii) \(x1 = x\) (if \(R\) has a unity 1), for all \(x \in G\) and \(a, b \in R\). Sometimes, we denote an \(R\)-group \(G\) simply by \(G_R\).

We note that \(R\) itself is an \(R\)-group called the regular group.

Moreover, naturally, every group \(G\) has an \(M(G)\)-group structure, from the representation of \(M(G)\) on \(G\) given by applying \(f \in M(G)\) to the \(x \in G\) as a scalar multiplication \(xf\).
An *R*-group $G$ with the property that for each $x, y \in G$ and $a \in R$, $(x + y)a = xa + ya$ is called a distributive *R*-group, and also an *R*-group $G$ with $(G, +)$ is abelian is called an abelian *R*-group. For example, if $(G, +)$ is abelian, then $M(G)$ is an abelian near-ring and moreover, $G$ is an abelian $M(G)$-group, on the other hand, every distributive near-ring $R$ is a distributive *R*-group. We can seek a distributive abelian *R*-groups at lemma 2.22, in section 2.

We denote that the neutral element of $G$ by $e$, this is different from the neutral element $0$ of the near-ring $R$, also we write the trivial groups (or ideals) of $G$ and $R$ as $\{e\} = e$ and $\{0\} = 0$ respectively.

A representation $\theta$ of $R$ on $G$ is called faithful if $\text{Ker}\theta = \{0\}$ In this case, we say that $G$ is a faithful *R*-group, or that $R$ acts faithfully on $G$.

For an *R*-group $G$, a subgroup $T$ of $G$ such that $TR \subseteq T$ is called an *R*-subgroup of $G$, and an *R*-ideal of $G$ is a normal subgroup $N$ of $G$ such that $(N + x)a - xa \subseteq N$ for all $x \in G, a \in R$. The *R*-ideals of the regular group $R$ are precisely the right ideals of $R$. Also, a subset $V$ of $G$ such that $VR \subseteq V$ is called an *R*-subset of $G$.

Let $G$, $T$ be two additive groups (not necessarily abelian). Then the set

$$M(G, T) := \{f \mid f : G \to T\}$$

of all maps from $G$ to $T$ becomes an additive group under pointwise addition of maps. Since $M(T)$ is a near-ring of self maps on $T$, we obtain that $M(G, T)$ is an $M(T)$-group with a scalar multiplication

$$M(G, T) \times M(T) \to M(G, T)$$

defined by $(f, g) \mapsto f \cdot g$, where $x(f \cdot g) = (x f)g$ for all $x \in G$.

Let $G$ and $T$ be two *R*-groups. Then a mapping $f : G \to T$ is called a *R*-group homomorphism if for all $x, y \in G$ and $a \in R$, (i) $(x + y)f = xf + yf$ and (ii) $(xa)f = (xf)a$.

Also, we can replace *R*-group homomorphism by *R*-group monomorphism, *R*-group epimorphism, *R*-group isomorphism, *R*-group endomorphism and *R*-group automorphism, if these terms have their usual meanings as for modules ([1]).
A near-ring $R$ is called *distributively generated* (briefly, *D.G.*) by $S$ if

$$(R, +) = gp < S > = gp < R_d >$$

where $S$ is a semigroup of distributive elements in $R$.

In particular, $S = R_d$ (this is motivated by the fact that the set of all distributive elements of $R$ is multiplicatively closed and contains the unity of $R$ if it exists), where $gp < S >$ is a group generated by $S$, we denote this D.G. near-ring $R$ which is generated by $S$ is $(R, S)$.

We also note that the set of all distributive elements of $M(G)$ are obviously the set $End(G)$ of all endomorphism of the group $G$, that is,

$$(M(G))_d = End(G)$$

which is a semigroup under composition, but not yet a near-ring. Here we denote $E(G)$ is the D.G. near-ring generated by $End(G)$, that is,

$$E(G) = (M(G)), End(G).$$

Obviously, $E(G)$ is a subnear-ring of $(M_0(G), +, \cdot)$. Thus we say that $E(G)$ is the *endomorphism near-ring* of the group $G$.

Let $(R, S)$ and $(T, U)$ be D.G. near-rings. Then a near-ring homomorphism

$$\theta : (R, S) \rightarrow (T, U)$$

is called a *D.G. near-ring homomorphism* if $S\theta \subseteq U$. Clearly, any near-ring epimorphism $\theta : (R, S) \rightarrow (T, U)$ is a D.G. near-ring homomorphism.

Note that a semigroup homomorphism $\theta : S \rightarrow U$ is a D.G. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$ ([5], [6]).

For any group $G$, $M(G)$-group $G$ and $M_0(G)$-group $G$ are strongly monogenic which are appeared in Pilz [9].

For the remainder concepts and results on near-rings and $R$-groups, we refer to Meldrum [8], and Pilz [9].
2. Some results of faithful and related $R$-groups

There is a module like concept as follows: Let $(R, S)$ be a D.G. near-ring. Then an additive group $G$ is called a D.G. $(R, S)$-group if there exists a D.G. near-ring homomorphism

$$\theta : (R, S) \rightarrow (M(G), End(G)) = E(G)$$

such that $S\theta \subseteq End(G)$. If we write $xr$ instead of $x(\theta r)$ for all $x \in G$ and $r \in R$, then an D.G $(R, S)$-group is an additive group $G$ satisfying the following conditions:

$$x(rs) = (xr)s$$

and

$$x(r + s) = xr + xs,$$

for all $x \in G$ and all $r, s \in R$,

$$(x + y)s = xs + ys,$$

for all $x, y \in G$ and all $s \in S$.

Such a homomorphism $\theta$ is called a D.G. representation of $(R, S)$ on $G$. This D.G. representation is said to be faithful if $Ker\theta = \{0\}$. In this case, we also say that $G$ is a faithful D.G $(R, S)$-group.

Let $R$ be a near-ring and let $G$ be an $R$-group. If there exists an $x$ in $G$ such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then $G$ is called a monogenic $R$-group and the element $x$ is called a generator of $G$, more specially, if $G$ is monogenic and for each $x \in G$, $xR = 0$ or $xR = G$, then $G$ is called a strongly monogenic $R$-group. It is clear that $G \neq 0$ if and only if $GR \neq 0$ for any monogenic or strongly monogenic $R$-group $G$.

**Lemma 2.1.** Let $R$ be a near-ring and $G$ an $R$-group. Then we have the basic concepts:

1. If $I$ is a right ideal of $R$, then $IR_0 \subseteq I$.
2. If $A$ is an $R$-ideal of $G$, then $A$ is an $R_0$-subgroup of $G$.

From this useful lemma, we obtain the following several properties.
Proposition 2.2. For a near-ring $R$, the following are equivalent:

1. $R$ is a zero symmetric near-ring;
2. Every right ideal of $R$ is an $R$-subgroup of $R$.

Proof. (1) $\implies$ (2) is obtained from Lemma 2.1 (1).

(2) $\implies$ (1) Suppose that every right ideal of $R$ is an $R$-subgroup of $R$. Since $0$ is a right ideal of $R$, $0$ is an $R$-subgroup of $R$. Thus $0R = 0$. This implies that $R = R_0$. $\square$

Lemma 2.3 ([9]). For an $R$-group $G$, we have the following.

1. For any $x$ in $G$, $xR$ is an $R$-subgroup of $G$.
2. For any $R$-subgroup $A$ of $G$, we have that $oR = oR_c \subseteq A$.

In Lemma 2.3 (2), $oR$ is the smallest $R$-subgroup of $G$, so throughout this paper, we will write that

$$oR = oR_c =: \Omega.$$

We note that if $R$ is zero symmetric, then $\Omega = \{o\} =: o$, and $\Omega = xR_c$ for all $x \in G$.

From Lemma 2.3 (2), we define the following concepts: An $R$-group $G$ is called simple if $G$ has no non-trivial ideal, that is, $G$ has no ideals except $o$ and $G$. Similarly, we can define simple nearring as ring case. Also, $R$-group $G$ is called $R$-simple if $G$ has no $R$-subgroups except $\Omega$ and $G$.

We can explain the previous concepts elementwise: for example, a subgroup $A$ of $G$ such that $ar \in A$ for all $a \in A, r \in R$, is an $R$-subgroup of $G$, and an $R$-ideal of $G$ is a normal subgroup $N$ of $G$ such that

$$(x + g)a - ga \in N$$

for all $x \in N$, $g \in G$ and $a \in R$ (Meldrum [8]).

Lemma 2.4. For an $R$-group $G$ and a subgroup $A$ of $G$, we have the following:

1. $A$ is an $R$-ideal of $G$ if and only if $A$ is an $R_0$-ideal of $G$. 

(2) $A$ is an $R$-subgroup of $G$ if and only if $A$ is an $R_0$-subgroup of $G$ and $\Omega \subseteq A$.

Proof. (1) Necessity is obvious. Suppose $A$ is an $R_0$-ideal of $G$. Let $a \in A$, $x \in G$ and $r \in R$. Then since $R = R_0 \oplus R_c$, we rewrite that $r = s + t$, where $s \in R_0$ and $t \in R_c$. Thus we have

$$(a+x)r - xr = (a+x)(s+t) - x(s+t) = (a+x)s + (a+x)t - xt - xs.$$ 

Here, since $t \in R_c$, $(a + x)t - xt = t - t = 0$ so that $(a + x)r - xr = (a + x)s - xs$. Also since $s \in R_0$ and $A$ is an $R_0$-ideal of $G$, $(a + x)s - xs \in A$, that is $(a + x)r - xr \in A$. Consequently, $A$ is an $R$-ideal of $G$.

(2) This statement can be proved as a similar method of the proof of (1). □

Lemma 2.1 (2) and Lemma 2.4 imply the following proposition.

Proposition 2.5. For an $R$-group $G$ with $\Omega \neq 0$, we have the following:

(1) $G = \Omega$ if and only if $G$ is strongly monogenic.

(2) $R_0$-simplicity implies simplicity for $G$.

Lemma 2.6 ([7]). Let $(R, S)$ be a D.G. near-ring. Then all $R$-subgroups and all $R$-homomorphic images of a $(R, S)$-group are also $(R, S)$-groups.

Let $G$ be an $R$-group and $K$, $K_1$ and $K_2$ be subsets of $G$. Define

$$(K_1 : K_2) := \{a \in R; K_2 a \subseteq K_1\}.$$ 

We abbreviate that for $x \in G$

$$([x] : K_2) =: (x : K_2).$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the annihilator of $K$, sometimes denoted by $A(K)$. Easily, we can drive that $G$ is a faithful $R$-group, that is, $R$ acts faithfully on $G$ if $A(G) = \{0\}$, that is, $(0 : G) = \{0\}$. 
Lemma 2.7 ([3]). Let $G$ be an $R$-group and $K_1$ and $K_2$ subsets of $G$. Then we have the following conditions:

(1) If $K_1$ is a normal subgroup of $G$, then $(K_1 : K_2)$ is a normal subgroup of a near-ring $R$.

(2) If $K_1$ is an $R$-subgroup of $G$, then $(K_1 : K_2)$ is an $R$-subgroup of $R$ as an $R$-group.

(3) If $K_1$ is an $R$-ideal of $G$ and $K_2$ is an $R$-subset of $G$, then $(K_1 : K_2)$ is a two-sided ideal of $R$.

Proof. (1) and (2) are proved by Pilz [9] and Meldrum [8]. Now, we prove only (3): Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of $R$. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2r)a \subseteq K_2a \subseteq K_1,$$

also, since $K_2$ is an $R$-subset of $G$, $K_2r \subseteq K_2$ we have $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of $R$.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a + r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subseteq K_1$ and $K_1$ is an ideal of $G$. Thus $(K_1 : K_2)$ is a right ideal of $R$. Consequently, $(K_1 : K_2)$ is a two-sided ideal of $R$.

\[ \square \]

Corollary 2.8 ([8], [9]). Let $R$ be a near-ring and $G$ an $R$-group.

(1) For any $x \in G$, $(0 : x)$ is a right ideal of $R$.

(2) For any $R$-subset $K$ of $G$, $(0 : K)$ is a two-sided ideal of $R$.

(3) For any subset $K$ of $G$, $(0 : K) = \bigcap_{x \in K} (0 : x)$.

Remark 2.9. For any $R$-group homomorphism $f : G \rightarrow T$, we have $(0 : G) \subseteq (0 : f(G))$. So every monomorphic image of a faithful $R$-group is also faithful. Moreover, for any $R$-group isomorphism $f : G \rightarrow T$, we have $(0 : G) = (0 : T))$. In this case, $G$ is faithful iff $T$ is faithful.

The following statement can be proved easily, but it is important later.
Lemma 2.10 [9]. Let $G$ be a faithful $R$-group. Then we have the following conditions:

(1) If $(G, +)$ is abelian, then $(R, +)$ is abelian.
(2) If $G$ is distributive, then $R$ is distributive.

From this Lemma, we get the following proposition:

Proposition 2.11. If $G$ is a distributive abelian faithful $R$-group, then $R$ is a ring.

Proposition 2.12. Let $R$ be a near-ring and $G$ an $R$-group. Then we have the following conditions:

(1) $A(G)$ is a two-sided ideal of $R$. Moreover $G$ is a faithful $R/A(G)$-group.
(2) For any $x \in G$, we get $xR \cong R/(0 : x)$ as $R$-groups.

Proof. (1) By Corollary 2.8 and Lemma 2.7, $A(G)$ is a two-sided ideal of $R$. We now make $G$ an $R/A(G)$-group by defining, for $x \in R$, $A(G) + r \in R/A(G)$, by $x(A(G) + r) = xr$. If $A(G) + r = A(G) + s$, then $r - s \in A(G)$ hence $x(r - s) = 0$ for all $x$ in $G$, that is, $xr = xs$. This tells us that

$$x(A(G) + r) = xr = xs = x(A(G) + s)$$

Thus the action of $R/A(G)$ on $G$ has been shown to be well defined. The verification of the structure of an $R/A(G)$-group is a routine triviality. Finally, to see that $G$ is a faithful $R/A(G)$-group, we note that if $x(A(G) + r) = 0$ for all $x \in G$, then by the definition of $R/A(G)$-group structure, we have $xr = 0$. Hence $r \in A(G)$, that is,

$$A(G) + r = A(G)$$

This says that only the zero element of $R/A(G)$ annihilates all of $G$. Thus $G$ is a faithful $R/A(G)$-group.

(2) For any $x \in G$, clearly $xR$ is an $R$-subgroup of $G$. The map $\phi : R \to xR$ defined by $\phi(r) = xr$ is an $R$-group epimorphism, so
that from the isomorphism theorem for $R$-groups, since the kernel of $\phi$ is $(0: x)$, we deduce that

$$xR \cong R/(0 : x)$$

as $R$-groups.

**Proposition 2.13.** If $R$ is a near-ring and $G$ an $R$-group, then $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$.

*Proof.* Let $a \in R$. We define $\tau_a : G \to G$ by $x\tau_a = xa$ for each $x \in G$. Then $\tau_a$ is in $M(G)$. Consider the mapping $\phi : R \to M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, $\phi$ is a near-ring homomorphism from $R$ to $M(G)$.

Next, we must show that $\text{Ker } \phi = A(G)$ : Indeed, if $a \in \text{Ker } \phi$, then $\tau_a = 0$, which implies that $Ga = G\tau_a = 0$, that is, $a \in A(G)$. On the other hand, if $a \in A(G)$, then by the definition of $A(G)$, $Ga = 0$ hence $0 = \tau_a = \phi(a)$, this implies that $a \in \text{Ker } \phi$. Therefore from the first isomorphism theorem for $R$-groups, the image of $R$ is a near-ring isomorphic to $R/A(G)$. Consequently, $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$. \[\square\]

**Corollary 2.14.** If $G$ is a faithful $R$-group, then $R$ is embedded in $M(G)$. Furthermore, $G$ is a faithful $R$-group iff $G$ is both faithful $R_0$-group and faithful $R_e$-group.

**Proposition 2.15.** If $(R, S)$ is a D.G. near-ring, then every monogenic $R$-group is an $(R, S)$-group.

*Proof.* Let $G$ be a monogenic $R$-group with $x$ as a generator. Then the map $\phi : r \mapsto xr$ is an $R$-group epimorphism from $R$ to $G$. We see that by Proposition 2.12 (2),

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = \text{Ker } \phi$. From Lemma 2.6, we obtain that $G$ is an $(R, S)$-group. \[\square\]
LEMMA 2.16. Let $G$ be an $R$-group. Then $G$ is faithful iff for each $x \in G$, $R \cong xR$.

Proof. Suppose $G$ is a faithful $R$-group. Then we can define the map $f : a \mapsto xa$ which is an $R$-group epimorphism from $R$ to $xR$ as $R$-groups for each $x \in G$.

To show that $f$ is one-one, if $f(a) = f(b)$ for $a, b \in R$, then $xa = xb$, that is, $x(a - b) = 0$ for all $x \in G$. This implies that $a - b \in \bigcap_{x \in G}(0 : x)$, which is equal to $(0 : G) = A(G)$ from Corollary 2.8 (3). Since $G$ is faithful, $a - b = 0$. Hence for all $x \in G$, $R \cong xR$.

Conversely, assume the condition that $R \cong xR$ for all $x \in G$. Consider the map $f : R \rightarrow xR$ given by $a \mapsto xa$ is an $R$-group isomorphism. To show that $G$ is faithful, take any element $a \in A(G)$, that is, $Ga = 0$. This implies that for all $x \in G$, $xa = 0$, that is, $f(a) = 0$. Since $f$ is an $R$-group isomorphism, $a = 0$. Consequently, $G$ is faithful.

The following statement can be easily proved from Lemma 2.16 and Corollary 2.14.

PROPOSITION 2.17. Let $A$ be a right $R$-subgroup of a near-ring $R$. Then the following statements are equivalent:

(1) $A$ is faithful;
(2) $A$ is a base (of equality);
(3) $A$ is embedded in $M(A)$;
(4) For all $x \in G$, $R \cong xR$.

The following statement is a generalization of Proposition 2.11.

PROPOSITION 2.18 [3]. Let $(R, S)$ be a D.G. near-ring. If $G$ is an abelian faithful D.G. $(R, S)$-group, then $R$ is a ring.

As an immediate consequence of Proposition 2.18, we have the following important corollary.

COROLLARY 2.19. Let $(R, S)$ be an abelian D.G. near-ring. Then $R$ is a ring.
Lemma 2.20 ([2], [4]). If $R$ is a zero symmetric near-ring and $A$, $B$, $K$ are $R$-ideals of an $R$-group $G$, then we have the following $R$-group:

$$G' := [(A + K) \cap (B + K)]/[(A \cap B) + K]$$

which is abelian, and for any $x, y \in G'$, and $r \in R$, we have $(x+y)r = xr + yr$.

Proposition 2.22. Let $R$ be a zero symmetric near-ring and $G'$ be an $R$-group as in the above lemma. Then $G'$ is a faithful $R/(0 : G')$-group. Moreover, $R' := R/(0 : G')$ becomes a ring.

Proof. We can define the scalar multiplition as following: For $I = (o : G')$,

$$G' \times R/I \rightarrow G'$$

defined by $(x, I + a) \mapsto xa$, for all $x \in G'$ and all $I + a \in R/I$. Since $G'I = o$, this scalar multiplication is well defined, and it is easily proved that $G'$ is faithful $R/I$-group. Hence the Lemma 2.10 (1) and (2) implies that $R/(o : G') = R/I$ becomes a ring.

Proposition 2.22. Let $G$ be a faithful monogenic $R$-group with generator $x$, where $R$ is a zero symmetric near-ring. If $I$ and $J$ are right ideals of $R$ and $I \cap J \subseteq (0 : x)$, then $R$ is a ring.

Proof. From Proposition 2.12 (2) and 2.20, we have that

$$G = xR \cong R/(0 : x) = [(I+(0 : x) \cap J+(0 : x)]/[(I \cap J)+(0 : x)] = G'.$$

On the other hand, since $G$ is faithful, by the definition, we see that

$$(0 : G') \cong (0 : G) = A(G) = 0.$$

Consequently, the Lemma 2.22 implies that $R$ is a ring.
REFERENCES


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