KYUNGPOOK Math. J. 45(2005), 603-607

A Note on an Analogous Continued Fraction of Ramanujan

Bhaskar Srivastava
Department of Mathematics, Lucknow University, Lucknow, India
e-mail : bhaskarsrivastav@yahoo.com

Abstract. We give an integral representation for an analogous continued fraction of Ramanujan, for this we first prove an interesting identity.

1. Preliminaries

One of the most celebrated theorems given by Ramanujan is the Rogers-Ramanujan continued fraction

$$C(q) = 1 + \frac{q}{1 + \frac{q^2}{1+\frac{q^3}{\cdots}}} = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{1}$$

In his first letter to Hardy, Ramanujan besides this result, also gave several astounding corollaries of this result. Hardy found the results very interesting and was not able to prove them. This may be termed as the starting point of the unique relation between Hardy and Ramanujan.

Ramanujan gave, in the words of Andrews [1, p.188], "the following *outrageous* formula".

$$\frac{q^{\frac{1}{5}}}{C(q)} = \frac{\sqrt{5}-1}{2} \exp\left\{ -\frac{1}{5} \int_0^1 \frac{(1-t)^5(1-t^2)^5 \cdots}{(1-t^5)(1-t^{10})\cdots} \frac{dt}{t} \right\}. \tag{2}$$

Andrews [1, p.206] has given a proof of (2), by using the following result, also due to Ramanujan,

$$\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} = 1 - 5\left[\frac{q}{1-q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1-q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} \right]. \tag{3}$$

This can be written as

$$\frac{1}{5q} - \sum_{m=0}^{\infty} \left[\frac{(5m+1)q^{5m}}{1-q^{5m+1}} + \frac{(5m+4)q^{5m+3}}{1-q^{5m+4}} - \frac{(5m+2)q^{5m+1}}{1-q^{5m+2}} - \frac{(5m+3)q^{5m+2}}{1-q^{5m+3}} \right] = \frac{1}{5q} \frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}. \tag{4}$$

Received November 2, 2004, and, in revised form, March 7, 2005.
2000 Mathematics Subject Classification: 11A55.
Key words and phrases: q-hypergeometric series, continued fraction.

Taking the logarithmic derivative of (2) we have (4). This simple proof was given by Andrews [1,p. 206]. Now it only remains to prove (4). The proof of this identity was given by Bailey [3, pp 29-32].

We consider an analogous continued fraction

$$C(-q,q) = 1 + \cfrac{(1+1/q)q}{1+\cfrac{q^2}{1+\cfrac{(1+1/q^2)q^3}{1+\cfrac{q^4}{\cdots}}}}. \tag{5}$$

In this paper we give an identity analogous to (4) and give an integral representation for $C(-q,q)$ analogous to (2).

2. Notations

We shall use the following usual basic hypergeometric notations:
For $|q^k| < 1$,

$$\begin{aligned}
&(a)_0 = 1\\
&(a;q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)}), \quad 0 < n < \infty\\
&(a_1,a_2,\cdots a_r;q^k)_n = (a_1;q^k)_n(a_2;q^k)_n\cdots(a_k;q^k)_n,\\
&{}_r\psi_r\left[\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_r\end{matrix};q;z\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2\cdots,a_r;q)_n}{(b_1,b_2\cdots,b_r;q)_n}z^n,
\end{aligned}$$

where $|b_1b_2\cdots b_r/a_1a_2\cdots a_r| < |z| < 1$.

3. The identity

$$1 - 8\sum_{m=0}^{\infty}\left[\frac{(4m+1)q^{4m+1}}{1-q^{4m+1}} - \frac{2(4m+2)q^{4m+2}}{1-q^{4m+2}} - \frac{(4m+3)q^{4m+3}}{1-q^{4m+3}}\right] \tag{6}$$

$$= \frac{(-q^2;q^2)_\infty (q;q)_\infty^2 (q;q)_\infty^4}{(q^2;q^2)_\infty(-q;q)_\infty^2(q^4;q^4)_\infty}$$

$$= \frac{(q,q)_\infty^4}{(-q,q)_\infty^4}. \tag{7}$$

This identity is analogue to the identity of Ramanujan [4].

Proof. Bailey [4] proved that

$${}_6\psi_6\left[\begin{matrix} qa^{\frac12}, -qa^{\frac12}, b, c, d, e \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d, aq/e\end{matrix};q;qa^2/bcde\right] \tag{8}$$

$$= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a;q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde;q)_\infty}.$$

Taking $e = -a^{\frac{1}{2}}$, $b = a^{\frac{1}{2}}w$, $c = a^{\frac{1}{2}}w^2$, $d = a^{\frac{1}{2}}w^3$ where $w = e^{\frac{2\pi i}{4}}$ in (8), we have

$$(9)\qquad {}_4\psi_4\left[\begin{matrix} qa^{\frac{1}{2}}, wa^{\frac{1}{2}}, w^2a^{\frac{1}{2}}, w^3a^{\frac{1}{2}} \\ a^{\frac{1}{2}}, qw^3a^{\frac{1}{2}}, qw^2a^{\frac{1}{2}}, qwa^{\frac{1}{2}} \end{matrix} ;q;q\right]$$
$$= \frac{(aq, qw, q, -qw^3, qw^3, q, -qw, q, q/a; q)_\infty}{(qa^{\frac{1}{2}}w^3, qa^{\frac{1}{2}}w^2, qa^{\frac{1}{2}}w, -qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}w^3, qa^{-\frac{1}{2}}w^2, qa^{-\frac{1}{2}}w, -qa^{-\frac{1}{2}}, q; q)_\infty}.$$

The left side of (9)

$$= 1 + \sum_{n=-\infty}^{\infty} \frac{(qa^{\frac{1}{2}};q)_n(a^{\frac{1}{2}}w;q)_n(-a^{\frac{1}{2}};q)_n(-a^{\frac{1}{2}}w;q)_nq^n}{(a^{\frac{1}{2}};q)_n(-qa^{\frac{1}{2}}w;q)_n(-qa^{\frac{1}{2}};q)_n(qa^{\frac{1}{2}}w;q)_n}$$
$$= 1 + \sum_{n=-\infty}^{\infty} \frac{(1-a^{\frac{1}{2}}q^n)(1+a)(1+a^{\frac{1}{2}})q^n}{(1-a^{\frac{1}{2}})(1+aq^{2n})(1+a^{\frac{1}{2}})q^n}$$
$$= 1 + \frac{(1+a)(1+a^{\frac{1}{2}})^2}{1-a} \sum_{n=-\infty}^{\infty} \frac{(1-a^{\frac{1}{2}}q^n)^2q^n}{1-a^2q^{4n}}$$
$$= 1 + \frac{(1+a)(1+a^{\frac{1}{2}})^2}{1-a} \sum_{n=1}^{\infty} \left[\frac{(1-2a^{\frac{1}{2}}q^n+aq^{2n})q^n}{1-a^2q^{4n}} - \frac{(q^{2n}-2a^{\frac{1}{2}}q^n+a)q^n}{a^2-q^{4n}}\right]$$
$$= 1 + \frac{(1+a)(1+a^{\frac{1}{2}})^2}{1-a} \times \sum_{n=1}^{\infty} \left[\frac{q^na(a-1)-2q^{2n}(a^{\frac{1}{2}}-a^{\frac{5}{2}})}{(1-a^2q^{4n})(a^2-q^{4n})}\right.$$
$$\left. + \frac{q^{3n}(a^3-1)+q^{5n}(a^3-1)+2q^{6n}(a^{\frac{1}{2}}-a^{\frac{5}{2}})+q^{7n}(-a+a^2)}{(1-a^2q^{4n})(a^2-q^{4n})}\right]$$
$$= 1 - (1+a)(1+a^{\frac{1}{2}})^2 \times \sum_{n=1}^{\infty} \left[\frac{aq^n-2a^{\frac{1}{2}}(a+1)q^{2n}}{(1-a^2q^{4n})(a^2-q^{4n})}\right.$$
$$\left. + \frac{q^{3n}(a^2+a+1)+q^{5n}(a^2+a+1)-2a^{\frac{1}{2}}(a+1)q^{6n}+aq^{7n}}{(1-a^2q^{4n})(a^2-q^{4n})}\right].$$

Taking limit as $a \to 1$, the left side of (9)

$$= 1 - 8\sum_{n=1}^{\infty} \left[\frac{q^n-4q^{2n}+3q^{3n}+3q^{5n}-4q^{6n}+q^{7n}}{(1-q^{4n})^2}\right].$$

Now

$$(10)\qquad \sum_{n=1}^{\infty} \frac{q^{an}}{(1-q^{4n})^2} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{(4m+\alpha)n}$$
$$= \sum_{m=0}^{\infty} \frac{(m+1)q^{4m+\alpha}}{1-q^{4m+\alpha}}.$$

Using (10), the left side of (9)

$$1-8\sum_{m=0}^{\infty}\left[\frac{(4m+1)q^{4m+1}}{1-q^{4m+1}}-\frac{2(4m+2)q^{4m+2}}{1-q^{4m+2}}+\frac{(4m+3)q^{4m+3}}{1-q^{4m+3}}\right]. \tag{11}$$

The right side of (9)

$$=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-aq^n)(1-q^n/a)(1-q^{4n})(1+q^{2n})(1-\sqrt{a}q^n)(1-q^n/\sqrt{a})}{(1-a^2q^{4n})(1-q^{4n}/a)(1+\sqrt{a}q^n)(1+q^n/\sqrt{a})(1-q^{2n})}.$$

Taking the limit as $a\to 1$, the right side of (9)

$$=\prod_{n=1}^{\infty}\frac{(1-q^n)^4(1+q^{2n})(1-q^n)^2}{(1-q^{4n})(1-q^{2n})(1+q^n)^2}=\frac{(q;q)_\infty^4}{(-q;q)_\infty^4},$$

which proves (7).

4. Integral representation

Theorem. *We shall prove*

$$\frac{q^{\frac{1}{8}}}{[C(-q,q)-1]}=A\exp\left\{-\frac{1}{8}\int_q^1\frac{(t;t)_\infty^4}{(-t;t)_\infty^4}\frac{dt}{t}\right\}.$$

Proof. Now

$$C(-q,q)=1+\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}.$$

Taking logarithmic derivative

$$\begin{aligned}\frac{C'(-q,q)}{[C(-q,q)-1]} &= -\sum_{n=0}^{\infty}\left[\frac{2(4n+2)q^{4n+1}}{1-q^{4n+2}}-\frac{(4n+1)q^{4n}}{1-q^{4n+1}}-\frac{(4n+3)q^{4n+2}}{1-q^{4n+3}}\right]\\ &= \frac{1}{8q}-\frac{1}{8q}\frac{(q;q)_\infty^4}{(-q;q)_\infty^4}.\end{aligned}$$

So

$$\frac{1}{8q}-\frac{C'(-q,q)}{[C(-q,q)-1]}=\frac{1}{8q}\frac{(q;q)_\infty^4}{(-q;q)_\infty^4}. \tag{12}$$

Hence by (8)

$$\frac{q^{\frac{1}{8}}}{[C(-q,q)-1]}=A\exp\left\{-\frac{1}{8}\int_q^1\frac{(t;t)_\infty^4}{(-t;t)_\infty^4}\frac{dt}{t}\right\}, \tag{13}$$

which proves the theorem and is the integral representation of the function.

In the end we give another identity:

$$(q^4;q^4)_\infty[C(-q,q)-1]=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q;q^4)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+4n+2}}{(q^3;q^4)_{n+1}}. \tag{14}$$

Using the quintuple identity in the form

$$\begin{aligned}&\frac{(x^2;q)_\infty(q/x^2;q)_\infty(q;q)_\infty}{(x;q)_\infty(q/x;q)_\infty} \\ =\ & \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}x^{2n}}{(x;q)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+3n+2)/2}x^{-2n-2}}{(q/x;q)_{n+1}}\end{aligned} \tag{15}$$

and making $q\to q^4$ and taking $x=q$ in (15) we have immediately (14).

The analogous identity of Ramanujan on the base 5

$$(q^5;q^5)_\infty C(q)\ =\ \sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2-n)/2}}{(q;q^5)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2+11n+6)/2}}{(q^4;q^5)_{n+1}}$$

was proved by Andrews [1, p.200]. It can also be proved directly by making $q\to q^5$ and taking $x=q$ in (15).

5. Conclusion

In this paper we have given the analogue of the 'astounding' identity of Ramanujan and 'outrageous' integral representation. In a subsequent paper we shall find the analogue of other remarkable identities of Ramanujan [5].

References

[1] G. E. Andrews, *Ramanujans Lost Notebook III: The Rogers-Ramanujan Continued Fraction*, Adv. in Math., **41**(1981), 186-208.

[2] G. E. Andrews, B. C. Berndt, L. Jacobsen and R. L. Lamphere, *The continued fractions found in the Unorganised portions of Ramanujans notebooks*, Memoir, **477**(1992), American Mathematical Society, Providence.

[3] W. N. Bailey, *Series of Hypergeometric type which are infinite in both directions*, Quart. J. Math. (Oxford), **7**(1936), 105-115.

[4] W. N. Bailey, *A note on two of Ramanujans formulae*, Quart. J. Math. (Oxford), **3**(1952), 29-31.

[5] B. C. Berndt, H. H. Chan, S. S. Huang, S. Kang, J. Sohn and S. H. Son, *The Rogers-Ramanujan continued fraction*, J. Comput. Appl. Math., **105**(1999), 9-24.