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TRAVELING WAVE SOLUTIONS FOR HIGHER DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS USING THE $(\frac{G'}{G})$- EXPANSION METHOD

E. M. E. ZAYED

ABSTRACT. In the present paper, we construct the traveling wave solutions involving parameters of nonlinear evolution equations in the mathematical physics via the (3+1)- dimensional potential- YTSF equation, the (3+1)- dimensional generalized shallow water equation, the (3+1)- dimensional Kadomtsev- Petviashvili equation, the (3+1)- dimensional modified KdV- Zakharov- Kuznetsev equation and the (3+1)- dimensional Jimbo-Miwa equation by using a simple method which is called the $(\frac{G'}{G})-$ expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation . When the parameters are taken special values, the solitary waves are derived from the travelling waves. The travelling wave solutions are expressed by hyperbolic, trigonometric and rational functions.

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1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors(see for example [1-47]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform [3], the Backlund transform [14,15], the generalized Riccati equation [17,28], the Jacobi elliptic function expansion [7,12,25,27,29,33,36], the extended tanh- function method [1,8,34,35,44], the F-expansion method [2,18-20], the exp-function expansion method [6,9,30,42,43],

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the sub- ODE method [13,21], the extended sinh-cosh and sine-cosine methods [22], the complex hyperbolic function method [37], the truncated Painleve expansion [40] and so on.

In the present paper, we shall use a simple method which is called the $(\frac{G'}{G})$−expansion method [5,24,38,39,46,47]. This method is firstly proposed by the Chinese Mathematicians Wang et al [24] for which the traveling wave solutions of nonlinear equations are obtained. The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by a polynomial in $(\frac{G'}{G})$,where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x + y + z - Vt$ while λ, μ and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method . This new method will play an important role in expressing the traveling wave solutions for nonlinear evolution equations via the (3+1)- dimensional potential- YTSF equation, the (3+1)- dimensional generalized shallow water equation, the (3+1)-dimensional Kadomtsev- Petviashvili equation, the (3+1)- dimensional modified KdV- Zakharov- Kuznetsev equation and the (3+1)- dimensional Jimbo-Miwa equation in terms of hyperbolic, trigonometric and rational functions.

2. Description of the $(\frac{G'}{G})$-expansion method

Suppose we have the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{xx}, u_{xy}, u_{yy}, u_{yt}, u_{zz}, u_{zt}, u_{zx}, u_{zy}, ...) = 0, \quad (2.1)$$

where $u = u(x, y, z, t)$ is an unknown function, P is a polynomial in $u(x, y, z, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps [24] of the $(\frac{G'}{G})$-expansion method:

Step 1 . The traveling wave variable

$$u(x, y, z, t) = u(\xi) , \qquad \xi = x + y + z - V\,t, \quad (2.2)$$

where V is a constant, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u', u'', u''', ...) = 0. \quad (2.3)$$

Step 2. Suppose the solution of Eq(2.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i , \quad (2.4)$$

where $G = G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \qquad (2.5)$$

where α_i,V,λ and μ are constants to be determined later provided $\alpha_n \neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq (2.3).

Step 3. Substituting (2.4) into (2.3) and using Eq (2.5), collecting all terms with the same power of $(\frac{G'}{G})$ together and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for α_i, V, λ and μ.

Step 4. Since the general solution of Eq (2.5) has been well known for us, then substituting α_i, V and the general solution of Eq (2.5) into (2.4) we have more traveling wave solutions of the nonlinear partial differential equation (2.1).

3. Some applications

In this section, we apply the $(\frac{G'}{G})$ - expansion method to construct the traveling wave solutions for the (3+1)- dimensional potential- YTSF equation, the (3+1)- dimensional generalized shallow water equation, the (3+1)- dimensional Kadomtsev- Petviashvili equation, the (3+1)- dimensional modified KdV-Zakharov- Kuznetsev equation and the (3+1)- dimensional Jimbo-Miwa equation which are very important nonlinear evolution equations in the mathematical physics and have been paid attention by many researchers.

3.1. Example 1. The (3+1)- dimensional potential- YTSF equation. We start with the (3+1)- dimensional potential- YTSF equation [4,32] in the form

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx}u_z + 3u_{yy} = 0. \qquad (3.1)$$

The (3+1)- dimensional evolution equation (3.1) was recently derived by Yu et al [32]. This equation was called the potential- YTSF equation and it was developed by using the strong symmetry. Let us now solve the system (3.1) by $(\frac{G'}{G})$- expansion method. To this end, we see that the following traveling wave variable

$$u(x,y,z,t) = u(\xi), \quad \xi = x + y + z - Vt, \qquad (3.2)$$

where V is a constant, permits us converting (3.1) into the following ODE:

$$C - 4Vu' + u''' + 3u'^2 + 3u' = 0, \qquad (3.3)$$

where C is an integration constant. Suppose that the solutions of ODE (3.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \qquad (3.4)$$

where α_i are arbitrary constants, while $G(\xi)$ satisfies the second order linear ODE (2.5).

Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.3), we deduce that $n = 1$. Thus, we get

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \tag{3.5}$$

where α_0 and α_1 are constants provided $\alpha_1 \neq 0$. Substituting (3.5) into (3.3) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned}
\alpha_1\lambda[4V - (\lambda^2 + 8\mu) + 6\alpha_1\mu - 3] &= 0,\\
\alpha_1[4V - (7\lambda^2 + 8\mu) + 3\alpha_1\lambda^2 + 6\alpha_1\mu - 3] &= 0,\\
-12\alpha_1\lambda + 6\alpha_1^2\lambda &= 0,\\
-6\alpha_1 + 3\alpha_1^2 &= 0,\\
C + 4\alpha_1\mu V - \mu\alpha_1(\lambda^2 + 2\mu) + 3\alpha_1^2\mu^2 - 3\alpha_1\mu &= 0,
\end{aligned}$$

which can be solved, to get

$$\alpha_1 = 2, \quad V = \frac{1}{4}(\lambda^2 - 4\mu + 3), \quad C = 0. \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$u(\xi) = 2\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.7}$$

where

$$\xi = x + y + z - \frac{t}{4}(\lambda^2 - 4\mu + 3). \tag{3.8}$$

Solving Eq. (2.5), we deduce for $\lambda^2 - 4\mu > 0$ that

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi) + B\sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi) + B\cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi)}\right) - \frac{\lambda}{2}, \tag{3.9}$$

where A and B are arbitrary constants.

Substituting (3.9) into (3.7) we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi) + B\sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi) + B\cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ \xi)}\right) + \alpha_0 - \lambda, \tag{3.10}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \sqrt{4\mu - \lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}\right) + \alpha_0 - \lambda, \tag{3.11}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = 2\left(\frac{B}{A + B\ \xi}\right) + \alpha_0 - \lambda. \tag{3.12}$$

In particular, if $A = 0, B \neq 0, \lambda > 0, \mu = 0$, then we deduce from (3.10) that

$$u(\xi) = \lambda\tanh(\frac{\lambda}{2}\xi) + \alpha_0 - \lambda, \tag{3.13}$$

which represents the solitary wave solutions of the (3+1)- dimensional potential-YTSF equation (3.1) .

3.2. Example 2. The (3+1)- dimensional generalized shallow water equation. In this subsection, we study the following (3+1)- dimensional generalized shallow water equation [16]:

$$u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + u_{yt} - u_{xz} = 0. \tag{3.14}$$

Let us now solve Eq. (3.14) by the proposed method. To this end, we see that the traveling wave variable (3.2) permits us converting (3.14) into the following ODE:

$$C + u''' - 3u'^2 - (V+1)u' = 0, \tag{3.15}$$

where C is a constant of integration. Considering the homogeneous balance between highest order derivative and nonlinear term in (3.15), we get

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \tag{3.16}$$

Substituting (3.16) into (3.15) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned}
\alpha_1\lambda[V - (\lambda^2 + 8\mu) - 6\alpha_1\mu + 1] &= 0,\\
\alpha_1[V - (7\lambda^2 + 8\mu) - 3\alpha_1(\lambda^2 + 2\mu) + 1] &= 0,\\
12\alpha_1\lambda + 6\alpha_1^2\lambda &= 0,\\
6\alpha_1 + 3\alpha_1^2 &= 0,\\
C + \alpha_1\mu V - \mu\alpha_1(\lambda^2 + 2\mu) - 3\alpha_1^2\mu^2 + \alpha_1\mu &= 0,
\end{aligned}$$

which can be solved, to get

$$\alpha_1 = -2, \qquad V = \lambda^2 - 4\mu - 1, \qquad C = 0, \tag{3.17}$$

Substituting (3.17) into (3.16) yields

$$u(\xi) = -2\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.18}$$

where $\xi = x+y+z-t(\lambda^2-4\mu-1)$. On substituting (3.9) into (3.18), we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = -\sqrt{(\lambda^2-4\mu)}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi) + B\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi) + B\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi)}\right) + \alpha_0 + \lambda, \tag{3.19}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = -\sqrt{(4\mu-\lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}\right) + \alpha_0 + \lambda, \tag{3.20}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = -2\left(\frac{B}{A+B\ \xi}\right) + \alpha_0 + \lambda, \tag{3.21}$$

In particular, if $A = 0, B \neq 0, \lambda > 0, \mu = 0$, then we deduce from (3.19) that

$$u(\xi) = -\lambda\ \tanh(\frac{\lambda}{2}\xi) + \alpha_0 + \lambda, \tag{3.22}$$

which represents the solitary wave solutions of the (3+1)- dimensional generalized shallow water equation (3.14).

3.3. Example 3. The (3+1)- dimensional Kadomtsev- Petviashvili equation. In this subsection, we consider the (3+1)- dimensional Kadomtsev-Petviashvili equation [11,42,45] in the form:

$$u_{xt} + 6(u_x)^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0, \tag{3.23}$$

which describes the dynamics of solitons and nonlinear wave in plasmas and superfluids. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable (3.2) permits us converting (3.23) into the following ODE :

$$-Vu + 3u^2 - u'' - 2u = 0, \tag{3.24}$$

where the constants of integrations are assumed to be zero. Considering the homogeneous balance between highest order derivative and nonlinear term in (3.24), we get

$$u(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.25}$$

where α_i $(i = 0, 1, 2)$ are constants provided $\alpha_2 \neq 0$. Substituting (3.25) into (3.24) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned}
-\alpha_1 V + 6\alpha_0\alpha_1 - (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2) - 2\alpha_1 &= 0,\\
-\alpha_2 V + 3(\alpha_1^2 + 2\alpha_0\alpha_2) - (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) - 2\alpha_2 &= 0,\\
6\alpha_1\alpha_2 - (2\alpha_1 + 10\alpha_2\lambda) &= 0,\\
-6\alpha_2 + 3\alpha_2^2 &= 0,\\
-\alpha_0 V - 2\alpha_2\mu^2 + 3\alpha_0^2 - \alpha_1\lambda\mu - 2\alpha_0 &= 0,
\end{aligned}$$

which can be solved, to get

$$\alpha_2 = 2, \quad \alpha_1 = 2\lambda, \quad V = 6\alpha_0 - \lambda^2 - 8\mu - 2, \tag{3.26}$$

Substituting (3.26) into (3.25) yields

$$u(\xi) = 2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.27}$$

where $\xi = x + y + z - t(6\alpha_0 - \lambda^2 - 8\mu - 2)$. On substituting (3.9) into (3.27), we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \frac{1}{2}(\lambda^2 - 4\mu)\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi) + B\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi) + B\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi)}\right)^2 + \alpha_0 - \frac{\lambda^2}{2}, \tag{3.28}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \frac{1}{2}(4\mu - \lambda^2)\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}\right)^2 + \alpha_0 - \frac{\lambda^2}{2}, \tag{3.29}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = 2\left(\frac{B}{A + B\,\xi}\right)^2 + \alpha_0 - \frac{\lambda^2}{2}, \tag{3.30}$$

In particular, if $A = 0, B \neq 0, \lambda > 0, \mu = 0$, then we deduce from (3.28) that

$$u(\xi) = -\frac{\lambda^2}{2}\,\mathrm{sech}^2(\frac{\lambda}{2}\xi) + \alpha_0, \tag{3.31}$$

which represents the solitary wave solutions of the (3+1)- dimensional Kadomtsev-Petviashvili equation(3.23).

3.4. Example 4. The (3+1)- dimensional modified KdV- Zakharov-Kuznetsev equation. In this subsection, we consider the (3+1)- dimensional modified KdV- Zakharov- Kuznetsev equation [27] in the form:

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{3.32}$$

where α is a nonzero constant. Xu [27] has discussed Eq. (3.32) using an elliptic equation method and found new types of elliptic function solutions. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable (3.2) permits us converting (3.32) into the following ODE :

$$C - Vu + \frac{1}{3}\alpha u^3 + 3u'' = 0, \tag{3.33}$$

where C is a constant of integration. Considering the homogeneous balance between highest order derivative and nonlinear term in (3.33), we get

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0. \tag{3.34}$$

Substituting (3.34) into (3.33) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned} -\alpha_1 V + \alpha\alpha_1\alpha_0^2 + 3\alpha_1(\lambda^2 + 2\mu) &= 0, \\ \alpha\alpha_0\alpha_1^2 + 9\alpha_1\lambda &= 0, \\ \frac{1}{3}\alpha\alpha_1^3 + 6\alpha_1 &= 0, \\ C - \alpha_0 V + \frac{1}{3}\alpha\alpha_0^3 + 3\mu\lambda\alpha_1 &= 0, \end{aligned}$$

which can be solved, to get

$$\alpha_1 = \pm\frac{6i}{\sqrt{2\alpha}}, \quad \alpha_0 = \pm\frac{3\,i\lambda}{\sqrt{2\alpha}}, \quad V = -\frac{3}{2}\lambda^2 + 6\mu, \qquad C = 0, \tag{3.35}$$

Substituting (3.35) into (3.34) yields

$$u(\xi) = \pm\frac{6i}{\sqrt{2\alpha}}\left(\frac{G'}{G}\right) \pm \frac{3\,i\lambda}{\sqrt{2\alpha}}, \tag{3.36}$$

where $\xi = x + y + z + \frac{3}{2}t(\lambda^2 - 4\mu)$. On substituting (3.9) into (3.36), we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \pm 3i\sqrt{\frac{(\lambda^2-4\mu)}{2\alpha}}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi) + B\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi) + B\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)}\right), \tag{3.37}$$

Case 2 . If $\lambda^2 - 4\mu < 0$, then we have

$u(\xi)$

$$= \pm 3i\sqrt{\frac{(4\mu-\lambda^2)}{2\alpha}}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}\right), \tag{3.38}$$

Case 3 . If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \pm\frac{6i}{\sqrt{2\alpha}}\left(\frac{B}{A+B\ \xi}\right), \tag{3.39}$$

In particular, if $A = 0, B \neq 0,\ \lambda > 0, \mu = 0$, then we deduce from (3.37) that

$$u(\xi) = \pm\frac{3\ i\lambda}{\sqrt{2\alpha}}\ \tanh(\frac{\lambda}{2}\xi), \tag{3.40}$$

which represents the solitary wave solutions of the (3+1)- dimensional modified KdV- Zakharov- Kuznetsev equation(3.32).

3.5. Example 5. The (3+1)- dimensional Jimbo-Miwa equation. In this subsection, we consider the(3+1)- dimensional Jimbo-Miwa equation [10,23] in the form:

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \tag{3.41}$$

This equation is firstly investigated by Jimbo-Miwa and its certain soliton solutions are obtained in [10] . Then its studied on many manifolds by several authors regarding its solutions, symmetries and integrability properties. Wazwaz [25,26] successfully studied one- soliton solution to equation (3.41) by means of the tanh-coth method. He also employed the Hirota's bilinear method to this equation and confirmed that it is completely integrable and it admits multiple - soliton solutions of any order. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable (3.2) permits us converting (3.41) into the following ODE :

$$C + u''' + 3u'^2 - (2V+3)u' = 0, \tag{3.42}$$

where C is a constant of integration. Considering the homogeneous balance between highest order derivative and nonlinear term in (3.42), we get

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \tag{3.43}$$

Substituting (3.43) into (3.42) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$\begin{aligned}
-\alpha_1\lambda(\lambda^2+8\mu)+6\alpha_1^2\lambda\mu+2V\lambda\alpha_1+3\lambda\alpha_1 &= 0,\\
-\alpha_1(7\lambda^2+8\mu)+3\alpha_1^2(\lambda^2+2\mu)+2V\alpha_1+3\alpha_1 &= 0,\\
-12\alpha_1\lambda+6\alpha_1^2\lambda &= 0,\\
-6\alpha_1+3\alpha_1^2 &= 0,\\
C+2\alpha_1\mu V-\mu\alpha_1(\lambda^2+2\mu)+3\alpha_1^2\mu^2+3\alpha_1\mu &= 0,
\end{aligned}$$

which can be solved, to get

$$\alpha_1=2, \qquad V=\frac{1}{2}(\lambda^2-4\mu-3), \qquad C=0, \tag{3.44}$$

Substituting (3.44) into (3.43) yields

$$u(\xi) \ =2\left(\frac{G'}{G}\right)+\alpha_0, \tag{3.45}$$

where $\xi = x+y+z-\frac{1}{2}(\lambda^2-4\mu-3)\ t$. On substituting (3.9) into (3.45), we deduce the following three types of traveling wave solutions:

Case 1 . If $\lambda^2-4\mu>0$, then we have

$$u(\xi) = \sqrt{(\lambda^2-4\mu)}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)+B\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)}{A\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)+B\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \ \xi)}\right)+\alpha_0-\lambda, \tag{3.46}$$

Case 2 . If $\lambda^2-4\mu<0$, then we have

$$u(\xi) = \sqrt{(4\mu-\lambda^2)}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)+B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)+B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi)}\right)+\alpha_0-\lambda, \tag{3.47}$$

Case 3 . If $\lambda^2-4\mu=0$, then we have

$$u(\xi) \ =2\left(\frac{B}{A+B\ \xi}\right)+\alpha_0-\lambda, \tag{3.48}$$

In particular, if $A=0, B\neq 0,\ \lambda>0, \mu=0$, then we deduce from (3.46) that

$$u(\xi) \ =\lambda\ \tanh(\frac{\lambda}{2}\xi)+\alpha_0-\lambda, \tag{3.49}$$

which represents the solitary wave solutions of the the(3+1)- dimensional Jimbo-Miwa equation (3.41).

4. Summary and conclusions

In this paper, we have seen that three types of traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the (3+1)- dimensional potential- YTSF equation, the (3+1)- dimensional generalized shallow water equation, the (3+1)- dimensional Kadomtsev- Petviashvili equation, the (3+1)- dimensional modified KdV- Zakharov- Kuznetsev equation and the (3+1)- dimensional Jimbo-Miwa equation are successfully found out by using the $(\frac{G'}{G})$- expansion method. This method is more powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The $(\frac{G'}{G})$- expansion method has more advantages: It is direct and concise. It is also a standard and computerizable method which allows us to solve complicated nonlinear evolution equations in the mathematical physics. We have noted that this method changes the given problems into simple problems which can be solved easily. Also, the models proposed in this article described important applications in physics and engineering.

References

1. M.A. Abdou, *The extended tanh-method and its applications for solving nonlinear physical models,* Appl.Math.Comput, 190 (2007) 988-996.
2. M.A. Abdou, *The extended F-expansion method and its applications for a class of nonlinear evolution equation,Chaos,* Solitons and Fractals 31(2007) 95 –104 .
3. M.J. Ablowitz and P.A. Clarkson, *Solitons, nonlinear Evolution Equations and Inverse Scattering Transform,* Cambridge Univ. Press, Cambridge, 1991.
4. C.L.Bai and H.Zhao, *Generalized method to construct the solitonic solutions to (3+1)-dimensional nonlinear equation,* Phys. Letters A, 354 (2006) 428-436.
5. A.Bekir, *Application of the* $(\frac{G'}{G})-$ *expansion method for nonlinear evolution equations,* Phys.Letters A, 372 (2008) 3400-3406.
6. A.Bekir and A.Boz, *Exact solutions for nonlinear evolution equations using Exp-function method,* Phys. Letters A, 372 (2008) 1619-1625.
7. Y.Chen and Q.Wang, *Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic functions solutions to (1+1) dimensional dispersive long wave equation,* Chaos, Solitons and Fractals, 24 (2005) 745-757 .
8. E.G. Fan, *Extended tanh- function method and its applications to nonlinear equations,* Phys.Letters A, 277 (2000) 212-218.
9. J.H.He and X.H.Wu, *Exp-function method for nonlinear wave equations,* Chaos, Solitons and Fractals, 30 (2006) 700-708.
10. M. Jimbo and T. Miwa, *solitons and infinite dimensional Lie algebra,* Publ.Res.Inst. Math. Sci., 19 (1983) 943-948.
11. H.Junqi, *An algebraic method exactly solving two high dimensional nonlinear evolution equations,* Chaos, Solitons and Fractals, 23 (2005) 391-398.
12. S.Liu, Z.Fu, S.D. Liu and Q.Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Letters A, 289 (2001) 69-74.
13. X.Z. Li and M.L.Wang, *A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with higher order nonlinear terms,* Phys.Letters A, 361 (2007) 115-118.
14. M.R.Miura, *Backlund Transformation,* Springer-Verlag, Berlin,1978.
15. C.Rogers and W.F.Shadwick, *Backlund Transformations,* Academic Press, New York,1982.

16. B.Tian and Y.T. Gao, *Beyond traveling waves: a new algorithm for solving nonlinear evolution equations,* Comput. Phys.Commu., 95 (1996) 139-142.
17. Z.Wang and H.Q.Zhang, *A new generalized Riccati equation rational expansion method to a class of nonlinear evolution equation with nonlinear terms of any order,* Appl.Math.Comput, 186 (2007) 693-704.
18. M.Wang and Y.Zhou, *The periodic wave equations for the Klein-Gordon-Schordinger equations,* Phys. Letters A, 318 (2003) 84-92 .
19. M.Wang and X. Li, *Extended F-expansion and periodic wave solutions for the generalized Zakharov equations,* Phys. Letters A, 343 (2005) 48-54 .
20. M.Wang and X.Li, *Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation,* Chaos, Solitons and Fractals 24 (2005) 1257- 1268.
21. M.L.Wang, X.Z.Li and J.L.Zhang, *Sub-ODE method and solitary wave solutions for higher order nonlinear Schrodinger equation,* Phys. Letters A, 363 (2007) 96-101.
22. D.S.Wang, Y.J.Ren and H.Q.Zhang, *Further extended sinh-cosh and sin-cos methods and new non traveling wave solutions of the (2+1)-dimensional dispersive long wave equations,* Appl. Math.E-Notes, 5 (2005) 157-163.
23. D.S.Wang, W. Sun, C.Kong and H. Zhang, *New extended rational expansion method and exact solutions of Boussinesq and Jimbo- Miwa equation,* Appl. Math. Comput., 189 (2007) 878-886.
24. M.Wang, X.Li and J.Zhang, *The* $(\frac{G'}{G})-$ *expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics,* Phys.Letters A, 372 (2008) 417-423.
25. A.M. Wazwaz, *New solutions of distinct physical structures to high - dimensional nonlinear evolution equations,* Appl. Math. Comput., 196 (2008) 363-368.
26. A.M. Wazwaz, *Multiple soliton solutions for the Calogero- Bogoyavlenskii- Schiff Jimbo-Miwa and YTSF equations*, Appl. Math. Comput., 203 (2008) 592-597.
27. G. Xu, *An elliptic equation method and its applications in nonlinear evolution equations,* Chaos, Solitons and Fractals, 29 (2006) 942-947.
28. Z.Yan and H.Zhang, *New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water,* Phys. Letters A, 285 (2001) 355-362.
29. Z.Yan, *Abundant families of Jacobi elliptic functions of the (2+1)-dimensional integrable Davey-Stawartson-type equation via a new method,* Chaos,Solitons and Fractals, 18 (2003) 299-309.
30. E.Yusufoglu, *New solitary solutions for the MBBM equations using Exp-function method,* Phys. Letters A, 372 (2008) 442-446.
31. E.Yusufoglu and A.Bekir, *Exact solution of coupled nonlinear evolution equations,* Chaos, Solitons and Fractals, 37 (2008) 842-848.
32. S.J. Yu, K.Toda, N. Sasa and T. Fukuyama, *N- soliton solutions to Bogoyavlenskii- Schiff equation and a guest for the soliton solutions in (3+1) dimensions,* J.Phys.A. Math. Gen., 31 (1998) 3337-3347.
33. E.M.E.Zayed, H.A. Zedan and K.A.Gepreel, *On the solitary wave solutions for nonlinear Hirota-Satsuma coupled KdV equations,* Chaos,Solitons and Fractals, 22 (2004) 285-303.
34. E.M.E.Zayed, H.A.Zedan and K.A.Gepreel, *On the solitary wave solutions for nonlinear Euler equations,* Appl. Anal., 83 (2004)1101-1132.
35. E.M.E.Zayed, H.A.Zedan and K.A.Gepreel, *Group analysis and modified tanh-function to find the invariant solutions and soliton solution for nonlinear Euler equations,* Int.J.nonlinear Sci. and Nume.Simul.5 (2004) 221-234.
36. E.M.E. Zayed, A.M. Abourabia, K.A.Gepreel and M.M. Horbaty, *Traveling solitary wave solutions for the nonlinear coupled KdV system,* Chaos, Solitons and Fractals, 34(2007) 292-306 .
37. E.M.E.Zayed, K.A.Gepreel and M.M.Horbaty, *Exact solutions for some nonlinear differential equations using complex hyperbolic function,* Appl. Anal., 87 (2008) 509-522.

38. E.M.E.Zayed and K.A.Gepreel, *The* $(\frac{G'}{G})-$ *expansion method for finding traveling wave solutions of nonlinear PDEs in mathematical physics,* J. Math. Phys., 50 (2009) 013502-013513.
39. E.M.E.Zayed, *The* $(\frac{G'}{G})-$ *expansion method and its applications to some nonlinear evolution equations in the mathematical physics,* J. Appl. Math. Computing, 31 (2009) 89-103.
40. S.L.Zhang, B. Wu and S.Y.Lou, *Painlevé analysis and special solutions of generalized Broer-Kaup equations,* Phys. Lett. A, 300 (2002) 40-48.
41. S. Zhang and T.C. Xia, *A generalized F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equations,* Appl. Math. Comput., 183 (2006) 1190-1200 .
42. S. Zhang, *Application of Exp-function method to higher dimensional nonlinear evolution equation,* Chaos, Solitons and Fractals 38 (2008) 270-276.
43. S. Zhang, *Application of Exp-function method to Riccati equation and new exact solutions with three arbitrary functions of Broer- Kaup- Kupershmidt equations,* Phys. Letters A, 372 (2008)1873-1880.
44. S. Zhang and T.C. Xia, *A further improved tanh-function method exactly solving the (2+1)-dimensional dispersive long wave equations,* Appl.Math.E-Notes, 8 (2008) 58-66.
45. S. Zhang and T.C. Xia, *Symbolic computation and new families of exact non-traveling wave solutions to (3+1)- dimensional Kadomtsev- Petviashvili equation,* Appl.Math. Comput., 181 (2006) 319-331.
46. S.Zhang, J.Tong and W.Wang, *A generalized* $(\frac{G'}{G})-$ *expansion method for the mKdV equation with variable coefficients,* Phys.Letters A, 372 (2008) 2254-2257.
47. J. Zhang, X.Wei and Y.Lu, *A generalized* $(\frac{G'}{G})-$ *expansion method and its applications,* Phys.Letters A, 372 (2008) 3653-3658.

E. M. E. Zayed ,
Mathematics Department,Faculty of Science, Zagazig University, Zagazig,Egypt
Email: `emezayed@@hotmail.com`