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EXTENSIONS OF EULER TYPE II TRANSFORMATION AND SAALSCHÜTZ'S THEOREM

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Abstract. In this research paper, motivated by the extension of the Euler type I transformation obtained very recently by Rathie and Paris, the authors aim at presenting the extensions of Euler type II transformation. In addition to this, a natural extension of the classical Saalschütz's summation theorem for the series ${}_3F_2$ has been investigated. Two interesting applications of the newly obtained extension of classical Saalschütz's summation theorem are given.

1. Introduction

Very recently, Rathie and Paris [3] have obtained an extension of Euler type I transformation viz.

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & \nu,\end{matrix}\ ;\ x\right] = (1-x)^{-\alpha}\,{}_2F_1\left[\begin{matrix}\alpha, & \nu-\beta \\ & \nu,\end{matrix}\ ;\ -\frac{x}{1-x}\right]$$

in the form

(1.1)

$$ {}_3F_2\left[\begin{matrix}d, & b-a-1, & f+1 \\ & b, & f\end{matrix}\ ;\ x\right] = (1-x)^{-d}\,{}_3F_2\left[\begin{matrix}d, & a, & c+1 \\ & b, & c\end{matrix}\ ;\ -\frac{x}{1-x}\right],$$

where, of course

$$f = \frac{c(1+a-b)}{a-c}. \tag{1.2}$$

On the other hand, the Euler type-II transformation is

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & \nu,\end{matrix}\ ;\ x\right] = (1-x)^{\nu-\alpha-\beta}\,{}_2F_1\left[\begin{matrix}\nu-\alpha, & \nu-\beta \\ & \nu,\end{matrix}\ ;\ x\right] \tag{1.3}$$

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and as mentioned in almost all the books on special functions that writing (1.3) in the form

$$(1.4)\quad {}_2F_1\left[\begin{matrix}\nu-\alpha, & \nu-\beta \\ & \nu,\end{matrix}\; ;\; x\right] = (1-x)^{-(\nu-\alpha-\beta)}\,{}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & \nu,\end{matrix}\; ;\; x\right]$$

and equating the coefficients of x^n on both sides, one can easily obtains the following Saalschütz's theorem for the series ${}_3F_2$ viz.

$$(1.5)\qquad {}_3F_2\left[\begin{matrix}\alpha, & \beta, & -n \\ \nu, & 1+\alpha+\beta-\nu-n & \end{matrix}\; ;\; 1\right] = \frac{(\nu-\alpha)_n(\nu-\beta)_n}{(\nu)_n(\nu-\alpha-\beta)_n}.$$

Using (1.5), Bailey, in his well known and very interesting popular research paper [1], had obtained the following interesting results viz.

(1.6)

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix}a, & b \\ a+b+\frac{1}{2} & \end{matrix}\; ;\; -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}2a, & a-b+\frac{1}{2} \\ a+b+\frac{1}{2} & \end{matrix}\; ;\; x\right]$$

and
(1.7)

$$(1-x)^{1-2a}\,{}_2F_1\left[\begin{matrix}a, & b \\ a+b+\frac{1}{2} & \end{matrix}\; ;\; -\frac{4x}{(1-x)^2}\right] = {}_3F_2\left[\begin{matrix}2a-1, & a+\frac{1}{2}, & a-b-\frac{1}{2} \\ a-\frac{1}{2}, & a+b+\frac{1}{2} & \end{matrix}\; ;\; x\right].$$

The aim of this research paper is to obtain the natural extensions of the results (1.4) and (1.5). As applications, two interesting results, which generalize (1.6) and (1.7) are also given.

Since the generalized hypergeometric functions ${}_pF_q$ appear ubiquitously as solutions to a plethora of problems in Mathematics, Statistics, Engineering and Mathematical Physics, the results given in this paper should be eventually prove useful in a wide range of applications.

2. Extensions of (1.4) and (1.5)

In this section, we shall establish the extensions of (1.4) and (1.5).

For this, if we put $x = -\frac{x}{1-x}$ in (1.1), we get after little simplification
(2.1)

$${}_3F_2\left[\begin{matrix}d, & a, & c+1 \\ & b, & c\end{matrix}\; ;\; x\right] = (1-x)^{-d}\,{}_3F_2\left[\begin{matrix}d, & b-a-1, & f+1 \\ & b, & f\end{matrix}\; ;\; -\frac{x}{1-x}\right].$$

Writing the right-hand side of (2.1) in the form

(2.2)
$$ {}_3F_2\left[\begin{matrix} d, & a, & c+1 \\ & b, & c \end{matrix} ; x\right] = (1-x)^{-d}\, {}_3F_2\left[\begin{matrix} b-a-1, & d, & f+1 \\ & b, & f \end{matrix} ; -\frac{x}{1-x}\right], $$

apply this result on the right-hand side of (2.2), we get at once

(2.3)
$$ {}_3F_2\left[\begin{matrix} d, & a, & c+1 \\ & b, & c \end{matrix} ; x\right] = (1-x)^{b-a-d-1}\, {}_3F_2\left[\begin{matrix} b-a-1, & b-d-1, & 1+g \\ & b, & g \end{matrix} ; x\right], $$

where

$$ g = \frac{f(1+d-b)}{d-f} \tag{2.4} $$

and f is the same as given in (1.2).

The result (2.3) is easily seen to be an extension of the Euler type II transformation (1.3) [by taking $b = 1 + c \Rightarrow f = c = g$].

Now we come to the extension of the Saalschütz's theorem (1.5). For this writing (2.3) in the form

$$ {}_3F_2\left[\begin{matrix} b-a-1, & b-d-1, & 1+g \\ & b, & g \end{matrix} ; x\right] = (1-x)^{-(b-a-d-1)}\, {}_3F_2\left[\begin{matrix} d, & a, & c+1 \\ & b, & c \end{matrix} ; x\right]. $$

Expressing the functions on both sides as series

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(b-a-1)_n (b-d-1)_n (1+g)_n}{(b)_n (g)_n n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(b-a-d-1)_n}{n!} x^n \sum_{m=0}^{\infty} \frac{(d)_m (a)_m (c+1)_m}{(b)_m (c)_m} \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b-a-d-1)_n (d)_m (a)_m (c+1)_m}{(b)_m (c)_m n! m!} x^{n+m} \end{aligned} $$

replace n by $n-m$, and using

$$ (\alpha)_{n-m} = \frac{(-1)^m (\alpha)_n}{(1-\alpha-n)_m} \quad \text{and} \quad (n-m)! = \frac{(-1)^m n!}{(-n)_m}, $$

we get

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(b-a-1)_n (b-d-1)_n (1+g)_n}{(b)_n (g)_n n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(b-a-d-1)_n}{n!} x^n \sum_{m=0}^{n} \frac{(-n)_m (d)_m (a)_m (c+1)_m}{(b)_m (c)_m (2+a-b+d-n)_m m!} \end{aligned} $$

summing the inner series, we get

$$= \sum_{n=0}^{\infty} \frac{(b-a-d-1)_n}{n!} x^n \, {}_4F_3 \left[\begin{matrix} -n, & d, & a, & c+1 \\ b, & c, & 2+a-b+d-n & \end{matrix} ; 1 \right].$$

Finally, equating the coefficients of x^n in both sides, we get, after some simplification

(2.5)

$${}_4F_3 \left[\begin{matrix} -n, & d, & a, & c+1 \\ b, & c, & 2+a-b+d-n & \end{matrix} ; 1 \right] = \frac{(b-a-1)_n (b-d-1)_n (1+g)_n}{(b)_n (g)_n (b-a-d-1)_n},$$

where g and f are the same as given in (2.4) and (1.2) respectively.

The result (2.5) is clearly seen to be an extension of the Saalschütz's theorem (1.5) [by taking $b = 1 + c \Rightarrow f = c = g$].

Remark. Our new summation theorem (2.5) for the series ${}_4F_3$ is Saalschützian.

3. Applications

As applications of the newly obtained extension of Saalschütz's theorem (2.5), we shall establish the following two interesting results in this section.

$$(1-x)^{-2a} \, {}_3F_2 \left[\begin{matrix} a, & b, & d+1 \\ a+b+\frac{3}{2}, & d & \end{matrix} ; -\frac{4x}{(1-x)^2} \right]$$

(3.1)

$$= {}_4F_3 \left[\begin{matrix} 2a, & a-b-\frac{1}{2}, & 1+a-A, & 1+a+A \\ a+b+\frac{3}{2}, & a-A, & a+A & \end{matrix} ; x \right],$$

where

(3.2)

$$A^2 = \frac{1}{b-d} \left(a^2 b - abd - \frac{1}{2} bd - \frac{1}{4} d \right)$$

and

$$(1-x)^{1-2a} \, {}_3F_2 \left[\begin{matrix} a, & b, & d+1 \\ a+b+\frac{3}{2}, & d & \end{matrix} ; -\frac{4x}{(1-x)^2} \right]$$

(3.3)

$$= {}_5F_4 \left[\begin{matrix} 2a-1, & a-b-\frac{3}{2}, & a+\frac{1}{2}, & a+\frac{1}{2}-A, & a+\frac{1}{2}+A \\ a-\frac{1}{2}, & a+b+\frac{3}{2}, & a-\frac{1}{2}+A, & a-\frac{1}{2}-A & \end{matrix} ; x \right],$$

where

(3.4)

$$A^2 = \frac{1}{b-d} \left(a^2 b - ab - d - abd - \frac{1}{2} bd + \frac{1}{4} d \right).$$

3.1. Derivations

In order to prove (3.1), we proceed as follows. Denoting the left-hand side of (3.1) by $S(x)$, expressing ${}_3F_2$ as a series, we get after a little simplification

$$S(x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (d+1)_m (-1)^m 2^{2m} x^m}{(a+b+\frac{3}{2})_m (d)_m m!} (1-x)^{-(2a+2m)}.$$

Using binomial theorem and appropriate Pochhammer's identities, we have after some simplification

$$S(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2a)_{2m+n} (b)_m (d+1)_m (-1)^m}{(a+\frac{1}{2})_m (a+b+\frac{3}{2})_m (d)_m \; m! \; n!} \; x^{m+n}$$

changing n into $n-m$ and using Bailey's transforms for double series, we have

$$S(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(2a)_{m+n} (b)_m (d+1)_m (-1)^m x^n}{(a+\frac{1}{2})_m (a+b+\frac{3}{2})_m (d)_m \; m! \; (n-m)!}.$$

Again using appropriate Pochhammer's identities, we have

$$S(x) = \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \sum_{m=0}^{n} \frac{(-n)_m (b)_m (2a+n)_m (d+1)_m}{(a+\frac{1}{2})_m (a+b+\frac{3}{2})_m (d)_m m!},$$

summing up the inner series, we have

$$S(x) = \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \; {}_4F_3 \left[\begin{matrix} -n, & b, & 2a+n, & d+1 \\ & & & & ; \; 1 \\ a+\frac{1}{2}, & a+b+\frac{3}{2}, & d & \end{matrix} \right].$$

Finally using our new summation formula (2.5) and after much simplification, we get

$$S(x) = \sum_{n=0}^{\infty} \frac{(2a)_n (1+a-A)_n (1+a+A)_n}{n! (a-A)_n (a+A)_n} \frac{(a-b-\frac{1}{2})_n}{(a+b+\frac{3}{2})_n} x^n,$$

where A is the same as given in (3.2). Summing up the series, we get the right-hand side of (3.1). This completes the proof of (3.1).

In exactly the same manner, the result (3.3) can also be established. Therefore we omit for the details.

4. Special cases

(a) In (2.5), if we take $n \to \infty$, we get

$$(4.1) \qquad {}_3F_2 \left[\begin{matrix} d, & a, & c+1 \\ & & & ; \; 1 \\ & b, & c & \end{matrix} \right] = \frac{\Gamma(b)\Gamma(g)\Gamma(b-a-d-1)}{\Gamma(1+g)\Gamma(b-a-1)\Gamma(b-d-1)}$$

provided that $\mathrm{Re}(b-a-d) > 1$. Of course the values of f and g are same as mentioned above.

Furthermore, if we simplify (4.1), we get

(4.2)
$$ {}_3F_2\left[\begin{matrix} d, & a, & c+1 \\ & b, & c \end{matrix} ; \; 1\right] = \frac{\Gamma(b)\Gamma(b-a-d-1)}{\Gamma(b-a)\Gamma(b-d)}\left[(b-a-d+1)+\frac{ad}{c}\right] $$

which is regarded as an extension of the classical Gauss summation theorem viz

(4.3)
$$ {}_2F_1\left[\begin{matrix} a, & b, \\ & c, \end{matrix} ; \; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

provided that $\mathrm{Re}(c-a-b) > 0$ and recorded in the literature [2].

(b) In (3.1) and (3.3), if we take $d = a + b + \frac{1}{2}$, we get Bailey's result (1.6) and (1.7) respectively. Thus our results (3.1) and (3.3) may be regarded as the natural extensions of Bailey's results (1.6) and (1.7).

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