Bull. Korean Math. Soc. **51** (2014), No. 4, pp. 1163–1173
http://dx.doi.org/10.4134/BKMS.2014.51.4.1163

ON 2-ABSORBING PRIMARY IDEALS IN COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with $1 \neq 0$. In this paper, we introduce the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. A number of results concerning 2-absorbing primary ideals and examples of 2-absorbing primary ideals are given.

1. Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is said to be proper if $I \neq R$. Let I be a proper ideal of R. Then $Z_I(R) = \{r \in R \mid rs \in I$ for some $s \in R \setminus I\}$. The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [3] and studied in [2], [8], and [4]. Various generalizations of prime ideals are also studied in [1] and [5]. Recall that a proper ideal I of R is called a 2-*absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this paper, we introduce the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is said to be a 2-*absorbing primary ideal* of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Note that a 2-absorbing ideal of a commutative ring R is a 2-absorbing primary ideal of R. However, these are different concepts. For instance, consider the ideal $I = (12)$ of $\mathbb{Z}$. Since $2 \cdot 2 \cdot 3 \in I$, but $2 \cdot 2 \notin I$ and $2 \cdot 3 \notin I$, I is not a 2-absorbing ideal of $\mathbb{Z}$. However, it is clear that I is a 2-absorbing primary ideal of $\mathbb{Z}$. It is also clear that every primary ideal of a ring R is a 2-absorbing primary ideal of R. However, the converse is not true. For example, (6) is a 2-absorbing primary ideal of $\mathbb{Z}$, but it is not a primary ideal of $\mathbb{Z}$.

Among many results in this paper, it is shown (Theorem 2.2) that the radical of a 2-absorbing primary ideal of a ring R is a 2-absorbing ideal of R. It is shown (Theorem 2.4) that if I_1 is a P_1-primary ideal of R for some prime ideal

Received September 23, 2013; Revised January 14, 2014.
2010 *Mathematics Subject Classification.* Primary 13A15; Secondary 13F05, 13G05.
Key words and phrases. primary ideal, prime ideal, 2-absorbing ideal, n-absorbing ideal.

P_1 of R and I_2 is a P_2-primary ideal of R for some prime ideal P_2 of R, then I_1I_2 and $I_1 \cap I_2$ are 2-absorbing primary ideals of R. It is shown (Theorem 2.8) that if I is a proper ideal of a ring R such that $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R. It is shown (Theorem 2.10) that every proper ideal of a divided ring is a 2-absorbing primary ideal. It is shown (Theorem 2.11) that a Noetherian domain R is a Dedekind domain if and only if a nonzero 2-absorbing primary ideal of R is either M^k for some maximal ideal M of R and some positive integer $k \geq 1$ or $M_1^k M_2^n$ for some distinct maximal ideals M_1, M_2 of R and some positive integers $k, n \geq 1$. It is shown (Theorem 2.19) that a proper ideal I of R is a 2-absorbing primary ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. Let $R = R_1 \times R_2$, where R_1, R_2 are commutative rings with $1 \neq 0$. It is shown (Theorem 2.23) that a proper ideal J of R is a 2-absorbing primary ideal of R if and only if either $J = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $J = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2 or $J = I_1 \times I_2$ for some primary ideal I_1 of R_1 and some primary ideal I_2 of R_2.

2. Properties of 2-absorbing primary ideals

Definition 2.1. A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Theorem 2.2. *If I is a 2-absorbing primary ideal of R, then $\sqrt{I}$ is a 2-absorbing ideal of R.*

Proof. Let $a, b, c \in R$ such that $abc \in \sqrt{I}$, $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Since $abc \in \sqrt{I}$, there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in I$. Since I is 2-absorbing primary and $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$, we conclude that $a^n b^n = (ab)^n \in I$, and hence $ab \in \sqrt{I}$. Thus $\sqrt{I}$ is a 2-absorbing ideal of R. □

Theorem 2.3. *Suppose that I is a 2-absorbing primary ideal of R. Then one of the following statements must hold.*

(1) *$\sqrt{I} = P$ is a prime ideal,*
(2) *$\sqrt{I} = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime ideals of R that are minimal over I.*

Proof. Suppose that I is a 2-absorbing primary ideal of R. Then $\sqrt{I}$ is a 2-absorbing ideal by Theorem 2.2. Since $\sqrt{\sqrt{I}} = \sqrt{I}$, the claim follows from [3, Theorem 2.4]. □

Theorem 2.4. *Let R be a commutative ring with $1 \neq 0$. Suppose that I_1 is a P_1-primary ideal of R for some prime ideal P_1 of R, and I_2 is a P_2-primary ideal of R for some prime ideal P_2 of R. Then the following statements hold.*

(1) *I_1I_2 is a 2-absorbing primary ideal of R.*
(2) *$I_1 \cap I_2$ is a 2-absorbing primary ideal of R.*

Proof. (1) Suppose that $abc \in I_1I_2$ for some $a,b,c \in R$, $ac \notin \sqrt{I_1I_2}$, and $bc \notin \sqrt{I_1I_2} = P_1 \cap P_2$. Then $a,b,c \notin \sqrt{I_1I_2} = P_1 \cap P_2$. Since $\sqrt{I_1I_2} = P_1 \cap P_2$, we conclude that $\sqrt{I_1I_2}$ is a 2-absorbing ideal of R. Since $\sqrt{I_1I_2}$ is a 2-absorbing ideal of R and $ac, bc \notin \sqrt{I_1I_2}$, we have $ab \in \sqrt{I_1I_2}$. We show that $ab \in I_1I_2$. Since $ab \in \sqrt{I_1I_2} \subseteq P_1$, we may assume that $a \in P_1$. Since $a \notin \sqrt{I_1I_2}$ and $ab \in \sqrt{I_1I_2} \subseteq P_2$, we conclude that $a \notin P_2$ and $b \in P_2$. Since $b \in P_2$ and $b \notin \sqrt{I_1I_2}$, we have $b \notin P_1$. If $a \in I_1$ and $b \in I_2$, then $ab \in I_1I_2$ and we are done. Thus assume that $a \notin I_1$. Since I_1 is a P_1-primary ideal of R and $a \notin I_1$, we have $bc \in P_1$. Since $b \in P_2$ and $bc \in P_1$, we have $bc \in \sqrt{I_1I_2}$, which is a contradiction. Thus $a \in I_1$. Similarly, assume that $b \notin I_2$. Since I_2 is a P_2-primary ideal of R and $b \notin I_2$, we have $ac \in P_2$. Since $ac \in P_2$ and $a \in P_1$, we have $ac \in \sqrt{I_1I_2}$, which is a contradiction. Thus $b \in I_2$. Hence $ab \in I_1I_2$.

(2)(Similar to the proof in (1)). Let $H = I_1 \cap I_2$. Then $\sqrt{H} = P_1 \cap P_2$. Suppose that $abc \in H$ for some $a,b,c \in R$, $ac \notin \sqrt{H}$, and $bc \notin \sqrt{H}$. Then $a,b,c \notin \sqrt{H} = P_1 \cap P_2$. Since $\sqrt{H} = P_1 \cap P_2$ is a 2-absorbing ideal of R and $ac, bc \notin \sqrt{H}$, $ab \in \sqrt{H}$. We show that $ab \in H$. Since $ab \in \sqrt{H} \subseteq P_1$, we may assume that $a \in P_1$. Since $a \notin \sqrt{H}$ and $ab \in \sqrt{H} \subseteq P_2$, we conclude that $a \notin P_2$ and $b \in P_2$. Since $b \in P_2$ and $b \notin \sqrt{H}, b \notin P_1$. If $a \in I_1$ and $b \in I_2$, then $ab \in H$ and we are done. Thus assume that $a \notin I_1$. Since I_1 is a P_1-primary ideal of R and $a \notin I_1$, we have $bc \in P_1$. Since $b \in P_2$ and $bc \in P_1$, we have $bc \in \sqrt{H}$, which is a contradiction. Thus $a \in I_1$. Similarly, assume that $b \notin I_2$. Since I_2 is a P_2-primary ideal of R and $b \notin I_2$, we have $ac \in P_2$. Since $ac \in P_2$ and $a \in P_1$, we have $ac \in \sqrt{H}$, which is a contradiction. Thus $b \in I_2$. Hence $ab \in H$. □

In view of Theorem 2.4, we have the following result.

Corollary 2.5. *Let R be a commutative ring with $1 \neq 0$, and let P_1, P_2 be prime ideals of R. If P_1^n is a P_1-primary ideal of R for some positive integer $n \geq 1$ and P_2^m is a P_2-primary ideal of R for some positive integer $m \geq 1$, then $P_1^nP_2^m$ and $P_1^n \cap P_2^m$ are 2-absorbing primary ideals of R. In particular, P_1P_2 is a 2-absorbing primary ideal of R.*

In the following example, we show that if P_1, P_2 are prime ideals of a ring R and n, m are positive integers, then $P_1^nP_2^m$ need not be a 2-absorbing primary ideal of R.

Example 2.6. Let $R = \mathbb{Z}[Y] + 3X\mathbb{Z}[Y,X]$. Then $P_1 = YR$ and $P_2 = 3X\mathbb{Z}[Y,X]$ are prime ideals of R. Let $I = P_1P_2^2$. Then $3X^2 \cdot Y \cdot 3 = 9X^2Y \in I$ and $3X^2 \cdot Y = 3X^2Y \notin I$. Clearly $3X^2 \cdot 3 = 9X^2 \notin \sqrt{I} = P_1 \cap P_2$ and $Y \cdot 3 = 3Y \notin \sqrt{I} = P_1 \cap P_2$. Hence I is not a 2-absorbing primary ideal of R.

In the following example, we show that if $I \subset J$ such that I is a 2-absorbing primary ideal of R and $\sqrt{I} = \sqrt{J}$, then J need not be a 2-absorbing ideal of R.

Example 2.7. Let $R = \mathbb{Z}[X,Y,Z]$. Then $P_1 = XR$, $P_2 = YR$ are prime ideals of R, and $I = P_1^3P_2^3$ is a 2-absorbing primary ideal of R by Corollary 2.5. Let

$J = (XYZ, Y^3, X^3)R$. Then $I \subset J$ and $\sqrt{I} = \sqrt{J} = P_1 \cap P_2 = (XY)R$. We show that J is not a 2-absorbing ideal of R. For $X \cdot Y \cdot Z = XYZ \in J$, but $X \cdot Y = XY \notin J$, $X \cdot Z = XZ \notin \sqrt{J}$, and $Y \cdot Z = YZ \notin \sqrt{J}$. Thus J is not a 2-absorbing ideal of R.

Let I be a proper ideal of a ring R. It is known that if $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R. In the following result, we show that if $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R.

Theorem 2.8. *Let I be an ideal of R. If $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R. In particular, if P is a prime ideal of R, then P^n is a 2-absorbing primary ideal of R for every positive integer $n \geq 1$.*

Proof. Suppose that $abc \in I$ and $ab \notin I$. Since $(ac)(bc) = abc^2 \in I \subseteq \sqrt{I}$ and $\sqrt{I}$ is a prime ideal of R, we have $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence I is a 2-absorbing primary ideal of R. □

In view of Theorem 2.2, Theorem 2.3, and Theorem 2.8, the following is an example of an ideal J of a ring R where $\sqrt{J}$ is a 2-absorbing ideal of R, but J is not a 2-absorbing primary ideal of R.

Example 2.9. Let $R = \mathbb{Z}[X, Y, Z]$ and let $J = (XYZ, Y^3, X^3)R$. Then $\sqrt{J} = YR \cap XR$ is a 2-absorbing ideal of R, but J is not a 2-absorbing primary ideal of R by Example 2.7. Also, see Example 2.6.

Recall that a commutative ring R with $1 \neq 0$ is called a *divided ring* if for every prime ideal P of R, we have $P \subseteq xR$ for every $x \in R \setminus P$. Every chained ring is a divided ring (recall that a commutative ring R with $1 \neq 0$ is called a *chained ring*, if $x \mid y (inR)$ or $y \mid x (inR)$ for every $x, y \in R$). It is known that the prime ideals of a divided ring are linearly ordered; i.e., if P_1, P_2 are prime ideals of R, then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. We have the following result.

Theorem 2.10. *Let R be a commutative divided ring with $1 \neq 0$. Then every proper ideal of R is a 2-absorbing primary ideal of R. In particular, every proper ideal of a chained ring is a 2-absorbing primary ideal.*

Proof. Let I be a proper ideal of R. Since the prime ideals of a divided ring are linearly ordered, we conclude that $\sqrt{I}$ is a prime ideal of R. Hence I is a 2-absorbing primary ideal of R by Theorem 2.8. □

Let R be an integral domain with $1 \neq 0$, and let K be the quotient field of R. If I is a nonzero proper ideal of R, then $I^{-1} = \{x \in K \mid xI \in R\}$. An integral domain R is said to be a *Dedekind domain* if $II^{-1} = R$ for every nonzero proper ideal I of R.

Theorem 2.11. *Let R be a Noetherian integral domain with $1 \neq 0$ that is not a field. Then the following statements are equivalent.*

(1) *R is a Dedekind domain.*

(2) *A nonzero proper ideal I of R is a* 2*-absorbing primary ideal of R if and only if either $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$.*
(3) *If I is a nonzero proper* 2*-absorbing primary ideal of R, then either $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$.*
(4) *A nonzero proper ideal I of R is a* 2*-absorbing primary ideal of R if and only if either $I = P^n$ for some prime ideal P of R and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some prime ideals P_1, P_2 of R and some positive integers $n, m \geq 1$.*
(5) *If I is a nonzero proper* 2*-absorbing primary ideal of R, then either $I = P^n$ for some prime ideal P of R and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some prime ideals P_1, P_2 of R and some positive integers $n, m \geq 1$.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is a Dedekind domain that is not a field. Then every nonzero prime ideal of R is maximal. Let I be a nonzero proper ideal of R. Then $I = M_1^{n_1} M_2^{n_2} \cdots M_k^{n_k}$ for some distinct maximal ideals $M_1, \ldots, M_k$ of R and some positive integers $n_1, \ldots, n_k \geq 1$. Suppose that I is a 2-absorbing primary ideal of R. Since every nonzero prime ideal of R is maximal and $\sqrt{I}$ is either a maximal ideal of R or $I_1 \cap I_2$ for some maximal ideals I_1, I_2 of R by Theorem 2.3, we conclude that either $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$. Conversely, suppose that $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$. Then I is a 2-absorbing primary ideal of R by Theorem 2.8 and Corollary 2.5.

(2) $\Rightarrow$ (3). It is clear.
(2) $\Rightarrow$ (4). It is clear.
(4) $\Rightarrow$ (5). It is clear.
(3) $\Rightarrow$ (5). It is clear.
(5) $\Rightarrow$ (1). Let M be a maximal ideal of R. Since every ideal between M^2 and M is an M-primary ideal, and hence a 2-absorbing primary ideal of R, the hypothesis in (5) implies that there are no ideals properly between M^2 and M. Hence R is a Dedekind domain by [6, Theorem 39.2, p. 470]. □

Since every principal ideal domain is a Dedekind domain, we have the following result as a consequence of Theorem 2.11.

Corollary 2.12. *Let R be a principal ideal domain and I be a nonzero proper ideal of R. Then I is a* 2*-absorbing primary ideal of R if and only if either $I = p^k R$ for some prime element p of R and $k \geq 1$ or $I = p_1^n p_2^m R$ for some* *distinct prime elements p_1, p_2 of R and some positive integers $n, m \geq 1$. In particular, if $R = \mathbb{Z}$ or $R = F[X]$ for some field F, then a proper ideal I of R is a 2-absorbing primary ideal of R if and only if either $I = p^k R$ for some prime element p of R and some positive integer $k \geq 1$ or $I = p_1^n p_2^m R$ for some distinct prime elements p_1, p_2 of R and some positive integers $n, m \geq 1$.*

The following is an example of a unique factorization domain that contains a 2-absorbing primary ideal not of the form $P_1^n P_2^m$ for some prime ideals P_1, P_2 of R and some positive integers $n, m \geq 1$.

Example 2.13. Let $R = K[X, Y]$, where K is a field. Consider the ideal $I = (X, Y^2)$ of R. Then I is a 2-absorbing primary ideal of R that is not of the form $P_1^n P_2^m$, where P_1, P_2 are prime ideals of R and $n, m \geq 1$.

Let R be a commutative Noetherian ring with $1 \neq 0$. It is well-known that every proper ideal of R has a primary decomposition. Since every primary ideal is a 2-absorbing primary ideal, we conclude that every proper ideal of R has a 2-absorbing primary decomposition. However, decomposition of an ideal of R into 2-absorbing primary ideals need not be unique. We have the following example.

Example 2.14. In light of Corollary 2.12, consider the ideal (60) of $\mathbb{Z}$. Then

$$(60) = (3) \cap (4) \cap (5) = (3) \cap (20) = (4) \cap (15) = (5) \cap (12).$$

Hence (60) has four distinct 2-absorbing primary decompositions. The ideal (210) of $\mathbb{Z}$ has exactly ten distinct 2-absorbing primary decompositions.

$$\begin{aligned}(210) &= (2) \cap (3) \cap (5) \cap (7) = (6) \cap (5) \cap (7) = (10) \cap (3) \cap (7) \\ &= (14) \cap (3) \cap (5) = (15) \cap (2) \cap (7) = (15) \cap (14) = (21) \cap (2) \cap (5) \\ &= (21) \cap (10) = (35) \cap (2) \cap (3) = (35) \cap (6).\end{aligned}$$

Definition 2.15. Let I be a 2-absorbing primary ideal of R. Then $P = \sqrt{I}$ is a 2-absorbing ideal by Theorem 2.2. We say that I is a P-2-absorbing primary ideal of R.

Theorem 2.16. *Let $I_1, I_2, \ldots, I_n$ be P-2-absorbing primary ideals of R for some 2-absorbing ideal P of R. Then $I = \bigcap_{i=1}^n I_i$ is a P-2-absorbing primary ideal of R.*

Proof. First observe that $\sqrt{I} = \bigcap_{i=1}^n \sqrt{I_i} = P$. Suppose that $abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Then $ab \notin I_i$ for some $1 \leq i \leq n$. Hence $bc \in \sqrt{I_i} = P$ or $ac \in \sqrt{I_i} = P$. □

If I_1, I_2 are 2-absorbing primary ideals of a ring R, then $I_1 \cap I_2$ need not be a 2-absorbing primary ideal of R. We have the following example.

Example 2.17. Let $I_1 = 50\mathbb{Z}$ and $I_2 = 75\mathbb{Z}$. Then I_1, I_2 are 2-absorbing primary ideals of $\mathbb{Z}$ by Corollary 2.12. Since $\sqrt{I_1 \cap I_2} = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$, $I_1 \cap I_2$ is not a 2-absorbing primary ideal of $\mathbb{Z}$ by Theorem 2.3.

In the following result, we show that a proper ideal I of a ring R is a 2-absorbing primary ideal of R if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$. But first we have the following lemma.

Lemma 2.18. *Let I be a 2-absorbing primary ideal of a ring R and suppose that $abJ \subseteq I$ for some elements $a, b \in R$ and some ideal J of R. If $ab \notin I$, then $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.*

Proof. Suppose that $aJ \not\subseteq \sqrt{I}$ and $bJ \not\subseteq \sqrt{I}$. Then $aj_1 \notin \sqrt{I}$ and $bj_2 \notin \sqrt{I}$ for some $j_1, j_2 \in J$. Since $abj_1 \in I$ and $ab \notin I$ and $aj_1 \notin \sqrt{I}$, we have $bj_1 \in \sqrt{I}$. Since $abj_2 \in I$ and $ab \notin I$ and $bj_2 \notin \sqrt{I}$, we have $aj_2 \in \sqrt{I}$. Now, since $ab(j_1 + j_2) \in I$ and $ab \notin I$, we have $a(j_1 + j_2) \in \sqrt{I}$ or $b(j_1 + j_2) \in \sqrt{I}$. Suppose that $a(j_1 + j_2) = aj_1 + aj_2 \in \sqrt{I}$. Since $aj_2 \in \sqrt{I}$, we have $aj_1 \in \sqrt{I}$, a contradiction. Suppose that $b(j_1 + j_2) = bj_1 + bj_2 \in \sqrt{I}$. Since $bj_1 \in \sqrt{I}$, we have $bj_2 \in \sqrt{I}$, a contradiction again. Thus $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$. □

Theorem 2.19. *Let I be a proper ideal of R. Then I is a 2-absorbing primary ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.*

Proof. Suppose that whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$. Then clearly I is a 2-absorbing primary ideal of R by definition.

Conversely, suppose that I is a 2-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, such that $I_1I_2 \not\subseteq I$. We show that $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. Suppose that neither $I_1I_3 \subseteq \sqrt{I}$ nor $I_2I_3 \subseteq \sqrt{I}$. Then there are $q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$. Since $q_1q_2I_3 \subseteq I$ and neither $q_1I_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we have $q_1q_2 \in I$ by Lemma 2.18.

Since $I_1I_2 \not\subseteq I$, we have $ab \notin I$ for some $a \in I_1$, $b \in I_2$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq \sqrt{I}$ or $bI_3 \subseteq \sqrt{I}$ by Lemma 2.18. We consider three cases. **Case one**: Suppose that $aI_3 \subseteq \sqrt{I}$, but $bI_3 \not\subseteq \sqrt{I}$. Since $q_1bI_3 \subseteq I$ and neither $bI_3 \subseteq \sqrt{I}$ nor $q_1I_3 \subseteq \sqrt{I}$, we conclude that $q_1b \in I$ by Lemma 2.18. Since $(a + q_1)bI_3 \subseteq I$ and $aI_3 \subseteq \sqrt{I}$, but $q_1I_3 \not\subseteq \sqrt{I}$, we conclude that $(a + q_1)I_3 \not\subseteq \sqrt{I}$. Since neither $bI_3 \subseteq \sqrt{I}$ nor $(a + q_1)I_3 \subseteq \sqrt{I}$, we conclude that $(a + q_1)b \in I$ by Lemma 2.18. Since $(a + q_1)b = ab + q_1b \in I$ and $q_1b \in I$, we conclude that $ab \in I$, a contradiction. **Case two**: Suppose that $bI_3 \subseteq \sqrt{I}$, but $aI_3 \not\subseteq \sqrt{I}$. Since $aq_2I_3 \subseteq I$ and neither $aI_3 \subseteq \sqrt{I}$ nor $q_2I_3 \subseteq \sqrt{I}$, we conclude that $aq_2 \in I$. Since $a(b + q_2)I_3 \subseteq I$ and $bI_3 \subseteq \sqrt{I}$, but $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b + q_2)I_3 \not\subseteq \sqrt{I}$. Since neither $aI_3 \subseteq \sqrt{I}$ nor $(b + q_2)I_3 \subseteq \sqrt{I}$, we conclude that $a(b + q_2) \in I$ by Lemma 2.18. Since $a(b + q_2) = ab + aq_2 \in I$ and $aq_2 \in I$, we conclude that $ab \in I$, a contradiction. **Case three**: Suppose that $aI_3 \subseteq \sqrt{I}$ and $bI_3 \subseteq \sqrt{I}$. Since $bI_3 \subseteq \sqrt{I}$ and $q_2I_3 \not\subseteq \sqrt{I}$, we conclude that $(b + q_2)I_3 \not\subseteq \sqrt{I}$. Since $q_1(b + q_2)I_3 \subseteq I$ and neither $q_1I_3 \subseteq \sqrt{I}$ nor

$(b+q_2)I_3 \subseteq \sqrt{I}$, we conclude that $q_1(b+q_2) = q_1b + q_1q_2 \in I$ by Lemma 2.18. Since $q_1q_2 \in I$ and $q_1b+q_1q_2 \in I$, we conclude that $bq_1 \in I$. Since $aI_3 \subseteq \sqrt{I}$ and $q_1I_3 \nsubseteq \sqrt{I}$, we conclude that $(a+q_1)I_3 \nsubseteq \sqrt{I}$. Since $(a+q_1)q_2I_3 \subseteq I$ and neither $q_2I_3 \subseteq \sqrt{I}$ nor $(a+q_1)I_3 \subseteq \sqrt{I}$, we conclude that $(a+q_1)q_2 = aq_2+q_1q_2 \in I$ by Lemma 2.18. Since $q_1q_2 \in I$ and $aq_2+q_1q_2 \in I$, we conclude that $aq_2 \in I$. Now, since $(a+q_1)(b+q_2)I_3 \subseteq I$ and neither $(a+q_1)I_3 \subseteq \sqrt{I}$ nor $(b+q_2)I_3 \subseteq \sqrt{I}$, we conclude that $(a+q_1)(b+q_2) = ab+aq_2+bq_1+q_1q_2 \in I$ by Lemma 2.18. Since $aq_2, bq_1, q_1q_2 \in I$, we have $aq_2+bq_1+q_1q_2 \in I$. Since $ab+aq_2+bq_1+q_1q_2 \in I$ and $aq_2+bq_1+q_1q_2 \in I$, we conclude that $ab \in I$, a contradiction. Hence $I_1I_3 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$. □

Theorem 2.20. *Let $f : R \to R'$ be a homomorphism of commutative rings. Then the following statements hold.*

1. *If I' is a 2-absorbing primary ideal of R', then $f^{-1}(I')$ is a 2-absorbing primary ideal of R.*
2. *If f is an epimorphism and I is a 2-absorbing primary ideal of R containing $Ker(f)$, then $f(I)$ is a 2-absorbing primary ideal of R'.*

Proof. (1) Let $a, b, c \in R$ such that $abc \in f^{-1}(I')$. Then $f(abc) = f(a)f(b)f(c) \in I'$. Hence we have $f(a)f(b) \in I'$ or $f(b)f(c) \in \sqrt{I'}$ or $f(a)f(c) \in \sqrt{I'}$, and thus $ab \in f^{-1}(I')$ or $bc \in f^{-1}(\sqrt{I'})$ or $ac \in f^{-1}(\sqrt{I'})$. By using the equality $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$, we conclude that $f^{-1}(I')$ is a 2-absorbing primary ideal of R.

(2) Let $a', b', c' \in R'$ and $a'b'c' \in f(I)$. Then there exist $a, b, c \in R$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(abc) = a'b'c' \in f(I)$. Since $Ker\ f \subseteq I$, we have $abc \in I$. It implies that $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. This means that $a'b' \in f(I)$ or $a'c' \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $b'c' \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Thus $f(I)$ is a 2-absorbing primary ideal of R'. □

Corollary 2.21. *Let R be a commutative ring with $1 \neq 0$. Suppose that I, J are distinct proper ideals of R. If $J \subseteq I$ and I is a 2-absorbing primary ideal of R, then I/J is a 2-absorbing primary ideal of R/J.*

Proof. The proof is clear by Theorem 2.20(2). □

Theorem 2.22. *Let R be a commutative ring with $1 \neq 0$, S be a multiplicatively closed subset of R, and I be a proper ideal of R. Then the following statements hold.*

1. *If I is a 2-absorbing primary ideal of R such that $I \cap S = \varnothing$, then $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$.*
2. *If $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$ and $S \cap Z_I(R) = \varnothing$, then I is a 2-absorbing primary ideal of R.*

Proof. (1) Let $a, b, c \in R$, $s, t, k \in S$ such that $\frac{a}{s}\frac{b}{t}\frac{c}{k} \in S^{-1}I$. Then there exists $u \in S$ such that $uabc \in I$. Since I is a 2-absorbing primary ideal, we get

$uab \in I$ or $bc \in \sqrt{I}$ or $uac \in \sqrt{I}$. If $uab \in I$, then $\frac{a}{s}\frac{b}{t} = \frac{uab}{ust} \in S^{-1}I$. If $bc \in \sqrt{I}$, then $\frac{b}{t}\frac{c}{k} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$. If $uac \in \sqrt{I}$, then $\frac{a}{s}\frac{c}{k} = \frac{uac}{usk} \in \sqrt{S^{-1}I}$.

(2) Let $a, b, c \in R$ such that $abc \in I$. Then $\frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$. It follows $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ or $\frac{b}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$ or $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$. If $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in S^{-1}I$, then $uab \in I$, for some $u \in S$. Since $u \in S$ and $S \cap Z_I(R) = \varnothing$, we conclude $ab \in I$. If $\frac{b}{1}\frac{c}{1} = \frac{bc}{1} \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$, then there exists $v \in S$ and a positive integer n such that $(vbc)^n = v^n b^n c^n \in I$. Since $v \in S$, we have $v^n \notin Z_I(R)$. Thus $b^n c^n \in I$, and so $bc \in \sqrt{I}$. If $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$, then similarly we obtain $ac \in \sqrt{I}$, and it completes the proof. □

Theorem 2.23. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following statements are equivalent.*

(1) *J is a 2-absorbing primary ideal of R.*
(2) *Either $J = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $J = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2 or $J = I_1 \times I_2$ for some primary ideal I_1 of R_1 and some primary ideal I_2 of R_2.*

Proof. (1) ⇒ (2). Assume that J is a 2-absorbing primary ideal of R. Then $J = I_1 \times I_2$ for some ideal I_1 of R_1 and some ideal I_2 of R_2. Suppose that $I_2 = R_2$. Since J is a proper ideal of R, $I_1 \neq R_1$. Let $R' = \frac{R}{\{0\} \times R_2}$. Then $J' = \frac{J}{\{0\} \times R_2}$ is a 2-absorbing primary ideal of R' by Corollary 2.21. Since R' is ring-isomorphic to R_1 and $I_1 \cong J'$, I_1 is a 2-absorbing primary ideal of R_1. Suppose that $I_1 = R_1$. Since J is a proper ideal of R, $I_2 \neq R_2$. By a similar argument as in the previous case, I_2 is a 2-absorbing primary ideal of R_2. Hence assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Then $\sqrt{J} = \sqrt{I_1} \times \sqrt{I_2}$. Suppose that I_1 is not a primary ideal of R_1. Then there are $a, b \in R_1$ such that $ab \in I_1$ but neither $a \in I_1$ nor $b \in \sqrt{I_1}$. Let $x = (a, 1), y = (1, 0)$, and $c = (b, 1)$. Then $xyc = (ab, 0) \in J$ but neither $xy = (a, 0) \in J$ nor $xc = (ab, 1) \in \sqrt{J}$ nor $yc = (b, 0) \in \sqrt{J}$, which is a contradiction. Thus I_1 is a primary ideal of R_1. Suppose that I_2 is not a primary ideal of R_2. Then there are $d, e \in R_2$ such that $de \in I_2$ but neither $d \in I_2$ nor $e \in \sqrt{I_2}$. Let $x = (1, d), y = (0, 1)$, and $c = (1, e)$. Then $xyc = (0, de) \in J$ but neither $xy = (0, d) \in J$ nor $xc = (1, de) \in \sqrt{J}$ nor $yc = (0, e) \in \sqrt{J}$, which is a contradiction. Thus I_2 is a primary ideal of R_2.

(2) ⇒ (1). If $J = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $J = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2, then it is clear that J is a 2-absorbing primary ideal of R. Hence assume that $J = I_1 \times I_2$ for some primary ideal I_1 of R_1 and some primary ideal I_2 of R_2. Then $I_1' = I_1 \times R_2$ and $I_2' = R_1 \times I_2$ are primary ideals of R. Hence $I_1' \cap I_2' = I_1 \times I_2 = J$ is a 2-absorbing primary ideal of R by Theorem 2.4. □

Theorem 2.24. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 \leq n < \infty$, and $R_1, R_2, \ldots, R_n$ are commutative rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following statements are equivalent.*

(1) *J is a 2-absorbing primary ideal of R.*
(2) *Either $J = \times_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, I_k is a 2-absorbing primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \times_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, I_k is a primary ideal of R_k, I_m is a primary ideal of R_m, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.*

Proof. We use induction on n. Assume that $n = 2$. Then the result is valid by Theorem 2.23. Thus let $3 \leq n < \infty$ and assume that the result is valid when $K = R_1 \times \cdots \times R_{n-1}$. We prove the result when $R = K \times R_n$. By Theorem 2.23, J is a 2-absorbing primary ideal of R if and only if either $J = L \times R_n$ for some 2-absorbing primary ideal L of K or $J = K \times L_n$ for some 2-absorbing primary ideal L_n of R_n or $J = L \times L_n$ for some primary ideal L of K and some primary ideal L_n of R_n. Observe that a proper ideal Q of K is a primary ideal of K if and only if $Q = \times_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, I_k is a primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n-1\} \setminus \{k\}$. Thus the claim is now verified. □

Acknowledgement. We would like to thank the referee for his/her great effort in proofreading the manuscript.

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