KYUNGPOOK Math. J. 55(2015), 507-514
http://dx.doi.org/10.5666/KMJ.2015.55.3.507
pISSN 1225-6951 eISSN 0454-8124

A Note on S-Noetherian Domains

Jung Wook Lim
Department of Mathematics, Kyungpook National University, Daegu, 702-701, Republic of Korea
e-mail : jwlim@knu.ac.kr

Abstract. Let D be an integral domain, t be the so-called t-operation on D, and S be a (not necessarily saturated) multiplicative subset of D. In this paper, we study the Nagata ring of S-Noetherian domains and locally S-Noetherian domains. We also investigate the t-Nagata ring of t-locally S-Noetherian domains. In fact, we show that if S is an anti-archimedean subset of D, then D is an S-Noetherian domain (respectively, locally S-Noetherian domain) if and only if the Nagata ring $D[X]_N$ is an S-Noetherian domain (respectively, locally S-Noetherian domain). We also prove that if S is an anti-archimedean subset of D, then D is a t-locally S-Noetherian domain if and only if the polynomial ring $D[X]$ is a t-locally S-Noetherian domain, if and only if the t-Nagata ring $D[X]_{N_v}$ is a t-locally S-Noetherian domain.

1. Introduction

1.1 Star-operations

To help readers better understanding this paper, we briefly review some definitions and notation related to star-operations. Let D be an integral domain with quotient field K, and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For an $I \in \mathbf{F}(D)$, set $I^{-1} := \{x \in K \mid xI \subseteq D\}$. The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$ is called the *v-operation* on D, and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t := \bigcup\{J_v \mid J$ is a nonzero finitely generated fractional subideal of $I\}$ is called the *t-operation* on D; and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w := \{a \in K \mid Ja \subseteq I$ for some finitely generated ideal J of D with $J_v = D\}$ is called the *w-operation* on D. It is easy to see that $I \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$; and if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. An $I \in \mathbf{F}(D)$ is called a *t-ideal* (respectively, *w-ideal*) of D if $I_t = I$ (respectively, $I_w = I$). A

Received September 10, 2014; accepted November 7, 2014.
2010 Mathematics Subject Classification: 13A15, 13B25, 13E99, 13G05.
Key words and phrases: S-Noetherian domain, (t-)locally S-Noetherian domain, (t-) Nagata ring, finite (t-)character.

maximal t-ideal means a t-ideal which is maximal among proper integral t-ideals. It is well known that a maximal t-ideal of D always exists if D is not a field. We say that D is *of finite character* (respectively, *of finite t-character*) if each nonzero nonunit in D belongs to only finitely many maximal ideals (respectively, maximal t-ideals) of D.

1.2 S-Noetherian domains

Let D be an integral domain and S a (not necessarily saturated) multiplicative subset of D. In [4], the authors introduced the concept of "almost finitely generated" to study Querre's characterization of divisorial ideals in integrally closed polynomial rings. Later, the authors in [2] generalized the concept of (almost) finitely generatedness and defined a general notion of Noetherian domains. (Recall that D is a *Noetherian domain* if it satisfies the ascending chain condition on integral ideals of D, or equivalently, every (prime) ideal of D is finitely generated.) To do this, they first built the notion of S-finiteness. Let I be an ideal of D. Then I is said to be *S-finite* if there exist an element $s \in S$ and a finitely generated ideal J of D such that $sI \subseteq J \subseteq I$. Also, D is called an *S-Noetherian domain* if each ideal of D is S-finite. As mentioned above, the concept of S-Noetherian domains can be regarded as a slight generalization of that of Noetherian domains, because two notions precisely coincide when S consists of units. Hence the results on S-Noetherian domains can recover known facts for Noetherian domains.

Among other results in [2], Anderson and Dumitrescu proved the Hilbert basis theorem for S-Noetherian domains, which states that if S is an anti-archimedean subset of an S-Noetherian domain D, then the polynomial ring $D[X]$ is also an S-Noetherian domain [2, Proposition 9]. (Recall that a multiplicative subset S of D is *anti-archimedean* if $\bigcap_{n\geq 1} s^n D \cap S \neq \emptyset$ for all $s \in S$. For example, if V is a valuation domain with no height-one prime ideals, then $V \setminus \{0\}$ is an anti-archimedean subset of V [3, Proposition 2.1].) After the paper by Anderson and Dumitrescu, more properties of S-Noetherian domains have been studied further. In [14], Liu found an equivalent condition for the generalized power series ring to be an S-Noetherian domain. In [12], the authors studied the S-Noetherian properties in special pullbacks which are the so-called composite ring extensions $D + E[\Gamma^*]$ and $D + [\![E^{\Gamma^*,\leq}]\!]$. As a continuation of [12], the same authors investigated when the amalgamated algebra along an ideal has the S-Noetherian property [13]. For more results, the readers can refer to [2, 12, 13, 14].

Let $\mathbf{P}$ denote one of the properties "Noetherian" or "S-Noetherian". We say that D is *locally* $\boldsymbol{P}$ (respectively, *t-locally* $\boldsymbol{P}$) if D_M is $\mathbf{P}$ for all maximal ideals (respectively, maximal t-ideals) M of D.

The purpose of this paper is to study the Nagata ring of S-Noetherian domains and locally S-Noetherian domains, and to investigate the t-Nagata ring of t-locally S-Noetherian domains. (The concepts of Nagata rings and t-Nagata rings will be reviewed in Section .) More precisely, we show that if S is an anti-archimedean subset of D, then D is an S-Noetherian domain (respectively, locally S-Noetherian domain) if and only if the Nagata ring $D[X]_N$ is an S-Noetherian domain (respectively, locally S-Noetherian domain); a locally S-Noetherian domain with finite character is an S-Noetherian domain; and if S is an anti-archimedean subset of D, then D is a t-locally S-Noetherian domain if and only if the polynomial ring $D[X]$ is a t-locally S-Noetherian domain, if and only if the t-Nagata ring $D[X]_{N_v}$ is a t-locally S-Noetherian domain.

2. Main Results

We start this section with a simple result for a quotient ring of S-Noetherian domains. This also recovers the fact that any quotient ring of a Noetherian domain is Noetherian [5, Proposition 7.3].

Lemma 1. *Let D be an integral domain and S a (not necessarily saturated) multiplicative subset of D. If D is an S-Noetherian domain and T is a (not necessarily saturated) multiplicative subset of D, then D_T is an S-Noetherian domain.*

Proof. Let A be an ideal of D_T. Then $A = ID_T$ for some ideal I of D. Since D is an S-Noetherian domain, there exist an element $s \in S$ and a finitely generated ideal J of D such that $sI \subseteq J \subseteq I$. Therefore we obtain

$$sA = sID_T \subseteq JD_T \subseteq ID_T = A,$$

and hence A is S-finite. Thus D_T is an S-Noetherian domain. □

The next result is an S-Noetherian version of well-known facts that a Noetherian domain is locally Noetherian; and a locally Noetherian domain with finite character is Noetherian [5, Section 7, Exercise 9].

Theorem 2. *The following statements hold.*

(1) *An S-Noetherian domain is locally S-Noetherian.*

(2) *A locally S-Noetherian domain with finite character is S-Noetherian.*

Proof. (1) This is an immediate consequence of Lemma 1.

(2) Assume that D is a locally S-Noetherian domain which is of finite character, and let I be an ideal of D. If $I \cap S \neq \emptyset$, then for any $s \in I \cap S$, $sI \subseteq (s) \subseteq I$; so I is S-finite. Next, we consider the case when I does not intersect S. Choose any $0 \neq a \in I$. Since D has finite character, a belongs to only a finite number of maximal ideals of D, say $M_1, \ldots, M_n$. Fix an $i \in \{1, \ldots, n\}$. Since D_{M_i} is S-Noetherian, there exist an element $s_i \in S$ and a finitely generated subideal F_i of I such that $s_iID_{M_i} \subseteq F_iD_{M_i}$. By letting $s = s_1 \cdots s_n$ and setting $C = (a) + F_1 + \cdots + F_n$, we obtain that $sID_{M_i} \subseteq CD_{M_i}$. Let M' be a maximal ideal of D which is distinct from $M_1, \ldots, M_n$. Then a is a unit in $D_{M'}$; so $ID_{M'} = D_{M'} = CD_{M'}$. Therefore

$sID_M \subseteq CD_M$ for all maximal ideals M of D. Hence we have

$$\begin{aligned} sI &= \bigcap_{M \in \mathrm{Max}(D)} sID_M \\ &\subseteq \bigcap_{M \in \mathrm{Max}(D)} CD_M \\ &= C, \end{aligned}$$

where $\mathrm{Max}(D)$ denotes the set of maximal ideals of D and the equalities follow from [9, Proposition 2.8(3)]. Note that C is a finitely generated subideal of I. Therefore I is S-finite, and thus D is an S-Noetherian domain. □

Recall that an integral domain D is an *almost Dedekind domain* if D_M is a Noetherian valuation domain for all maximal ideals M of D.

Remark 3. The converse of Theorem 2(1) does not generally hold. (This also indicates that the condition being finite character in Theorem 2(2) is essential.) For example, if D is an almost Dedekind domain which is not Noetherian, then D is a locally S-Noetherian domain which is not S-Noetherian. (This is the case when S consists of units in D.) For a concrete illustration, see [8, Example 42.6].

Let D be an integral domain and $D[X]$ be the polynomial ring over D. For an $f \in D[X]$, $c(f)$ denotes the content ideal of f, *i.e.*, the ideal of D generated by the coefficients of f, and for an ideal I of $D[X]$, $c(I)$ stands for the ideal of D generated by the coefficients of polynomials in I, *i.e.*, $c(I) = \sum_{f \in I} c(f)$. Let $N = \{f \in D[X] \mid c(f) = D\}$. Then N is a saturated multiplicative subset of $D[X]$ and the quotient ring $D[X]_N$ is called the *Nagata ring* of D. It was shown that D is a Noetherian domain if and only if $D[X]$ is a Noetherian domain [5, Theorem 7.5 (Hilbert basis theorem)] (or [10, Theorem 69]), if and only if $D[X]_N$ is a Noetherian domain (cf. [1, Theorem 2.2(2)]). We now give the S-Noetherian analogue of these equivalences.

Theorem 4. *Let D be an integral domain, S an anti-archimedean subset of D, and $N := \{f \in D[X] \mid c(f) = D\}$. Then the following statements are equivalent.*

(1) *D is an S-Noetherian domain.*

(2) *$D[X]$ is an S-Noetherian domain.*

(3) *$D[X]_N$ is an S-Noetherian domain.*

Proof. (1) ⇒ (2) This implication appears in [2, Proposition 9].

(2) ⇒ (3) This was shown in Lemma 1.

(3) ⇒ (1) Let I be an ideal of D. Then $ID[X]_N$ is an ideal of $D[X]_N$. Since $D[X]_N$ is an S-Noetherian domain, we can find an element $s \in S$ and a finitely generated subideal J of $ID[X]$ such that $sID[X]_N \subseteq JD[X]_N$; so $sID[X]_N \subseteq c(J)D[X]_N$. Let $a \in I$. Then $sag \in c(J)D[X]$ for some $g \in N$; so $sa \in c(J)$. Hence $sI \subseteq c(J)$. Note that $c(J)$ is a finitely generated subideal of I. Therefore I is S-finite, and thus D is an S-Noetherian domain. □

Let D be an integral domain and let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Then N_v is a saturated multiplicative subset of $D[X]$ and the quotient ring $D[X]_{N_v}$ is called the *t-Nagata ring* of D. It was shown that D is t-locally Noetherian if and only if $D[X]$ is t-locally Noetherian, if and only if $D[X]_{N_v}$ is t-locally Noetherian [6, Theorem 1.4]. To investigate the (t-)Nagata ring of (t-)locally S-Noetherian domains, we need the following lemma.

Lemma 5. *Let D be a quasi-local domain with unique maximal ideal M, S a (not necessarily saturated) multiplicative subset of D, and I an ideal of D. Then I is S-finite if and only if $ID[X]_{MD[X]}$ is S-finite.*

Proof. If I is S-finite, then there exist an element $s \in S$ and a finitely generated subideal J of I such that $sI \subseteq J$; so we obtain

$$sID[X]_{MD[X]} \subseteq JD[X]_{MD[X]} \subseteq ID[X]_{MD[X]}.$$

Thus $ID[X]_{MD[X]}$ is S-finite. Conversely, if $ID[X]_{MD[X]}$ is S-finite, then there exist suitable elements $s \in S$ and $f_1, \ldots, f_n \in ID[X]$ such that $sID[X]_{MD[X]} \subseteq (f_1, \ldots, f_n)D[X]_{MD[X]}$; so we obtain

$$sID[X]_{MD[X]} \subseteq (c(f_1) + \cdots + c(f_n))D[X]_{MD[X]}.$$

Note that $JD[X]_{MD[X]} \cap D = J$ for all ideals J of D, because D is quasi-local. Hence we obtain

$$\begin{aligned} sI &= sID[X]_{MD[X]} \cap D \\ &\subseteq (c(f_1) + \cdots + c(f_n))D[X]_{MD[X]} \cap D \\ &= c(f_1) + \cdots + c(f_n). \end{aligned}$$

Note that $c(f_1)+\cdots+c(f_n)$ is a finitely generated subideal of I. Thus I is S-finite.□

We are ready to study the polynomial extension and the t-Nagata ring of t-locally S-Noetherian domains.

Theorem 6. *Let D be an integral domain, S an anti-archimedean subset of D, and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following statements are equivalent.*

(1) *D is a t-locally S-Noetherian domain.*

(2) *$D[X]$ is a t-locally S-Noetherian domain.*

(3) *$D[X]_{N_v}$ is a locally S-Noetherian domain.*

(4) *$D[X]_{N_v}$ is a t-locally S-Noetherian domain.*

Proof. (1) ⇒ (2) Let M be a maximal t-ideal of $D[X]$ and let K be the quotient field of D. If $M \cap D = (0)$, then $D[X]_M$ is a quotient ring of $K[X]$; so $D[X]_M$ is a principal ideal domain. Hence $D[X]_M$ is an S-Noetherian domain. Next, we assume that $M \cap D \neq (0)$, and let $P = M \cap D$. Then $M = PD[X]$ and P is a maximal t-ideal of D [7, Proposition 2.2]. Since D is t-locally S-Noetherian, D_P is S-Noetherian. Also, since S is an anti-archimedean subset of D_P, $D_P[X]$ is S-Noetherian [2, Proposition 9]; so by Lemma 1, $D_P[X]_{PD_P[X]}$ is S-Noetherian. Note that $D[X]_M = D_P[X]_{PD_P[X]}$; so $D[X]_M$ is an S-Noetherian domain. From both cases, we conclude that $D[X]$ is a t-locally S-Noetherian domain.

(2) ⇒ (3) Let Q be a maximal ideal of $D[X]_{N_v}$. Then $Q = MD[X]_{N_v}$ for some maximal t-ideal M of D [9, Proposition 2.1(2)]. Note that $(D[X]_{N_v})_Q = (D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]}$ and $MD[X]$ is a maximal t-ideal of $D[X]$ [7, Proposition 2.2]. Since $D[X]$ is t-locally S-Noetherian, $D[X]_{MD[X]}$, and hence $(D[X]_{N_v})_Q$ is S-Noetherian. Thus $D[X]_{N_v}$ is a locally S-Noetherian domain.

(3) ⇒ (1) Let M be a maximal t-ideal of D. Then $MD[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [9, Proposition 2.1(2)]. Note that $(D[X]_{N_v})_{MD[X]_{N_v}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $D_M[X]_{MD_M[X]}$ is S-Noetherian, because $D[X]_{N_v}$ is locally S-Noetherian. Let I be an ideal of D_M. Then $ID_M[X]_{MD_M[X]}$ is S-finite. Since D_M is quasi-local, Lemma 5 forces I to be S-finite. Hence D_M is S-Noetherian, and thus D is a t-locally S-Noetherian domain.

(3) ⇔ (4) This equivalence follows directly from the fact that the set of maximal t-ideals of $D[X]_{N_v}$ is precisely the same as that of maximal ideals of $D[X]_{N_v}$ (cf. [9, Propositions 2.1(2) and 2.2(3)]). □

We next study locally S-Noetherian domains in terms of the Nagata ring.

Theorem 7. *Let D be an integral domain, S an anti-archimedean subset of D, and $N := \{f \in D[X] \mid c(f) = D\}$. Then the following statements are equivalent.*

(1) *D is a locally S-Noetherian domain.*

(2) *$D[X]_N$ is a locally S-Noetherian domain.*

Proof. (1) ⇒ (2) Let Q be a maximal ideal of $D[X]_N$. Then $Q = MD[X]_N$ for some maximal ideal M of D [9, Proposition 2.1(2)]. Since D is locally S-Noetherian, D_M is S-Noetherian. Also, since S is an anti-archimedean subset of D_M, $D_M[X]$ is S-Noetherian [2, Proposition 9]. Hence by Lemma 1, $D_M[X]_{MD_M[X]}$ is an S-Noetherian domain. Note that $(D[X]_N)_Q = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $(D[X]_N)_Q$ is S-Noetherian. Thus $D[X]_N$ is a locally S-Noetherian domain.

(2) ⇒ (1) Let M be a maximal ideal of D. Then $MD[X]_N$ is a maximal ideal of $D[X]_N$ [9, Proposition 2.1(2)]. Since $D[X]_N$ is locally S-Noetherian, $(D[X]_N)_{MD[X]_N}$ is S-Noetherian. Note that $(D[X]_N)_{MD[X]_N} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$; so $D_M[X]_{MD_M[X]}$ is S-Noetherian. Let I be an ideal of D_M. Then $ID_M[X]_{MD_M[X]}$ is S-finite. Since D_M is quasi-local, I is S-finite by Lemma 5. Hence D_M is S-Noetherian, and thus D is a locally S-Noetherian domain. □

We are closing this article by comparing our results with recent researches related to S-Noetherian domains. In [11], the authors defined an integral domain D to be an *S-strong Mori domain* (S-SM-domain) if for each nonzero ideal I of D, there exist an element $s \in S$ and a finitely generated ideal J of D such that

$sI \subseteq J_w \subseteq I_w$. This concept generalizes the notions of both S-Noetherian domains and strong Mori domains. (Recall from [15, Definition 4] that D is a *strong Mori domain* (SM-domain) if it satisfies the ascending chain condition on integral w-ideals of D, or equivalently, for each (prime) w-ideal I of D, $I = J_w$ for some finitely generated ideal J of D [15, Theorem 4.3].) It was shown that if D is a t-locally S-Noetherian domain with finite t-character, then D is an S-SM-domain [11, Proposition 2.1(2)]; and that if S is an anti-archimedean subset of D, then D is an S-SM-domain if and only if $D[X]_{N_v}$ is an S-SM-domain [11, Theorem 2.10].

Lemma 8. *Let D be an integral domain, $N := \{f \in D[X] \mid c(f) = D\}$, and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following assertions hold.*

(1) *D is of finite character if and only if $D[X]_N$ is of finite character.*

(2) *D is of finite t-character if and only if $D[X]_{N_v}$ is of finite character.*

Proof. The equivalence is an immediate consequence of the fact that $\{MD[X]_N \mid M$ is a maximal ideal of $D\}$ (respectively, $\{MD[X]_{N_v} \mid M$ is a maximal t-ideal of $D\}$) is the set of maximal ideals of $D[X]_N$ (respectively, $D[X]_{N_v}$) [9, Proposition 2.1(2)].□

By Theorems 6 and 7 and Lemma 8, we obtain

Corollary 9. *Let D be an integral domain, S an anti-archimedean subset of D, $N := \{f \in D[X] \mid c(f) = D\}$, and $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then the following assertions hold.*

(1) *D is a locally S-Noetherian domain with finite character if and only if $D[X]_N$ is a locally S-Noetherian domain with finite character.*

(2) *D is a t-locally S-Noetherian domain with finite t-character if and only if $D[X]_{N_v}$ is a locally S-Noetherian domain with finite character.*

Acknowledgments. The author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2014R1A1A1002478).

References

[1] D. D. Anderson, D. F. Anderson, and R. Markanda, *The ring $R(X)$ and $R\langle X\rangle$*, J. Algebra, **95**(1985), 96-115.

[2] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra, **30**(2002), 4407-4416.

[3] D. D. Anderson, B. G. Kang, and M. H. Park, *Anti-archimedean rings and power series rings*, Comm. Algebra, **26**(1998), 3223-3238.

[4] D. D. Anderson, D. J. Kwak, and M. Zafrullah, *Agreeable domains*, Comm. Algebra, **23**(1995), 4861-4883.

[5] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1969.

[6] G. W. Chang, *Strong Mori domains and the ring* $D[\mathbf{X}]_{N_v}$, J. Pure Appl. Algebra, **197**(2005), 293-304.

[7] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra, **26**(1998), 1017-1039.

[8] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure Appl. Math., **90**, Queen's University, Kingston, Ontario, 1992.

[9] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra, **123**(1989), 151-170.

[10] I. Kaplansky, *Commutative Rings*, Polygonal Publishing House, Washington, New Jersey, 1994.

[11] H. Kim, M. O. Kim, and J. W. Lim, *On S-strong Mori domains*, J. Algebra, **416**(2014), 314-332.

[12] J. W. Lim and D. Y. Oh, *S-Noetherian properties of composite ring extensions*, Comm. Algebra, **43**(2015), 2820-2829.

[13] J. W. Lim and D. Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra, **218**(2014), 1075-1080.

[14] Z. Liu, *On S-Noetherian rings*, Arch. Math. (Brno), **43**(2007), 55-60.

[15] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra, **25**(1997), 1285-1306.