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ASYMPTOTIC BEHAVIORS OF FUNDAMENTAL SOLUTION AND ITS DERIVATIVES TO FRACTIONAL DIFFUSION-WAVE EQUATIONS

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Abstract. Let $p(t,x)$ be the fundamental solution to the problem

$$\partial_t^\alpha u = -(-\Delta)^\beta u, \quad \alpha \in (0,2),\, \beta \in (0,\infty).$$

If $\alpha, \beta \in (0,1)$, then the kernel $p(t,x)$ becomes the transition density of a Lévy process delayed by an inverse subordinator. In this paper we provide the asymptotic behaviors and sharp upper bounds of $p(t,x)$ and its space and time fractional derivatives

$$D_x^n(-\Delta_x)^\gamma D_t^\sigma I_t^\delta p(t,x), \quad \forall\, n \in \mathbb{Z}_+,\ \gamma \in [0,\beta],\ \sigma,\delta \in [0,\infty),$$

where D_x^n is a partial derivative of order n with respect to x, $(-\Delta_x)^\gamma$ is a fractional Laplace operator and D_t^σ and I_t^δ are Riemann-Liouville fractional derivative and integral respectively.

1. Introduction

Let $\alpha \in (0,2)$, $\beta \in (0,\infty)$ and $p(t,x)$ be the fundamental solution to the space-time fractional differential equation

$$\partial_t^\alpha u = \Delta^\beta u, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{1.1}$$

with $u(0,x) = u_0(x)$ (and $\partial_t u(0,x) = 0$ if $\alpha > 1$). Here ∂_t^α denotes Caputo fractional derivative and $\Delta^\beta := -(-\Delta)^\beta$ is the fractional Laplacian. The aim of this paper is to present rigorous and self-contained exposition of fundamental solution $p(t,x)$. More precisely, we provide asymptotic behaviors and sharp upper bounds of

$$D_x^n(-\Delta_x)^\gamma D_t^\sigma I_t^\delta p(t,x), \quad \forall\, n \in \mathbb{Z}_+,\ \gamma \in [0,\beta],\ \sigma,\delta \in [0,\infty), \tag{1.2}$$

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where I_t^δ and D_t^σ denotes the Riemann-Liouville fractional integral and derivative respectively.

Equation (1.1) has been an important topic in the mathematical physics related to non-Markovian diffusion processes with a memory [26, 27, 28, 29], in the probability theory related to jump processes [5, 6] and in the theory of differential equations [7, 8, 17, 32, 37]. If $\alpha \in (0,1)$, then the fractional time derivative of order α can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena and the fractional spatial derivative describes long range jumps of particles. In particular, if $\beta \in (0,1)$, then $p(t,x)$ is the transition density of the process

$$Y_t := X_{L_t},$$

where X_t is an $\mathbb{R}^d$-valued process with the characteristic function $\mathbb{E}\exp\{i\xi \cdot X_t\} = \exp\{-t|\xi|^{2\beta}\}$ and $L_t := \inf\{\tau > 0 : S_\tau > t\}$, where S_t is an increasing Lévy process independent of X_t having the Laplace transform $\mathbb{E}\exp\{-\lambda S_t\} = \exp\{-t\lambda^\alpha\}$. See [23] and references therein for detail. Also, if $\alpha = 1$, then $p(t,x)$ is the transition density function of the process X_t. If $\alpha \in (1,2)$, the fractional wave equation (1.1) governs the propagation of mechanical diffusive waves in viscoelastic media (see [24, 34]).

There are certainly considerable results dealing with explicit formula for the fundamental solutions and their asymptotic behaviors (see e.g. [3, 9, 10, 11, 12, 13, 20, 22]). However, only few of them cover the derivative estimates of the fundamental solutions. In [9, 10, 17, 20], upper bounds of (1.2) were obtained for $\beta = 1$, $\sigma = 1-\alpha$, and $\gamma = 0$. Also, in [13] asymptotic behavior for the case $\alpha = 1, \beta \in (0,1)$ and $\sigma = 0$ was obtained. Note that it is assumed that either $\alpha = 1$ or $\beta = 1$ in [9, 10, 13, 17, 20], and moreover spatial fractional derivative $\Delta^\gamma p$ and time fractional derivative $D_t^\sigma p$ are not obtained in [9, 10, 17, 20] and [13] respectively. Our result substantially improves these results because we only assume $\alpha \in (0,2)$ and $\beta \in (0,\infty)$ and we provide two sides estimates of both space and time fractional derivatives of arbitrary order.

Our approach relies also on the properties of some special functions including the Fox H functions. However, unlike most of previous works, we do not use the method of Mellin, Laplace and inverse Fourier transforms which, due to the non-exponential decay at infinity of the Fox H functions, require restrictions on the space dimension d and other parameters β, δ, γ, and σ (see Remark 5.2).

Below we give two main applications of our results. First, consider the non-homogeneous fractional evolution equation

$$\partial_t^\alpha u = \Delta^\beta u + f, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{1.3}$$

with $u(0,x) = 0$ and additionally $\partial_t u(0,x) = 0$ if $\alpha > 1$. One can show (see e.g. [9, 17, 20]) that the solution to the problem is given by

$$\mathcal{G}f(t,x) = \int_0^t \int_{\mathbb{R}^d} Q_\alpha(t-s, x-y) f(s,y)\, dy ds$$

where

$$Q_\alpha(t,x) := \begin{cases} D_t^{1-\alpha}p(t,x) & : \alpha \in (0,1) \\ I_t^{\alpha-1}p(t,x) & : \alpha \in (1,2). \end{cases}$$

It turns out (see [18, 19]) that to obtain the $L_q(L_p)$-estimate

$$\|\Delta^\beta \mathcal{G} f\|_{L_q((0,T),L_p(\mathbb{R}^d))} \leq N\|f\|_{L_q((0,T),L_p(\mathbb{R}^d))}, \quad p,q>1 \tag{1.4}$$

it is sufficient to show that for $a,b>0$

$$\int_0^a \int_{|x|\geq b} |\Delta^\beta Q_\alpha(t,x)| dxdt \leq N\frac{a^\alpha}{b^{2\beta}},$$

$$\int_a^\infty \int_{\mathbb{R}^d} \left|\partial_t \Delta^\beta Q_\alpha(t,x)\right| dxdy \leq \frac{N}{a}, \quad \int_a^\infty \int_{\mathbb{R}^d} \left|\nabla_x \Delta^\beta Q_\alpha(t,x)\right| dxdt \leq Na^{-\frac{\alpha}{2\beta}}.$$

One can use our estimates to prove the above three inequalities, and therefore (1.4) can be obtained as a corollary. Our second application is the L_p-theory of the stochastic partial differential equations of the type

$$\partial_t^\alpha u = \Delta u + \partial_t^{\alpha+\sigma}\int_0^t g(s,x)dW_s,$$

where $\sigma < \frac{1}{2}$ and W_t is a Wiener process defined on a probability space (Ω, dP). One can show (see [5]) that the solution to this problem is given by the formula

$$u(t,x) = \int_0^t \left(\int_{\mathbb{R}^d} P_\sigma(t-s,x-y)g(s,y)dy\right)dW_s.$$

Here $P_\sigma(t,x)$ is defined as

$$P_\sigma(t,x) := \begin{cases} I_t^{|\sigma|}p(t,x) & : \sigma \leq 0 \\ D_t^{|\sigma|}p(t,x) & : \sigma > 0. \end{cases}$$

As has been shown for the case $\alpha = 1$ (see [15, 16, 21, 36]), sharp estimates of $D_x^n\Delta^\gamma P_\sigma$ can be used to obtain L_p-estimate

$$\|\Delta^\gamma u\|_{L_p(\Omega\times(0,T)\times\mathbb{R}^d)} \leq N\|g\|_{L_p(\Omega\times(0,T)\times\mathbb{R}^d)}. \tag{1.5}$$

The detail of (1.5) will be given for $\gamma \leq (2 \wedge \frac{1-2\sigma}{\alpha})$ in a subsequent paper.

The rest of the article is organized as follows. In Section 2 we state our main results, Theorems 2.1, 2.3, and 2.4. In Section 3 we present the definition of the Fox H functions and their several properties. For the convenience of the reader, we repeat the relevant material and demonstration in [14] and [17], thus making our exposition self-contained. Section 4 contains asymptotic behaviors at zero and infinity of the Fox H function. In Section 5 we present explicit representation of fundamental solutions and their fractional and classical derivatives. Finally, in Section 6 we prove our main results.

We finish the introduction with some notion used in this article. We write $f \lesssim g$ for $|x| \leq \delta$ (resp. $|x| \geq \delta$) if there exists a positive constant C independent of x such that $f(x) \leq Cg(x)$ for $|x| \leq \delta$ (resp. $|x| \geq \delta$), and $f \sim g$ for $|x| \leq \delta$ (resp. $|x| \geq \delta$) if $f \lesssim g \lesssim f$ for $|x| \leq \delta$ (resp. $|x| \geq \delta$). We say $f \sim g$ as $|x| \to 0$

(resp. $|x| \to \infty$) if there exists $\varepsilon \in (0,1)$ such that $f \sim g$ for $|x| \le \varepsilon$ (resp. $|x| \ge \varepsilon^{-1}$). We write $f(x) = O(g(|x|))$ as $|x| \to 0$ (resp. $|x| \to \infty$) if there exists $\delta > 0$ such that $|f(x)| \lesssim |g(|x|)|$ for $|x| < \delta$ (resp. $|x| \ge \delta$). We use ":=" to denote a definition. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$, and $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. For multi-indices $\mathfrak{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_+$, $n \in \mathbb{N}$ and functions $u(x)$ we set

$$D_i u = \frac{\partial u}{\partial x^i}, \quad D^{\mathfrak{a}}_x u = D^{a_1}_1 \cdots D^{a_d}_d u, \quad D^n_x := \{D^{\mathfrak{a}}_x : |\mathfrak{a}| = n\}.$$

$\lfloor a \rfloor$ is the biggest integer which is less than or equal to a. By $\mathcal{F}$ we denote the d-dimensional Fourier transform, that is,

$$\mathcal{F}\{f\}(\xi) := \int_{\mathbb{R}^d} e^{-\mathrm{i}(x,\xi)} f(x) dx.$$

For a complex number z, $\Re[z]$ and $\Im[z]$ are the real part and imaginary part of z respectively.

2. Main results

We first introduce some definitions related to the fractional calculus. Let $\beta \ge 0$. For a function $u \in L_1(\mathbb{R}^d)$, we write $\Delta^\beta u = f$ if there exists a function $f \in L_1(\mathbb{R}^d)$ such that

$$\mathcal{F}\left\{f(\cdot)\right\}(\xi) = |\xi|^{2\beta} \mathcal{F}\left\{u\right\}(\xi).$$

For $u \in L_1((0,T))$, the Riemann-Liouville fractional integral of the order $\alpha \in (0,\infty)$ is defined as

$$I^\alpha_t u := \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \le T.$$

One can easily check

$$I^\alpha_t I^\beta_t = I^{\alpha+\beta}_t, \quad \forall\, \alpha, \beta \ge 0. \tag{2.1}$$

Let $n \in \mathbb{N}$ and $n-1 \le \alpha < n$. The Riemann-Liouville fractional derivative D^α_t and the Caputo fractional derivative ∂^α_t are defined as

$$D^\alpha_t u := \left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}_t u\right), \tag{2.2}$$

$$\partial^\alpha_t u := D^{\alpha-(n-1)}_t \left(u^{(n-1)}(t) - u^{(n-1)}(0)\right).$$

By (2.2) for any $\alpha \ge 0$ and $u \in L_1((0,T))$, we have

$$D^\alpha_t I^\alpha_t u = u. \tag{2.3}$$

Using (2.1)-(2.3), one can check

$$\partial^\alpha_t u = D^\alpha_t \left(u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0)\right), \qquad n-1 \le \alpha < n.$$

Thus $D_t^\alpha u = \partial_t^\alpha u$ if $u(0) = u^{(1)}(0) = \cdots = u^{(n-1)}(0) = 0$. For more information on the fractional derivatives, we refer the reader to [30, 33].

For $\sigma \in \mathbb{R}$ we define Riemann-Liouville fractional operator $\mathbb{D}_t^\sigma$ as

$$\mathbb{D}_t^\sigma := \begin{cases} D_t^{|\sigma|} & : \sigma > 0 \\ I_t^{|\sigma|} & : \sigma < 0. \end{cases}$$

Then by (2.1) and (2.2), for any $\alpha, \beta \geq 0$

$$D_t^\alpha I_t^\beta = \mathbb{D}_t^{\alpha-\beta}.$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\ \Re[\alpha] > 0 \tag{2.4}$$

for $z \in \mathbb{C}$ and we write $E_\alpha(z) = E_{\alpha,1}(z)$ for short. Using the equality

$$\mathbb{D}_t^\sigma t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\sigma)} t^{\alpha-\sigma}, \quad \forall \sigma \in \mathbb{R},\ \alpha \geq 0$$

one can check that for $t > 0$

$$\mathbb{D}_t^\sigma E_\alpha(-\lambda t^\alpha) = t^{-\sigma} E_{\alpha,1-\sigma}(-\lambda t^\alpha), \quad \sigma \in \mathbb{R},\ \lambda \geq 0, \tag{2.5}$$

and that for any constant λ,

$$\varphi(t) := E_\alpha(-\lambda t^\alpha)$$

satisfies $\varphi(0) = 1$ (also $\varphi'(0) = 0$ if $\alpha > 1$) and

$$\partial_t^\alpha \varphi = -\lambda \varphi, \quad t > 0.$$

Let $\alpha \in (0,2)$ and $\beta \in (0,\infty)$. By taking the Fourier transform to the equation

$$\partial_t^\alpha u = \Delta^\beta u, \quad t > 0, \quad u(0,x) = u_0(x), \quad (\text{and } u'(0,x) = 0 \text{ if } \alpha > 1)$$

one can formally get

$$\mathcal{F}\{u(t,\cdot)\} = E_\alpha(-|\xi|^{2\beta} t^\alpha) \mathcal{F}\{u_0\}.$$

Therefore, to obtain the fundamental solution, it is needed to find an integrable function $p(t,x) \in L_1(\mathbb{R}^d)$ satisfying

$$\mathcal{F}\{p(t,\cdot)\} = E_\alpha(-|\xi|^{2\beta} t^\alpha). \tag{2.6}$$

Let

$$\mathbf{M}(t,x) := |x|^{2\beta} t^{-\alpha}.$$

In the following theorems we give the asymptotic behaviors of $D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)$ as $\mathbf{M} \to 0$ and $\mathbf{M} \to \infty$. We also provide upper bounds when $\mathbf{M} \leq 1$ and $\mathbf{M} \geq 1$.

Firstly, we consider the case $\mathbf{M} \to \infty$.

Theorem 2.1. *Let $\alpha \in (0,2)$, $\beta \in (0,\infty)$, $\gamma \in [0,\infty)$, $\sigma \in \mathbb{R}$ and $n \in \mathbb{N}$. There exists a function $p(t,\cdot) \in L_1(\mathbb{R}^d)$ satisfying* (2.6)*. Furthermore, the following asymptotic behaviors hold as $\mathbf{M} \to \infty$:*

(i) *If* $\beta \in \mathbb{N}$*, then for some constant* $c > 0$ *depending only on* $d, n, \alpha, \beta, \sigma$

$$|\mathbb{D}_t^\sigma p(t,x)| \lesssim |x|^{-d} t^{-\sigma} \exp\left\{-c|x|^{\frac{2\beta}{2\beta-\alpha}} t^{-\frac{\alpha}{2\beta-\alpha}}\right\} \tag{2.7}$$

and

$$|D_x^n \mathbb{D}_t^\sigma p(t,x)| \lesssim |x|^{-d-n} t^{-\sigma} \exp\left\{-c|x|^{\frac{2\beta}{2\beta-\alpha}} t^{-\frac{\alpha}{2\beta-\alpha}}\right\}. \tag{2.8}$$

(ii) *If* $\alpha = 1$*,* $\beta \notin \mathbb{N}$*, and* $\sigma = 0$*,*

$$|\Delta^\gamma p(t,x)| \sim \begin{cases} t|x|^{-d-2\gamma-2\beta} & : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-2\gamma} & : \gamma \in [0,\infty) \setminus \mathbb{Z}_+ \end{cases} \tag{2.9}$$

and

$$|D_x^n \Delta^\gamma p(t,x)| \lesssim \begin{cases} t|x|^{-d-2\gamma-2\beta-n} & : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-2\gamma-n} & : \gamma \in [0,\infty) \setminus \mathbb{Z}_+. \end{cases} \tag{2.10}$$

(iii) *If* $\gamma \in (0,\beta) \setminus \mathbb{N}$*,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \sim |x|^{-d-2\gamma} t^{-\sigma} \tag{2.11}$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \lesssim |x|^{-d-2\gamma-n} t^{-\sigma}. \tag{2.12}$$

(iv) *If* $\beta \notin \mathbb{N}$ *and* $\gamma \in [0,\beta) \cap \mathbb{Z}_+$*,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \sim |x|^{-d-2\gamma-2\beta} t^{-\sigma+\alpha} \tag{2.13}$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \lesssim |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha}. \tag{2.14}$$

(v) *If* $\gamma = \beta \notin \mathbb{N}$ *and* $d \geq 2$*,*

$$\left|\Delta^\beta \mathbb{D}_t^\sigma p(t,x)\right| \sim \begin{cases} |x|^{-d-4\beta} t^{-\sigma+\alpha} & : \sigma \in \mathbb{N} \\ |x|^{-d-2\beta} t^{-\sigma} & : \textit{otherwise} \end{cases} \tag{2.15}$$

and

$$\left|D_x^n \Delta^\beta \mathbb{D}_t^\sigma p(t,x)\right| \lesssim \begin{cases} |x|^{-d-4\beta-n} t^{-\sigma+\alpha} & : \sigma \in \mathbb{N} \\ |x|^{-d-2\beta-n} t^{-\sigma} & : \textit{otherwise.} \end{cases} \tag{2.16}$$

(vi) *If* $\gamma = \beta \notin \mathbb{N}$*,* $\sigma + \alpha \in \mathbb{N}$*, and* $d = 1$*, then* (2.15) *and* (2.16) *hold.*

Remark 2.2. (i) Note that $D_x^n \mathbb{D}_t^\sigma p(t,x)$ has exponential decay as $\mathbf{M} \to \infty$ only when β is a positive integer.

(ii) Let $x \neq 0$. Then by (2.8) and (2.13) with $\gamma = 0$, $D_x^n \mathbb{D}_t^\sigma p(t,x) \to 0$ as $t \to 0$ if either β is a positive integer or $\sigma < \alpha$.

(iii) If $\gamma = \beta$ and $d = 1$, then we additionally assumed $\sigma + \alpha \in \mathbb{N}$. Without this extra condition we had a trouble in using Fubini's theorem in our proof.

(iv) Note that we have only upper bounds of $D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)$ unless $n = 0$. This is because, for instance, $D_i p(t,x)$ is of type $x^i g(t,x)$ (see (5.1)) and becomes zero if $x^i = 0$. Hence we can not have positive lower bound of $D_i p(t,x)$ for such x.

Secondly, we consider the case $\mathbf{M} \to 0$.

Theorem 2.3. *Let $\alpha, \beta, \gamma, \sigma$ be given as in Theorem 2.1 and $n \in \mathbb{N}$. Then the following asymptotic behaviors hold as $\mathbf{M} \to 0$:*

(i) *If $\gamma \in [0, \beta)$ and $\sigma + \alpha \notin \mathbb{N}$,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < \beta - \frac{d}{2} \\ t^{-\sigma-\alpha}\left(1 + |\ln|x|^{2\beta} t^{-\alpha}|\right) & : \gamma = \beta - \frac{d}{2} \\ |x|^{-d-2\gamma+2\beta} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} \end{cases} \tag{2.17}$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-\alpha}(1 + |\ln|x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+2\beta-n} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} - 1. \end{cases} \tag{2.18}$$

(ii) *If $\gamma \in [0, \beta)$ and $\sigma + \alpha \in \mathbb{N}$,*

$$|\Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ t^{-\sigma-2\alpha}\left(1 + \left|\ln|x|^{2\beta} t^{-\alpha}\right|\right) & : \gamma = 2\beta - \frac{d}{2} \\ |x|^{-d-2\gamma+4\beta} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} \end{cases} \tag{2.19}$$

and

$$|D_x^n \Delta^\gamma \mathbb{D}_t^\sigma p(t,x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{2-n} t^{-\sigma-2\alpha}(1 + \left|\ln|x|^{2\beta} t^{-\alpha}\right|) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+4\beta-n} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} - 1. \end{cases} \tag{2.20}$$

(iii) *If $\alpha = 1$ and $\sigma = 0$,*

$$|\Delta^\gamma p(t,x)| \sim t^{-\frac{d+2\gamma}{2\beta}} \tag{2.21}$$

and

$$|D_x^n \Delta^\gamma p(t,x)| \lesssim |x|^{2-n} t^{-\frac{d+2\gamma+2}{2\beta}}. \tag{2.22}$$

(iv) *If $\gamma = \beta$ and $d \geq 2$,*

$$\left|\Delta^\beta \mathbb{D}_t^\sigma p(t,x)\right| \sim \begin{cases} t^{-\sigma-\alpha-\frac{\alpha d}{2\beta}} & : \frac{d}{2} < \beta \\ t^{-\sigma-2\alpha}\left(1 + \left|\ln|x|^{2\beta} t^{-\alpha}\right|\right) & : \frac{d}{2} = \beta \\ |x|^{-d+2\beta} t^{-\sigma-2\alpha} & : \frac{d}{2} > \beta \end{cases} \tag{2.23}$$

and

$$\left|D_x^n \Delta^{\beta} \mathbb{D}_t^{\sigma} p(t,x)\right| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\alpha-\frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2}+1<\beta \\ |x|^{2-n} t^{-\sigma-2\alpha}(1+\left|\ln |x|^{2\beta} t^{-\alpha}\right|) & : \frac{d}{2}+1=\beta \\ |x|^{-d+2\beta-n} t^{-\sigma-2\alpha} & : \frac{d}{2}+1>\beta. \end{cases} \tag{2.24}$$

(v) *If $\gamma=\beta$, $\sigma+\alpha\in\mathbb{N}$, and $d=1$, then* (2.23) *and* (2.24) *hold.*

Next we give the upper estimates when $\mathbf{M}\geq 1$, i.e., $|x|^{2\beta}\geq t^{\alpha}$.

Theorem 2.4. *Under the same assumption of Theorem 2.1, the following hold for $\mathbf{M}\geq 1$:*

(i) *If $\beta\in\mathbb{N}$, then* (2.7) *and* (2.8) *hold.*
(ii) *If $\alpha=1$, $\beta\notin\mathbb{N}$, and $\sigma=0$, then* (2.10) *holds and additionally* (2.9) *follows if "$\sim$" is replaced by "$\lesssim$".*
(iii) *If $\gamma\in(0,\beta)\setminus\mathbb{N}$, then* (2.12) *holds and additionally* (2.11) *follows if "$\sim$" is replaced by "$\lesssim$".*
(iv) *If $\beta\notin\mathbb{N}$ and $\gamma\in[0,\beta)\cap\mathbb{Z}_+$, then* (2.14) *holds and additionally* (2.13) *follows if "$\sim$" is replaced by "$\lesssim$".*
(v) *If $\gamma=\beta\notin\mathbb{N}$ and $d\geq 2$, then* (2.16) *holds and additionally* (2.15) *follows if "$\sim$" is replaced by "$\lesssim$".*
(vi) *If $\gamma=\beta\notin\mathbb{N}$, $\sigma+\alpha\in\mathbb{N}$, and $d=1$, then* (2.16) *holds and additionally* (2.15) *follows if "$\sim$" is replaced by "$\lesssim$".*

Finally we provide the upper estimates when $\mathbf{M}\leq 1$, i.e., $|x|^{2\beta}\leq t^{\alpha}$.

Theorem 2.5. *Under the same assumption of Theorem 2.3, the following hold for $\mathbf{M}\leq 1$:*

(i) *If $\gamma\in[0,\beta)$ and $\sigma+\alpha\notin\mathbb{N}$, then* (2.18) *holds and additionally* (2.17) *follows if "$\sim$" is replaced by "$\lesssim$".*
(ii) *If $\gamma\in[0,\beta)$ and $\sigma+\alpha\in\mathbb{N}$, then* (2.20) *holds and additionally* (2.19) *follows if "$\sim$" is replaced by "$\lesssim$".*
(iii) *If $\alpha=1$ and $\sigma=0$, then* (2.21) *holds and additionally* (2.22) *follows if "$\sim$" is replaced by "$\lesssim$".*
(iv) *If $\gamma=\beta$ and $d\geq 2$, then* (2.24) *holds and additionally* (2.23) *follows if "$\sim$" is replaced by "$\lesssim$".*
(v) *If $\gamma=\beta$, $\sigma+\alpha\in\mathbb{N}$, and $d=1$, then* (2.24) *holds and additionally* (2.23) *follows if "$\sim$" is replaced by "$\lesssim$".*

The proofs of Theorems 2.1, 2.3, 2.4, and 2.5 are given in Section 6.

Remark 2.6. Let $\alpha=1$ and $\beta\in(0,\infty)$, and $\sigma=0$. Then by Theorem 2.4,

$$|D_x^n \Delta^{\gamma} p(t,x)| \lesssim \begin{cases} t|x|^{-d-2\beta-2\gamma-n} & : \gamma\in\mathbb{Z}_+ \\ |x|^{-d-2\gamma-n} & : \gamma\notin\mathbb{Z}_+ \end{cases}$$

holds for $|x|^2\geq t$. Also, by Theorem 2.4, for $|x|^2\leq t$,

$$|\Delta^{\gamma} p(t,x)| \lesssim t^{-\frac{d+2\gamma}{2\beta}}, \quad |D_x^n \Delta^{\gamma} p(t,x)| \lesssim |x|^{2-n} t^{-\frac{d+2\gamma+2}{2\beta}}.$$

These estimates cover the results of [15, Lemma 3.1, 3.3] and [13, Corollary 1].

Remark 2.7. Theorem 2.4 also implies the results of [9, Proposition 5.1, 5.2]. Let $\beta = 1$ and take a $|n| \geq 1$. For $|x|^2 \geq t^\alpha$.

$$\begin{aligned}|D_x^n \mathbb{D}_t^\sigma p(t,x)| &\lesssim |x|^{-d-n} t^{-\sigma} \exp\left\{-ct^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\right\} \\ &\lesssim t^{-\frac{\alpha(d+n)}{2}-\sigma} \exp\left\{-ct^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}}\right\}.\end{aligned}$$

Also, (cf. (2.17), (2.18), and (2.20)),

$$|p(t,x)| \lesssim \begin{cases} t^{-\frac{\alpha d}{2}} & : d \geq 3 \\ t^{-\alpha}(1 + |\ln |x|^2 t^{-\alpha}|) & : d = 2 \\ |x|^{-d+2} t^{-\alpha} & : d = 1 \end{cases}$$

and

$$|D_x^n p(t,x)| \lesssim |x|^{-d+2-n} t^{-\alpha}, \quad |D_x^n \Delta p(t,x)| \lesssim |x|^{-d+2-n} t^{-2\alpha},$$

$$\left|D_x^n \mathbb{D}_t^{1-\alpha} p(t,x)\right| \lesssim \begin{cases} |x|^{-d+4-n} t^{-\alpha-1} & : d \geq 3 \\ |x|^{2-n} t^{-1}(1 + |\ln |x|^2 t^{-\alpha}|) & : d = 2 \\ |x|^{1-n} t^{-1} & : d = 1 \end{cases}$$

hold for $|x|^2 \leq t^\alpha$.

3. Preliminaries to the Fox H function

In this section, we introduce the definition and some properties of the Fox H function. We refer to [14] for further information of the Fox H function.

3.1. Definition

Let $\Gamma(z)$ denote the gamma function which can be defined (see [2, Section 1.1]) for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

Note that $\Gamma(z)$ is a meromorphic function with simple poles at the nonpositive integers. From the definition, for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$z\Gamma(z) = \Gamma(z+1), \tag{3.1}$$

and it holds that

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz). \tag{3.2}$$

One can easily check that for $k \in \mathbb{Z}_+$,

$$\mathrm{Res}_{z=-k}[\Gamma(z)] = \lim_{z\to -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!},$$

where $\mathrm{Res}_{z=z_0}[f(z)]$ denotes the residue of $f(z)$ at $z = z_0$. From the Stirling's approximation

$$\Gamma(z) \sim \sqrt{2\pi} e^{(z-\frac{1}{2})\log z} e^{-z}, \quad |z| \to \infty,$$

it follows that

$$|\Gamma(a + \mathrm{i}b)| \sim \sqrt{2\pi} |a|^{a-\frac{1}{2}} e^{-a-\pi[1-\mathrm{sign}(a)]b/2}, \quad |a| \to \infty \tag{3.3}$$

and

$$|\Gamma(a + \mathrm{i}b)| \sim \sqrt{2\pi} |b|^{a-\frac{1}{2}} e^{-a-\frac{\pi|b|}{2}}, \quad |b| \to \infty. \tag{3.4}$$

Let m, n, ν, μ be fixed integers satisfying $0 \leq m \leq \mu$, $0 \leq n \leq \nu$. Assume that real parameters $\mathfrak{c}_1, \ldots, \mathfrak{c}_\nu$, $\mathfrak{d}_1, \ldots, \mathfrak{d}_\mu$ and positive real parameters $\gamma_1, \ldots, \gamma_\nu$, $\delta_1, \ldots, \delta_\mu$ are given such that

$$\max_{1 \leq j \leq m} \left(-\frac{\mathfrak{d}_j}{\delta_j} \right) < \min_{1 \leq j \leq n} \left(\frac{1 - \mathfrak{c}_j}{\gamma_j} \right). \tag{3.5}$$

For each $k \in \mathbb{Z}_+$, we set

$$\mathfrak{c}_{j,k} = \frac{1 - \mathfrak{c}_j + k}{\gamma_j}, \quad \mathfrak{d}_{j,k} = -\frac{\mathfrak{d}_j + k}{\delta_j}, \tag{3.6}$$

which constitute

$$P_1 = \{\mathfrak{d}_{j,k} \in \mathbb{R} : j \in \{1, \ldots, m\}, k \in \mathbb{Z}_+\},$$
$$P_2 = \{\mathfrak{c}_{j,k} \in \mathbb{R} : j \in \{1, \ldots, n\}, k \in \mathbb{Z}_+\}.$$

Note that $P_1 \cap P_2 = \emptyset$ by (3.5). We arrange the elements of P_1 and P_2 as follows:

$$P_1 = \left\{\hat{\mathfrak{d}}_0 > \hat{\mathfrak{d}}_1 > \hat{\mathfrak{d}}_2 > \cdots\right\}, \quad P_2 = \{\hat{\mathfrak{c}}_0 < \hat{\mathfrak{c}}_1 < \hat{\mathfrak{c}}_2 < \cdots\}. \tag{3.7}$$

For the above parameters, define

$$\mathcal{H}(z) := \frac{\prod_{j=1}^{m} \Gamma(\mathfrak{d}_j + \delta_j z) \prod_{j=1}^{n} \Gamma(1 - \mathfrak{c}_j - \gamma_j z)}{\prod_{j=n+1}^{\nu} \Gamma(\mathfrak{c}_j + \gamma_j z) \prod_{j=m+1}^{\mu} \Gamma(1 - \mathfrak{d}_j - \delta_j z)}. \tag{3.8}$$

Note that P_1 and P_2 are sets of poles of $\mathcal{H}(z)$. To describe the behavior of $\mathcal{H}(z)$ as $|z| \to \infty$, we set

$$\alpha^* := \sum_{i=1}^{n} \gamma_i - \sum_{i=n+1}^{\nu} \gamma_i + \sum_{j=1}^{m} \delta_j - \sum_{j=m+1}^{\mu} \delta_j, \quad \omega := \sum_{j=1}^{\mu} \delta_j - \sum_{j=1}^{\nu} \gamma_i \tag{3.9}$$

and

$$\Lambda := \sum_{j=1}^{\mu} \mathfrak{d}_j - \sum_{j=1}^{\nu} \mathfrak{c}_j + \frac{\nu - \mu}{2}, \quad \eta := \prod_{j=1}^{\nu} \gamma_j^{-\gamma_j} \prod_{j=1}^{\mu} \delta_j^{\delta_j}. \tag{3.10}$$

Due to (3.3),

$$|\mathcal{H}(a + \mathrm{i}b) r^{-a-\mathrm{i}b}| \sim \left(\frac{e}{a}\right)^{-\omega a} \left(\frac{\eta}{r}\right)^{a} a^{\Lambda}, \quad r \in (0, \infty) \tag{3.11}$$

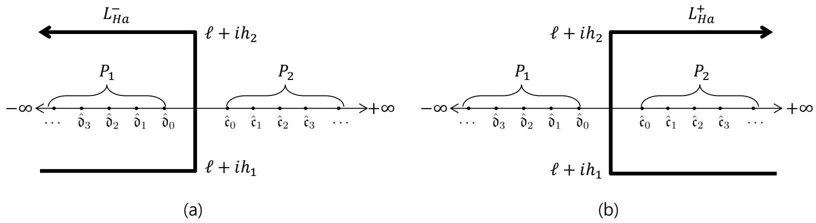

FIGURE 3.1. (a): Left Hankel contour, (b): Right Hankel contour

as $a \to \infty$ and

$$|\mathcal{H}(a+\mathrm{i}b)r^{-a-\mathrm{i}b}| \sim \left(\frac{e}{|a|}\right)^{\omega|a|}\left(\frac{r}{\eta}\right)^{-|a|}|a|^{\Lambda}, \quad r \in (0,\infty) \tag{3.12}$$

as $a \to -\infty$. By (3.4), it follows that

$$|\mathcal{H}(a+\mathrm{i}b)r^{-a-\mathrm{i}b}| \sim \left(\frac{e}{|b|}\right)^{-\omega a}\left(\frac{\eta}{r}\right)^{a}|b|^{\Lambda}e^{-\alpha^*|b|\pi/2}, \quad r \in (0,\infty) \tag{3.13}$$

as $|b| \to \infty$.

The Fox H function $\mathrm{H}^{mn}_{\nu\mu}(r)$ $(r>0)$ is defined via Mellin-Barnes type integral in the form

$$\begin{aligned}\mathrm{H}^{mn}_{\nu\mu}(r) &:= \mathrm{H}^{mn}_{\nu\mu}\left[r \;\middle|\; \begin{matrix}[\mathfrak{c},\gamma]\\ [\mathfrak{d},\delta]\end{matrix}\right]\\ &:= \mathrm{H}^{mn}_{\nu\mu}\left[r \;\middle|\; \begin{matrix}(\mathfrak{c}_1,\gamma_1) & \cdots & (\mathfrak{c}_\nu,\gamma_\nu)\\ (\mathfrak{d}_1,\delta_1) & \cdots & (\mathfrak{d}_\mu,\delta_\mu)\end{matrix}\right] := \frac{1}{2\pi\mathrm{i}}\int_L \mathcal{H}(z)r^{-z}dz.\end{aligned} \tag{3.14}$$

In (3.14), L is the infinite contour which separates all the poles in P_1 to the left and all the poles in P_2 to the right of L. Precisely, we choose L as follows:

(i) if $\omega > 0$, then $L = L^-_{Ha}$, which is a left loop situated in a horizontal strip (left Hankel contour, see Figure 3.1(a)), runs from $-\infty+\mathrm{i}h_1$ to $\ell+\mathrm{i}h_1$, and then to $\ell+\mathrm{i}h_2$ and finally terminates at the point $-\infty+\mathrm{i}h_2$ with $-\infty < h_1 < 0 < h_2 < \infty$ and

$$\hat{\mathfrak{d}}_0 < \ell < \hat{\mathfrak{c}}_0, \tag{3.15}$$

(ii) if $\omega < 0$, then $L = L^+_{Ha}$, which is a right loop situated in a horizontal strip (right Hankel contour, see Figure 3.1(b)), runs from $+\infty+\mathrm{i}h_1$ to $\ell+\mathrm{i}h_1$, and then to $\ell+\mathrm{i}h_2$ and finally terminates at the point $+\infty+\mathrm{i}h_2$ with $-\infty < h_1 < 0 < h_2 < \infty$ and (3.15),

(iii) if $\omega = 0$, then $L = L^-_{Ha}$ and $L = L^+_{Ha}$ and for $r \in (0,\eta)$ and $r \in (\eta,\infty)$ respectively.

The following proposition shows that integral (3.14) is well defined and independent of the choice of h_1, h_2, and $\ell \in (\hat{\mathfrak{d}}_0, \hat{\mathfrak{c}}_0)$. One can find more general one in [14, Theorems 1.2] which is considered as a corollary of residue theory. We provide its proof to make our paper self-contained and analytic.

Proposition 3.1. *Assume* (3.5) *and choose the contour* L *as above. Then Mellin-Barnes integral* (3.14) *makes sense and it is an analytic function of* $r \in (0,\infty)$ *and of* $r \in (0,\eta)\cup(\eta,\infty)$ *if* $\omega \neq 0$ *and* $\omega = 0$ *respectively. Furthermore,*

(i) *if* $\omega > 0$*, then*

$$\mathrm{H}^{mn}_{\nu\mu}(r) = \sum_{k=0}^{\infty} \mathrm{Res}_{z=\hat{\mathfrak{d}}_k} \left[\mathcal{H}(z) r^{-z}\right]. \tag{3.16}$$

(ii) *if* $\omega < 0$*, then*

$$\mathrm{H}^{mn}_{\nu\mu}(r) = -\sum_{k=0}^{\infty} \mathrm{Res}_{z=\hat{\mathfrak{c}}_k} \left[\mathcal{H}(z) r^{-z}\right]. \tag{3.17}$$

(iii) *if* $\omega = 0$*, then* (3.16) *and* (3.17) *hold for* $r \in (0,\eta)$ *and* $r \in (\eta,\infty)$ *respectively.*

Proof. (i) Let $\omega > 0$ and $r \in (0,\infty)$. Choose $h_1 < 0$, $h_2 > 0$, and $\ell \in \mathbb{R}$ so that (3.15) holds. Take a sufficiently large $p \in \mathbb{Z}_+$ so that $\hat{\mathfrak{d}}_p < 0$. Then there exists a real number $M = M(p) > 0$ such that

$$-\hat{\mathfrak{d}}_p < M < -\hat{\mathfrak{d}}_{p+1}. \tag{3.18}$$

Define a closed rectangular contour C^M which can be decomposed into four lines

$$C_M = L_1 \cup L_2 \cup L_3 \cup L_4,$$

where

$$\begin{aligned}
L_1 &:= \{z \in \mathbb{C} : \Re[z] = \ell, h_1 \leq \Im[z] \leq h_2\},\\
L_2 &:= \{z \in \mathbb{C} : \Re[z] = -M, h_1 \leq \Im[z] \leq h_2\},\\
L_3 &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_1\},\\
L_4 &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_2\}.
\end{aligned}$$

Note that $\mathcal{H}(z)r^{-z}$ is a meromorphic function on $z \in \mathbb{C} \setminus P_1 \cup P_2$ and

$$\int_{C^M} \left|\mathcal{H}(z) r^{-z}\right| |dz| < \infty. \tag{3.19}$$

Due to (3.12) ($i = 1, 2$)

$$\begin{aligned}
\left|\int_{-\infty}^{-M} \mathcal{H}(t + \mathrm{i}h_i) r^{-t-\mathrm{i}h_i} dt\right| &\leq \int_{-\infty}^{-M} \left|\mathcal{H}(t + \mathrm{i}h_i) r^{-t-\mathrm{i}h_i}\right| dt\\
&\lesssim \int_{M}^{\infty} \left(\frac{e}{t}\right)^{\omega t} \left(\frac{r}{\eta}\right)^{-t} t^{\Lambda} dt < \infty,
\end{aligned} \tag{3.20}$$

and

$$\left|\int_{L_2} \mathcal{H}(z) r^{-z} dz\right| = \left|\int_{h_1}^{h_2} \mathcal{H}\left(-M+\mathrm{i}t\right) r^{M-\mathrm{i}t} dt\right|$$

$$\leq \int_{h_1}^{h_2} \left|\mathcal{H}(-M+\mathrm{i}t) r^{M-\mathrm{i}t}\right| dt$$

$$\lesssim \left(\frac{e}{M}\right)^{\omega M} \left(\frac{r}{\eta}\right)^{M} M^{\Lambda}(h_2-h_1) < \infty. \tag{3.21}$$

By (3.19)-(3.21), integral (3.14) absolutely converges and makes sense with $L = L_{Ha}^{-}$.

We next show (3.16). Note that (3.20) and (3.21) converge to 0 as $M \to \infty$ and by the theory of residues,

$$\frac{1}{2\pi\mathrm{i}} \int_{C_M} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^{p} \mathrm{Res}_{z=\mathfrak{d}_k} \left[\mathcal{H}(z) r^{-z}\right].$$

Thus our claim follows immediately if we prove the convergence of the residue expansion. Let $q \in \mathbb{N}$ $(q > p)$ be arbitrarily given. Then we can choose a real number $N = N(q) > 0$ satisfying (3.18) where p and M are replaced by q and N respectively. Set

$$C'_M := L'_1 \cup L_2 \cup L'_3 \cup L'_4,$$

where

$$L'_1 := \{z \in \mathbb{C} : \Re[z] = -N, h_1 \leq \Im[z] \leq h_2\},$$
$$L'_3 := \{z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_1\},$$
$$L'_4 := \{z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_2\}.$$

Observe that

$$\frac{1}{2\pi\mathrm{i}} \int_{C'_M} \mathcal{H}(z) r^{-z} dz = \sum_{k=p+1}^{q} \mathrm{Res}_{z=\mathfrak{d}_k} \left[\mathcal{H}(z) r^{-z}\right]. \tag{3.22}$$

By replacing M by N in (3.20) and (3.21),

$$\lim_{N\to\infty} \left|\int_{L'_1} \mathcal{H}(z) r^{-z} dz\right| = \lim_{N,M\to\infty} \left|\int_{L'_3 \cup L'_4} \mathcal{H}(z) r^{-z} dz\right| = 0,$$

which implies

$$\lim_{p,q\to\infty} \sum_{k=p+1}^{q} \mathrm{Res}_{z=\mathfrak{d}_k} \left[\mathcal{H}(z) r^{-z}\right] = \lim_{N,M\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{C'_M} \mathcal{H}(z) r^{-z} dz = 0.$$

Thus (3.16) is proved.

(ii) The case $\omega < 0$ is an analogue of the case $\omega > 0$. By (3.11), for sufficiently large $M > 0$ $(i = 1, 2)$

$$\begin{aligned} \left| \int_M^\infty \mathcal{H}(t + \mathrm{i}h_i) r^{-t-\mathrm{i}h_i} dt \right| &\le \int_M^\infty \left| \mathcal{H}(t + \mathrm{i}h_i) r^{-t-\mathrm{i}h_i} \right| dt \\ &\lesssim \int_M^\infty \left(\frac{e}{t}\right)^{-\omega t} \left(\frac{\eta}{r}\right)^t t^\Lambda dt < \infty, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \left| \int_{L_2} \mathcal{H}(z) r^{-z} dz \right| &= \left| \int_{h_1}^{h_2} \mathcal{H}\left(M + \mathrm{i}t\right) r^{M-\mathrm{i}t} dt \right| \\ &\le \int_{h_1}^{h_2} \left| \mathcal{H}(M + \mathrm{i}t) r^{M-\mathrm{i}t} \right| dt \\ &\lesssim \left(\frac{e}{M}\right)^{-\omega M} \left(\frac{\eta}{r}\right)^M M^\Lambda (h_2 - h_1) < \infty. \end{aligned} \tag{3.24}$$

Note that both (3.23) and (3.24) converge to 0 as $M \to \infty$. Then we obtain our desired result by replacing $\hat{\mathfrak{d}}_k$ and M in the proof of the case $\omega > 0$ by $\hat{\mathfrak{c}}_k$ and $-M$ respectively.

(iii) Finally we consider $\omega = 0$. Note that (3.20) and (3.21) hold if $r < \eta$, consequently (3.16) follows. If $r > \eta$, then we have (3.23) and (3.24) which give (3.17) immediately. The proposition is proved. □

In the remainder of this section, we assume

$$\alpha^* > 0. \tag{3.25}$$

Under (3.25), we can choose a contour $L = L_{Br}$ in (3.14) due to (3.13) where L_{Br} is a vertical contour (or Bromwich contour) starting at the point $\ell - \mathrm{i}\infty$ and terminating at the point $\ell + \mathrm{i}\infty$ where ℓ satisfies (3.15). As a matter of fact, $\mathrm{H}^{mn}_{\nu\mu}(r)$ does not depend on the choice between Hankel contour and Bromwich contour due to the following proposition and it is an analytic function of $r \in (0, \infty)$ (it is a holomorphic function of $r \in \mathbb{C}$ in the sector $|\arg r| < \frac{\omega\pi}{2}$. See [14, Theorem 1.2.(iii)]).

Proposition 3.2. *Under* (3.25)*, Mellin-Barnes integral* (3.14) *makes sense with* $L = L_{Br}$*. Furthermore, for* $r \in (0, \infty)$*, if* $\omega > 0$*, then*

$$\frac{1}{2\pi\mathrm{i}} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \frac{1}{2\pi\mathrm{i}} \int_{L^-_{Ha}} \mathcal{H}(z) r^{-z} dz, \tag{3.26}$$

and if $\omega < 0$*, then*

$$\frac{1}{2\pi\mathrm{i}} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \frac{1}{2\pi\mathrm{i}} \int_{L^+_{Ha}} \mathcal{H}(z) r^{-z} dz. \tag{3.27}$$

If $\omega = 0$*, then* (3.26) *and* (3.27) *hold for* $r < \eta$ *and* $r > \eta$ *respectively.*

Proof. We only prove the case $\omega \geq 0$. The proof of the other case is almost same. Fix $r \in (0, \infty)$ and

$$L = L_{Br} = \{z \in \mathbb{C} : \Re[z] = \ell\},$$

where ℓ satisfies (3.15). Note that the relation

$$\left|\mathcal{H}(\ell + \mathrm{i}t) r^{-\ell-\mathrm{i}t}\right| \sim e^{-\omega}\eta^{\ell} r^{-\ell} |t|^{\omega\ell+\Lambda} \exp\left\{-\frac{|t|\alpha^*}{2}\pi\right\}$$

holds as $|t| \to \infty$ uniformly in ℓ by (3.13). Hence the contour integral

$$\frac{1}{2\pi\mathrm{i}} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz$$

absolutely converges and makes sense. Due to Proposition 3.1, we only need to show that

$$\frac{1}{2\pi\mathrm{i}} \int_{L_{Br}} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^{\infty} \operatorname{Res}_{z=\mathfrak{d}_k} \left[\mathcal{H}(z) r^{-z}\right]. \tag{3.28}$$

Given (sufficiently large) $p \in \mathbb{Z}_+$, we take M as in the proof of Proposition 3.1. Define a closed rectangular contour C^M which can be decomposed into four lines

$$C^M = L_v^M \cup L_v^{-M} \cup L_h^{-M} \cup L_h^M,$$

where

$$\begin{aligned}
L_v^M &:= \{z \in \mathbb{C} : \Re[z] = \ell, |\Im[z]| \leq M\}, \\
L_v^{-M} &:= \{z \in \mathbb{C} : \Re[z] = -M, |\Im[z]| \leq M\}, \\
L_h^M &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = M\}, \\
L_h^{-M} &:= \{z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = -M\}.
\end{aligned}$$

Then by the theorem of residues,

$$\frac{1}{2\pi\mathrm{i}} \int_{C^M} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^{p} \operatorname{Res}_{z=\mathfrak{d}_k} \left[\mathcal{H}(z) r^{-z}\right]. \tag{3.29}$$

By (3.12), for any $h \in \mathbb{R}$,

$$\left|\mathcal{H}(t + \mathrm{i}h) r^{-t-\mathrm{i}h}\right| \sim \left(\frac{e}{|t|}\right)^{\omega|t|} \left(\frac{r}{\eta}\right)^{|t|} |t|^{\Lambda}$$

as $t \to -\infty$. Thus

$$\begin{aligned}
\lim_{M\to\infty} \int_{L_v^{-M}} \left|\mathcal{H}(z) r^{-z}\right| |dz| &= \lim_{M\to\infty} \int_{-M}^{M} \left|\mathcal{H}(-M + \mathrm{i}h) r^{M-\mathrm{i}h}\right| dh \\
&\lesssim \lim_{M\to\infty} \int_{-M}^{M} \left(\frac{e}{M}\right)^{\omega M} \left(\frac{r}{\eta}\right)^{M} M^{\Lambda} dt \\
&\leq \lim_{M\to\infty} 2 \left(\frac{e}{M}\right)^{\omega M} \left(\frac{r}{\eta}\right)^{M} M^{\Lambda+1} = 0.
\end{aligned}$$

On the other hand, by (3.13),

$$\begin{aligned}
&\lim_{M\to\infty}\int_{L_h^M}\left|\mathcal{H}(z)r^{-z}\right||dz|\\
&=\lim_{M\to\infty}\int_{\ell}^{-M}\left|\mathcal{H}(t+\mathrm{i}M)r^{-t-\mathrm{i}M}\right|dt\\
&\lesssim\lim_{M\to\infty}\int_{\ell}^{-M}e^{-\omega t}\left(\frac{\eta}{r}\right)^{t}M^{\omega t+\Lambda}\exp\left\{-\frac{\alpha^*M\pi}{2}\right\}dt\\
&=\lim_{M\to\infty}\exp\left\{-\frac{\alpha^*M\pi}{2}\right\}M^{\Lambda}\int_{\ell}^{-M}\left(\frac{\eta M^{\omega}}{re^{\omega}}\right)^{t}dt\\
&=\lim_{M\to\infty}\exp\left\{-\frac{\alpha^*M\pi}{2}\right\}M^{\Lambda}\cdot\frac{\left(\frac{\eta M^{\omega}}{re^{\omega}}\right)^{-M}-\left(\frac{\eta M^{\omega}}{re^{\omega}}\right)^{\ell}}{\ln\eta-\ln r+\omega(\ln M-1)}=0
\end{aligned}$$

since $\alpha^*>0$, $\omega\geq 0$, and $r<\eta$. Similarly,

$$\lim_{M\to\infty}\int_{L_h^{-M}}\left|\mathcal{H}(z)r^{-z}\right||dz|=0.$$

Thus, by taking $p\to\infty$ in (3.29),

$$\begin{aligned}
\sum_{k=0}^{\infty}\mathrm{Res}_{z=\hat{\mathfrak{d}}_k}\left[\mathcal{H}(z)r^{-z}\right]&=\lim_{M\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{C^M}\mathcal{H}(z)r^{-z}dz\\
&=\lim_{M\to\infty}\frac{1}{2\pi\mathrm{i}}\left(\int_{L_v^M}+\int_{L_v^{-M}}+\int_{L_h^M}+\int_{L_h^{-M}}\right)\\
&=\lim_{M\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{L_v^M}\mathcal{H}(z)r^{-z}dz=\frac{1}{2\pi\mathrm{i}}\int_{L_{Br}}\mathcal{H}(z)r^{-z}dz.
\end{aligned}$$

Therefore (3.28) holds, and the proposition is proved. □

Remark 3.3. Fix $\alpha\in(0,2)$, $\sigma\in\mathbb{R}$, and consider

$$\mathcal{H}(z)=\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\sigma-\alpha z)}.$$

Note that $\alpha^*=2-\alpha$ and $\omega=\alpha$. Then by Propositions 3.1 and 3.2, (3.14) is well-defined with both L_{Br} and L_{Ha}^{-} and (3.26) holds. By (3.16) and (2.4),

$$\begin{aligned}
\mathrm{H}^{11}_{12}\left[t\,\middle|\,\begin{matrix}(0,1)&\\(0,1)&(\sigma,\alpha)\end{matrix}\right]&=\sum_{k=0}^{\infty}\mathrm{Res}_{z=-k}\left[\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\sigma-\alpha z)}t^{-z}\right]\\
&=\sum_{k=0}^{\infty}\frac{(-t)^k}{\Gamma(1-\sigma+\alpha k)}=E_{1-\sigma,\alpha}(-t).
\end{aligned}$$

Hence the Mittag-Leffler function is a special case of the Fox H function.

Proposition 3.4 ([14, (2.2.2)]).

$$\frac{d}{dr}\left\{ \mathrm{H}^{mn}_{\nu\mu}\left[r \;\middle|\; \begin{matrix} [\mathfrak{c},\gamma] \\ [\mathfrak{d},\delta] \end{matrix} \right] \right\} = -r^{-1}\mathrm{H}^{m+1\ n}_{\nu+1\mu+1}\left[r \;\middle|\; \begin{matrix} [\mathfrak{c},\gamma] & (0,1) \\ (1,1) & [\mathfrak{d},\delta] \end{matrix} \right].$$

Proof. Observe that ω and α^* of $\mathrm{H}^{mn}_{\nu\mu}(r)$ and of

$$\mathrm{H}^{m+1\ n}_{\nu+1\mu+1}\left[r \;\middle|\; \begin{matrix} [\mathfrak{c},\gamma] & (0,1) \\ (1,1) & [\mathfrak{d},\delta] \end{matrix} \right]$$

are same, respectively. Hence $\mathrm{H}^{m+1\ n}_{\nu+1\mu+1}(r)$ is well-defined with the same contour used for $\mathrm{H}^{mn}_{\nu\mu}(r)$. Therefore the proposition easily follows from (3.1), (3.11)-(3.13), and the definition of the Fox H function. $\square$

3.2. Algebraic asymptotic expansions of $\mathrm{H}^{mn}_{\nu\mu}(r)$ near zero and at infinity.

Proposition 3.1 gives explicit power and power-logarithmic expansion of $\mathrm{H}^{mn}_{\nu\mu}(r)$ near zero for $\omega \geq 0$ and at infinity for $\omega \leq 0$. For the reverse case i.e. near zero for $\omega < 0$ and at infinity for $\omega > 0$, the following asymptotic expansions hold.

Proposition 3.5. *Suppose* (3.25) *holds. Then for sufficiently large* $p \in \mathbb{Z}_+$, *if* $\omega > 0$

$$\mathrm{H}^{mn}_{\nu\mu}(r) = \sum_{k=0}^{p} \mathrm{Res}_{z=\hat{\mathfrak{c}}_k}\left[\mathcal{H}(r)r^{-z}\right] + O(r^{-M}), \quad \hat{\mathfrak{c}}_p < M < \hat{\mathfrak{c}}_{p+1} \tag{3.30}$$

as $r \to \infty$, *and if* $\omega < 0$

$$\mathrm{H}^{mn}_{\nu\mu}(r) = -\sum_{k=0}^{p} \mathrm{Res}_{z=\hat{\mathfrak{d}}_k}\left[\mathcal{H}(r)r^{-z}\right] + O(r^{M}), \quad -\hat{\mathfrak{d}}_p < M < -\hat{\mathfrak{d}}_{p+1} \tag{3.31}$$

as $r \to 0$.

Proof. Braaksma [4] proved (3.30) for $\omega \geq 0$. We follow Braaksma's method to prove (3.31) for the case $\omega < 0$.

Let $\omega < 0$. Given (sufficiently large) $p \in \mathbb{Z}_+$, take a constant $M > 0$ satisfying (3.18). Fix $h > 0$ and let L^{-M}_{Ha} be a right Hankel contour surrounding L^{+}_{Ha}. Precisely, L^{-M}_{Ha} is a right loop situated in a horizontal strip runs from $\infty - \mathrm{i}h$ to $-M - \mathrm{i}h$ and then to $-M + \mathrm{i}h$ and finally terminating at the point $\infty + \mathrm{i}h$. Let us denote by C a closed rectangular contour which encircles $\hat{\mathfrak{d}}_0, \ldots \hat{\mathfrak{d}}_p$ and satisfies

$$\int_{L^{+}_{Ha}} \mathcal{H}(z)r^{-z}dz + \int_{C} \mathcal{H}(z)r^{-z}dz = \int_{L^{-M}_{Ha}} \mathcal{H}(z)r^{-z}dz.$$

Then

$$\mathrm{H}^{mn}_{\nu\mu}(r) = \frac{1}{2\pi\mathrm{i}} \int_{L^{+}_{Ha}} \mathcal{H}(z)r^{-z}dz$$

$$= \frac{1}{2\pi \mathrm{i}}\left(-\int_C \mathcal{H}(z)r^{-z}dz + \int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz\right)$$

$$= -\sum_{k=0}^{p} \mathrm{Res}_{z=\hat{\mathfrak{d}}_k}\left[\mathcal{H}(z)r^{-z}\right] + \frac{1}{2\pi \mathrm{i}}\int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz. \tag{3.32}$$

Using (3.11), (3.12), and (3.25), and modifying the proof of Proposition 3.2,

$$\frac{1}{2\pi \mathrm{i}}\int_{L_{Ha}^{-M}} \mathcal{H}(z)r^{-z}dz = \sum_{k=0}^{p} \mathrm{Res}_{z=\hat{\mathfrak{d}}_k}\left[\mathcal{H}(z)r^{-z}\right] + \sum_{k=0}^{\infty} \mathrm{Res}_{z=\hat{\mathfrak{c}}_k}\left[\mathcal{H}(z)r^{-z}\right]$$
$$= \frac{1}{2\pi \mathrm{i}}\int_{L_{Br}^{-M}} \mathcal{H}(z)r^{-z}dz,$$

where L_{Br}^{-M} is a Bromwich contour starting at the point $-M-\mathrm{i}\infty$ and terminating at the point $-M+\mathrm{i}\infty$.

We estimate the upper bound of contour integral along L_{Br}^{-M}. Recall (3.13):

$$\left|\mathcal{H}(-M+\mathrm{i}t)r^{M-\mathrm{i}t}\right| \sim e^{\omega M}\eta^{-M}r^{M}|t|^{-\omega M+\Lambda}\exp\left\{-\frac{|t|\alpha^*}{2}\pi\right\}$$

as $t\to\infty$. Thus, for $r\le 1$,

$$\left|\frac{1}{2\pi \mathrm{i}}\int_{L_{Br}^{-M}} \mathcal{H}(z)r^{-z}dz\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathcal{H}(-M+\mathrm{i}t)r^{M-it}\right|dt$$
$$\lesssim r^M\left(1+\frac{e^{\omega M}}{\eta^M}\int_{|t|\ge 1}|t|^{-\omega M+\Lambda}\exp\left\{-\frac{|t|\alpha^*}{2}\pi\right\}dt\right)$$
$$\lesssim r^M.$$

The proposition is proved. □

If $\omega > 0$ and $n = 0$, i.e., $P_2 = \emptyset$, we have the following exponentially asymptotic behavior of $\mathrm{H}_{\nu\mu}^{m0}(r)$ (see [14, (1.7.13)]). For the proof we refer the reader to [4, Theorem 4].

Proposition 3.6. *Assume* (3.25) *and* $\omega > 0$. *Then*

$$\mathrm{H}_{\nu\mu}^{m0}(r) = O\left(r^{(\Lambda+\frac{1}{2})/\omega}\exp\left\{\cos\left(\frac{\alpha^*+\sum_{j=m+1}^{\mu}\delta_j}{\omega}\pi\right)\omega\left(\frac{r}{\eta}\right)^{1/\omega}\right\}\right)$$

as $r\to\infty$.

4. Asymptotic behaviors of particular Fox H functions

Throughout this section we fix

$$d\in\mathbb{N},\quad \alpha\in(0,2),\quad \beta\in(0,\infty).$$

For given $\gamma \in [0,\infty)$ and $\sigma \in \mathbb{R}$, we denote a function of $z \in \mathbb{C}$ as in (3.8) by

$$\mathcal{H}_{\sigma,\gamma}(z) := \frac{\Gamma(\frac{d}{2}+\gamma+\beta z)\Gamma(1+z)\Gamma(-z)}{\Gamma(-\gamma-\beta z)\Gamma(1-\sigma+\alpha z)}.$$

Here we set $m=2$, $n=1$, $\nu=2$, and $\mu=3$ in (3.8) so that

$$\mathfrak{c}_1 = 1, \quad \mathfrak{c}_2 = 1-\sigma, \quad \mathfrak{d}_1 = \frac{d}{2}+\gamma, \quad \mathfrak{d}_2 = 1, \quad \mathfrak{d}_3 = 1+\gamma,$$

$$\gamma_1 = 1, \quad \gamma_2 = \alpha, \quad \delta_1 = \beta, \quad \delta_2 = 1, \quad \delta_3 = \beta.$$

Obviously, (3.5) holds. Recall (3.9) and (3.10)

$$\alpha^* = 2-\alpha, \quad \Lambda = \frac{d}{2}+2\gamma+\sigma-\frac{1}{2}, \quad \omega = 2\beta-\alpha, \quad \eta = \alpha^{-\alpha}\beta^{2\beta}.$$

Hence (3.25) holds. Also recall (3.6) and for each $k \in \mathbb{Z}_+$ we write

$$\mathfrak{c}_{1,k} := k, \quad \mathfrak{d}_{1,k} := -\frac{\frac{d}{2}+\gamma+k}{\beta}, \quad \mathfrak{d}_{2,k} := -1-k$$

and thus $\mathfrak{d}_{1,k}$, $\mathfrak{d}_{2,k}$ and $\mathfrak{c}_{1,k}$ constitute P_1 and P_2 respectively.

Remark 4.1. (i) Note that $\mathcal{H}_{\sigma,\gamma}(z)$ has removable singularities at $z=0$ and at $z=-1$ if $\gamma=0$ and $\gamma=\beta$, respectively. Indeed, by (3.1),

$$\mathcal{H}_{\sigma,0}(z) = \frac{\beta\Gamma(\frac{d}{2}+\beta z)\Gamma(1+z)\Gamma(1-z)}{\Gamma(1-\beta z)\Gamma(1-\sigma+\alpha z)}.$$

Therefore, $\mathcal{H}_{\sigma,0}(z)$ has a removable singularity at $z=0$. Similarly,

$$\mathcal{H}_{\sigma,\beta}(z) = -\frac{\beta\Gamma(\frac{d}{2}+\beta+\beta z)\Gamma(2+z)\Gamma(-z)}{\Gamma(1-\beta-\beta z)\Gamma(1-\sigma+\alpha z)}.$$

Thus, $\mathcal{H}_{\sigma,\beta}(z)$ has a removable singularity at $z=-1$.

(ii) Assume that $\beta \in \mathbb{N}$ and $\gamma=0$, then by (3.2),

$$\mathcal{H}_{\sigma,0}(z) = (2\pi)^{\frac{\beta-1}{2}} \frac{\Gamma(\frac{d}{2}+\beta z)\Gamma(1+z)}{\prod_{k=1}^{\beta-1}\Gamma(\frac{k}{\beta}-z)\Gamma(1-\sigma+\alpha z)}\beta^{\beta z+\frac{1}{2}}. \tag{4.1}$$

Hence $P_2 = \emptyset$.

(iii) If $\alpha=1$ and $\sigma=0$, then

$$\mathcal{H}_{0,\gamma}(z) := \frac{\Gamma(\frac{d}{2}+\gamma+\beta z)\Gamma(-z)}{\Gamma(-\gamma-\beta z)}. \tag{4.2}$$

Thus, $\mathfrak{d}_{2,k}=0$ for all $k \in \mathbb{Z}_+$.

For $r \in (0,\infty)$, we define

$$\begin{aligned}\mathbb{H}_{\sigma,\gamma}(r) &:= \mathrm{H}^{21}_{23}\left[r\middle|\begin{matrix} (1,1) & (1-\sigma,\alpha) & \\ (\frac{d}{2}+\gamma,\beta) & (1,1) & (1+\gamma,\beta)\end{matrix}\right] \\ &= \frac{1}{2\pi \mathrm{i}}\int_L \mathcal{H}_{\sigma,\gamma}(z) r^{-z}\,dz. \end{aligned}\tag{4.3}$$

Here

$$L = L_{Br} = \{z \in \mathbb{C} : \Re[z] = \ell_0\}$$

and ℓ_0 is chosen to satisfy (3.15):

$$\max\left(-1, -\frac{\gamma}{\beta} - \frac{d}{2\beta}\right) < \ell_0 < 0$$

if $\gamma \notin \{0, \beta\}$,

$$\max\left(-1, -\frac{d}{2\beta}\right) < \ell_0 < 1$$

if $\gamma = 0$,

$$\max\left(-2, -1 - \frac{d}{2\beta}\right) < \ell_0 < 0$$

if $\gamma = \beta$. If $\alpha = 1$ and $\sigma = 0$, then we take ℓ_0 such that

$$-\frac{\gamma}{\beta} - \frac{d}{2\beta} < \ell_0 < 0.$$

Since (3.25) holds (i.e., $\alpha < 2$), by Propositions 3.1 and 3.2, the value of $\mathbb{H}_{\sigma,\gamma}(r)$ is independent of the choice of ℓ_0 as long as it is chosen as above.

By Proposition 3.5, we obtain the asymptotic behaviors of $\mathbb{H}_{\sigma,\gamma}(r)$ at infinity.

Lemma 4.2. *It holds that*

$$\mathbb{H}_{\sigma,\gamma}(r) = -\frac{\Gamma(\frac{d}{2} + \gamma)}{\Gamma(-\gamma)\Gamma(1-\sigma)} + O(r^{-1})$$

for $r \geq 1$. *In particular, if* $\gamma \in \mathbb{Z}_+$ *or* $\sigma \in \mathbb{N}$, *then*

$$\mathbb{H}_{\sigma,\gamma}(r) = \frac{\Gamma(\frac{d}{2} + \gamma + \beta)}{\Gamma(-\gamma - \beta)\Gamma(1 - \sigma + \alpha)} r^{-1} + O(r^{-2})$$

for $r \geq 1$. *If* $\beta \in \mathbb{N}$ *and* $\gamma = 0$, *then there exists a constant* $c = c(d, \alpha, \beta, \sigma) > 0$ *such that*

$$\mathbb{H}_{\sigma,0}(r) = O\left(\exp\left\{-cr^{1/(2\beta-\alpha)}\right\}\right)$$

as $r \to \infty$.

Proof. Observe that $\mathcal{H}_{\sigma,\gamma}(z)$ has simple poles at $z = \hat{\mathfrak{c}}_k = \mathfrak{c}_{1,k} = k$ for all $k \in \mathbb{Z}_+$. By (3.30), for sufficiently large $p \in \mathbb{Z}_+$,

$$\begin{aligned}
\mathbb{H}_{\sigma,\gamma}(r) &= \sum_{k=0}^{1} \mathrm{Res}_{z=\hat{\mathfrak{c}}_k}\left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] + \sum_{k=2}^{p} \mathrm{Res}_{z=\hat{\mathfrak{c}}_k}\left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^{-\hat{\mathfrak{c}}_p}) \\
&= \sum_{k=0}^{1} \mathrm{Res}_{z=k}\left[\mathcal{H}_{\sigma,\gamma}(z)\right] r^{-k} + O(r^{-2})
\end{aligned}$$

for $r \geq 1$. Additionally, if $\gamma \in \mathbb{Z}_+$ or $\sigma \in \mathbb{N}$, then

$$\mathrm{Res}_{z=0}\left[\mathcal{H}_{\sigma,\gamma}(z)\right] = \lim_{z \to 0}\left(\frac{z\Gamma(\frac{d}{2} + \gamma + \beta z)\Gamma(1+z)\Gamma(-z)}{\Gamma(-\gamma - \beta z)\Gamma(1 - \sigma + \alpha z)}\right)$$

$$= -\lim_{z\to 0}\left(\frac{\Gamma(\frac{d}{2}+\gamma+\beta z)\Gamma(1+z)\Gamma(1-z)}{\Gamma(-\gamma-\beta z)\Gamma(1-\sigma+\alpha z)}\right) = 0.$$

Hence

$$\begin{aligned}\mathbb{H}_{\sigma,\gamma}(r) &= \mathrm{Res}_{z=1}[\mathcal{H}_{\sigma,\gamma}(z)]r^{-1} + O(r^{-2})\\ &= \frac{\Gamma(\frac{d}{2}+\gamma+\beta)}{\Gamma(-\gamma-\beta)\Gamma(1-\sigma+\alpha)}r^{-1} + O(r^{-2}).\end{aligned}$$

Finally, assume $\beta\in\mathbb{N}$ and $\gamma = 0$. By (4.1),

$$\begin{aligned}&\mathbb{H}_{\sigma,0}(r)\\ &= \mathrm{H}^{21}_{23}\left[r\middle|\begin{matrix}(1,1) & (1-\sigma,\alpha) & \\ (\frac{d}{2},\beta) & (1,1) & (1,\beta)\end{matrix}\right]\\ &= \frac{(2\pi)^{\frac{\beta-1}{2}}\beta^{1/2}}{2\pi\mathrm{i}}\int_L \frac{\Gamma(\frac{d}{2}+\beta z)\Gamma(1+z)}{\prod_{k=1}^{\beta-1}\Gamma(\frac{k}{\beta}-z)\Gamma(1-\sigma+\alpha z)}\left(\frac{r}{\beta^\beta}\right)^{-z}dz\\ &= (2\pi)^{\frac{\beta-1}{2}}\beta^{1/2}\mathrm{H}^{20}_{1\beta+1}\left[\frac{r}{\beta^\beta}\middle|\begin{matrix}(1-\sigma,\alpha) & & & & \\ (\frac{d}{2},\beta) & (1,1) & (1+\frac{1}{\beta},1) & \cdots & (1+\frac{\beta-1}{\beta},1)\end{matrix}\right].\end{aligned}$$

For the above Fox H function, we have

$$\alpha^* = 2-\alpha,\quad \Lambda = \frac{d}{2}+\sigma+2\beta-\frac{3}{2},\quad \omega = 2\beta-\alpha,\quad \eta = \alpha^{-\alpha}\beta^\beta.$$

Note that

$$\frac{1}{2} < \frac{\beta+1-\alpha}{2\beta-\alpha} \le 1,\qquad 2\beta-\alpha \ge 2-\alpha > 0.$$

Hence by Proposition 3.6,

$$\begin{aligned}&\mathrm{H}^{20}_{1\beta+1}\left(\frac{r}{\beta^\beta}\right)\\ &= O\left(\left(\frac{r}{\beta^\beta}\right)^{(\Lambda+\frac{1}{2})/2\beta-\alpha}\exp\left\{\cos\left(\frac{\beta+1-\alpha}{2\beta-\alpha}\pi\right)(2\beta-\alpha)\left(\frac{r}{\eta\beta^\beta}\right)^{1/(2\beta-\alpha)}\right\}\right)\\ &= O\left(\exp\left\{-cr^{1/(2\beta-\alpha)}\right\}\right).\end{aligned}$$

The lemma is proved. □

Next we consider the asymptotic behavior of $\mathbb{H}_{\sigma,\gamma}(r)$ at zero.

Lemma 4.3. *It holds that*

$$\mathbb{H}_{\sigma,\gamma}(r) \sim \begin{cases} r^{\frac{d+2\gamma}{2\beta}} & : \gamma < \beta - \frac{d}{2}\\ r|\ln r| & : \gamma = \beta - \frac{d}{2}\\ r & : \gamma > \beta - \frac{d}{2}\end{cases}$$

as $r \to 0$. Additionally, if $\gamma - \beta \in \mathbb{Z}_+$ or $\sigma + \alpha \in \mathbb{N}$, then

$$\mathbb{H}_{\sigma,\gamma}(r) \sim \begin{cases} r^{\frac{d+2\gamma}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ r^2 |\ln r| & : \gamma = 2\beta - \frac{d}{2} \\ r^2 & : \gamma > 2\beta - \frac{d}{2} \end{cases}$$

as $r \to 0$. If $\alpha = 1$ and $\sigma = 0$, then

$$\mathbb{H}_{0,\gamma}(r) \sim r^{\frac{d+2\gamma}{2\beta}}$$

as $r \to 0$.

Proof. Due to (3.31), it is sufficient to compare the order of residues among $z = \mathfrak{d}_{1,0}, \mathfrak{d}_{1,1}, \mathfrak{d}_{2,0}$, and $\mathfrak{d}_{2,1}$.

First, let $\gamma \neq \beta - \frac{d}{2}$. Then $\mathcal{H}_{\sigma,\gamma}(z)$ has a simple pole at $z = \max\{\mathfrak{d}_{1,0}, \mathfrak{d}_{2,0}\}$. If $\gamma > \beta - \frac{d}{2}$, then $\mathfrak{d}_{1,0} < \mathfrak{d}_{2,0}$ so by (3.31)

$$\mathbb{H}_{\sigma,\gamma}(r) = \sum_{k=0}^{1} \mathrm{Res}_{z=\mathfrak{d}_{2,k}} \left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^{-\mathfrak{d}_{2,1}}) \sim r$$

as $r \to 0$. Similarly, if $\gamma < \beta - \frac{d}{2}$, then

$$\mathbb{H}_{\sigma,\gamma}(r) = \mathrm{Res}_{z=\mathfrak{d}_{1,0}} \left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^{-\mathfrak{d}_{1,0}}) \sim r^{\frac{d+2\gamma}{2\beta}}$$

as $r \to 0$.

Next, assume $\gamma = \beta - \frac{d}{2}$ (i.e., $\mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$). Then $\mathcal{H}_{\sigma,\gamma}(z)$ has a pole of order 2 at $z = \hat{\mathfrak{d}}_0 = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$ so that

$$\begin{aligned} \mathrm{Res}_{z=\hat{\mathfrak{d}}_0} \left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] &= \lim_{z\to -1} \frac{d}{dz} \left((z+1)^2 \mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right) \\ &= \left(\mathrm{Res}_{z=-1}\left[\mathcal{H}_{\sigma,\gamma}(z)\right] + \frac{|\ln r|}{\Gamma(\frac{d}{2})\Gamma(1-\sigma-\alpha)}\right) r. \end{aligned}$$

Then by (3.31), we obtain the first desired result.

Now we assume $\gamma - \beta \in \mathbb{Z}_+$ or $\sigma + \alpha \in \mathbb{N}$. Then one can easily see that $\mathrm{Res}_{z=\mathfrak{d}_{2,0}} \left[\mathcal{H}_{\sigma,\gamma}(z) r^{-z}\right] = 0$. Hence it remains to compare the order of residues at $z = \mathfrak{d}_{1,0}, \mathfrak{d}_{1,1}$, and $\mathfrak{d}_{2,1}$. Following the same argument of the above, we obtain the additional result.

Finally, we assume $\alpha = 1$ and $\sigma = 0$. Recall (4.2), and note $\mathfrak{d}_{2,k} = 0$ for all $k \in \mathbb{Z}_+$ which implies

$$\mathbb{H}_{0,\gamma}(r) = \sum_{k=0}^{\infty} \mathrm{Res}_{z=\mathfrak{d}_{1,k}} \left[\mathcal{H}_{0,\gamma}(z) r^{-z}\right] \sim r^{\frac{d+2\gamma}{2\beta}}$$

as $r \to 0$. The lemma is proved. □

For each $q \in \mathbb{Z}_+$ we define

$$\mathcal{H}^{(q)}_{\sigma,\gamma}(z) := \mathcal{H}_{\sigma,\gamma}(z) \left\{\frac{\Gamma(1+z)}{\Gamma(z)}\right\}^q \tag{4.4}$$

and

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) := \frac{1}{2\pi \mathrm{i}} \int_L \mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z} dz. \tag{4.5}$$

Note that $\mathbb{H}^{(0)}_{\sigma,\gamma}(r) = \mathbb{H}_{\sigma,\gamma}(r)$ and by Proposition 3.4,

$$\frac{d}{dr}\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \mathbb{H}^{(q+1)}_{\sigma,\gamma}(r)$$

holds and (4.5) is well-defined for each $q \in \mathbb{Z}_+$. By (3.1)

$$\mathcal{H}^{(q)}_{\sigma,\gamma}(z) = -\frac{\Gamma(\frac{d}{2}+\gamma+\beta z)\Gamma(1-z)\Gamma(1+z)}{\Gamma(-\gamma-\beta z)\Gamma(1-\sigma+\alpha z)} z^{q-1},$$

and thus $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ does not have a pole at $z = 0$ if $q \geq 1$. Furthermore, $\mathcal{H}^{(q)}_{\sigma,\gamma}$ has a pole at $z = -k-1$ of order at most 2 for each $k \in \mathbb{Z}_+$. Recall (3.6) and let us denote by

$$\mathfrak{c}_{1,k} := k+1, \quad \mathfrak{d}_{1,k} := -\frac{\frac{d}{2}+\gamma+k}{\beta}, \quad \mathfrak{d}_{2,k} := -1-k$$

the new elements of P_1 and P_2 for $\mathbb{H}^{(q)}_{\sigma,\gamma}(r)$. Then due to (3.30) again, we obtain an analogue of Lemma 4.2.

Lemma 4.4. *Let $q \in \mathbb{N}$. It holds that*

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) \sim r^{-1}$$

as $r \to \infty$. Additionally, if $\beta \in \mathbb{N}$ and $\gamma = 0$, then there exists a constant $c = c(d, \alpha, \beta, \sigma, |q|)$ such that

$$\mathbb{H}^{(q)}_{\sigma,0}(r) = O\left(\exp\left\{-cr^{1/(2\beta-\alpha)}\right\}\right)$$

as $r \to \infty$.

Proof. The proof is similar to the one of Lemma 4.2. The only difference is $\hat{\mathfrak{c}}_k = \mathfrak{c}_{1,k} = k+1$ which implies

$$\mathrm{Res}_{z=\hat{\mathfrak{c}}_k}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right] = \frac{(-1)^k \cdot \Gamma(\frac{d}{2}+\gamma+\beta+\beta k)}{\Gamma(-\gamma-\beta-\beta k)\Gamma(1-\sigma+\alpha+\alpha k)}(k+1)^q r^{-k-1}$$

for each $k \in \mathbb{Z}_+$. The lemma is proved. □

Lastly, we present another result which is necessary to obtain the upper estimates of classical derivatives of $p(t,x)$. Recall $\omega = 2\beta - \alpha$. Let κ_1, κ_2, $\hat{\kappa}_1$, and $\hat{\kappa}_2$ denote constants

$$\kappa_1 := \mathrm{Res}_{z=\mathfrak{d}_{1,0}}\left[\mathcal{H}_{\sigma,\gamma}(z)\right], \quad \kappa_2 := \mathrm{Res}_{z=\mathfrak{d}_{2,0}}\left[\mathcal{H}_{\sigma,\gamma}(z)\right]$$

$$\hat{\kappa}_1 := \lim_{z\to\mathfrak{d}_{1,0}}(z-\mathfrak{d}_{1,0})^2\mathcal{H}_{\sigma,\gamma}(z), \quad \hat{\kappa}_2 := \lim_{z\to\mathfrak{d}_{2,0}}(z-\mathfrak{d}_{2,0})^2\mathcal{H}_{\sigma,\gamma}(z)$$

which are independent of $q \geq 1$. Note that $\hat{\kappa}_2 = 0$ if $\sigma + \alpha \in \mathbb{N}$.

Lemma 4.5. *Let $q \in \mathbb{Z}_+$ and $\gamma \in [0,\infty)$.*

(i) *If $\gamma \notin \{\beta - \frac{d}{2}, \beta - \frac{d}{2} - 1, 2\beta - \frac{d}{2}\}$, then*

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r^{\min\left(1, \frac{d+2\gamma+2}{2\beta}\right)}), \quad \left(\frac{0}{0} := 1\right)$$

as $r \to 0$. Additionally, if $\gamma = \beta$ or $\sigma + \alpha \in \mathbb{N}$, then

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + \begin{cases} O(r^{\min\left(2, \frac{d+2\gamma+2}{2\beta}\right)}) & : \gamma \neq 2\beta - \frac{d}{2} - 1 \\ O(r^2 |\ln r|) & : \gamma = 2\beta - \frac{d}{2} - 1 \end{cases}$$

as $r \to 0$. If $\alpha = 1$ and $\sigma = 0$, then

$$\mathbb{H}^{(q)}_{0,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r^{\frac{d+2\gamma+2}{2\beta}})$$

as $r \to 0$.

(ii) *If $\gamma = \beta - \frac{d}{2}$, then*

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r + O(r)$$

as $r \to 0$. Additionally, if $\sigma + \alpha \in \mathbb{N}$, then

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = -\frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_2 r + \begin{cases} O(r^{\min(2, \frac{d+2\gamma+2}{2\beta})}) & : \beta \neq 1 \\ O(r^2 |\ln r|) & : \beta = 1 \end{cases}$$

as $r \to 0$.

(iii) *If $\gamma = \beta - \frac{d}{2} - 1$, then*

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r |\ln r|)$$

as $r \to 0$. Additionally, if $\sigma + \alpha \in \mathbb{N}$, then

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r)$$

as $r \to 0$.

(iv) *If $\gamma = 2\beta - \frac{d}{2}$, then*

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1)\, r^2 + O(r)$$

as $r \to 0$. Additionally, if $\sigma + \alpha \in \mathbb{N}$, then

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d+2\gamma}{2\beta}\right)^q \cdot (\hat{\kappa}_1 \ln r + \kappa_1)\, r^2 + O(r^2)$$

as $r \to 0$.

Proof. Due to Proposition 3.1, one can easily see our assertions hold if $\omega \geq 0$. Hence, we assume $\omega < 0$. Recall (3.31) and take $M \geq 2$ satisfying (3.18).

(i) If $\gamma \neq \beta - \frac{d}{2}$, $\gamma \neq \beta - \frac{d}{2} - 1$, and $\gamma \neq 2\beta - \frac{d}{2}$, then $\mathfrak{d}_{1,0} \neq \mathfrak{d}_{2,0}$, $\mathfrak{d}_{1,1} \neq \mathfrak{d}_{2,0}$, and $\mathfrak{d}_{1,0} \neq \mathfrak{d}_{2,1}$. Hence $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has simple poles at $\mathfrak{d}_{1,0} = -\frac{d+2\gamma}{2\beta}$ and $\mathfrak{d}_{2,0} = -1$. Therefore,

$$\begin{aligned}
\mathbb{H}^{(q)}_{\sigma,\gamma}(r) &= -\sum_{j=1}^{2} \mathrm{Res}_{z=\mathfrak{d}_{j,0}} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}\right] - \mathrm{Res}_{z=\mathfrak{d}_{2,1}} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^M) \\
&= -\left(\kappa_1 (\mathfrak{d}_{1,0})^q r^{-\mathfrak{d}_{1,0}} + \kappa_2 (\mathfrak{d}_{2,0})^q r^{-\mathfrak{d}_{2,0}}\right) \\
&\quad - \mathrm{Res}_{z=-2} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^{\min\left(2,\frac{d+2\gamma+2}{2\beta}\right)}) \\
&= -\left(-\frac{d+2\gamma}{2\beta}\right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} - (-1)^q \cdot \kappa_2 r \qquad (4.6) \\
&\quad - \mathrm{Res}_{z=-2} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}\right] + O(r^{\min\left(2,\frac{d+2\gamma+2}{2\beta}\right)})
\end{aligned}$$

as $r \to 0$. Note that $\kappa_2 = 0$ if $\gamma = \beta$ or $\sigma + \alpha \in \mathbb{N}$ so the second term of (4.6) vanishes. In that case, observe that

$$\mathrm{Res}_{z=-2}[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}] = \begin{cases} O(r^2) & : \gamma \neq 2\beta - \frac{d}{2} - 1 \\ O(r^2 |\ln r|) & : \gamma = 2\beta - \frac{d}{2} - 1. \end{cases}$$

Furthermore, if $\alpha = 1$ and $\sigma = 0$, then

$$\mathrm{Res}_{z=\mathfrak{d}_{2,k}}[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}] = 0$$

for all $k \in \mathbb{Z}_+$. Thus (i) is proved.

(ii) Suppose $\gamma = \beta - \frac{d}{2}$. Then $\hat{\mathfrak{d}}_0 = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,0}$, $\kappa_1 = \kappa_2$, and $\hat{\kappa}_1 = \hat{\kappa}_2$. Note that $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has a pole at

$$\hat{\mathfrak{d}}_0 = -\frac{d+2\gamma}{2\beta} = -1$$

of order 2. By (4.4),

$$\begin{aligned}
\mathrm{Res}_{z=-1} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z) r^{-z}\right] &= \lim_{z \to -1} \frac{d}{dz} \left((z+1)^2 \mathcal{H}_{\sigma,\gamma}(z) z^q r^{-z}\right) \\
&= (-1)^q \cdot \hat{\kappa}_2 r \ln r + \mathrm{Res}_{z=-1} \left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)\right] r \\
&= (-1)^q \cdot \hat{\kappa}_2 r \ln r + (-1)^q \cdot (\kappa_2 - q\hat{\kappa}_2)\, r \\
&= (-1)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r - q(-1)^q \hat{\kappa}_2 r. \qquad (4.7)
\end{aligned}$$

Therefore,

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = -(-1)^q \cdot (\hat{\kappa}_2 \ln r + \kappa_2) r + q(-1)^q \hat{\kappa}_2 r + \begin{cases} O(r^{\min(2,\frac{d+2\gamma+2}{2\beta})}) & : \beta \neq 1 \\ O(r^2 |\ln r|) & : \beta = 1 \end{cases}$$

as $r \to 0$. If we additionally assume $\sigma + \alpha \in \mathbb{N}$, then $\hat{\kappa}_2 = 0$. Thus (ii) is proved.

(iii) Now we let $\gamma = \beta - \frac{d}{2} - 1$. Then $\mathfrak{d}_{1,1} = \mathfrak{d}_{2,0}$ and $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has a pole at

$$-\frac{d+2\gamma+2}{2\beta} = \mathfrak{d}_{1,1} = \mathfrak{d}_{2,0} = -1$$

of order 2, and (4.7) holds. Also note that $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has a simple pole at

$$z = \mathfrak{d}_{1,0} = -\frac{d+2\gamma}{2\beta} = \frac{1}{\beta} - 1.$$

Therefore,

$$\begin{aligned}\mathbb{H}^{(q)}_{\sigma,\gamma}(r) &= -\mathrm{Res}_{z=-\frac{d+2\gamma}{2\beta}}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right] - \mathrm{Res}_{z=-1}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right] + O(r)\\ &= -\left(-\frac{d+2\gamma}{2\beta}\right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r|\ln r|)\end{aligned}$$

as $r \to 0$. If $\sigma + \alpha \in \mathbb{N}$, then $\hat{\kappa}_2 = 0$ and

$$\mathbb{H}^{(q)}_{\sigma,\gamma}(r) = -\left(-\frac{d+2\gamma}{2\beta}\right)^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r).$$

Thus (iii) is proved.

(iv) Finally, we assume $\gamma = 2\beta - \frac{d}{2}$. Then $\mathfrak{d}_{1,0} = \mathfrak{d}_{2,1}$ and $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has a pole at

$$-\frac{d+2\gamma}{2\beta} = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,1} = -2$$

of order 2. By (4.4),

$$\begin{aligned}&\mathrm{Res}_{z=-2}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right]\\ &= \lim_{z\to-2}\frac{d}{dz}\left((z+2)^2\mathcal{H}_{\sigma,\gamma}(z)z^q r^{-z}\right)\\ &= \left(-\frac{d+2\gamma}{2\beta}\right)\cdot\hat{\kappa}_1 r^2\ln r + \lim_{z\to-2}\frac{d}{dz}\left((z+2)^2\mathcal{H}_{\sigma,\gamma}(z)z^q\right)r^2\\ &= \left(-\frac{d+2\gamma}{2\beta}\right)^q\cdot(\hat{\kappa}_1\ln r + \kappa_1)\,r^2 + q\left(-\frac{d+2\gamma}{2\beta}\right)^{q-1}\hat{\kappa}_1 r^2.\end{aligned}$$

Note that $\mathcal{H}^{(q)}_{\sigma,\gamma}(z)$ has a simple pole at $z = \mathfrak{d}_{2,0} = -1$. Therefore,

$$\begin{aligned}\mathbb{H}^{(q)}_{\sigma,\gamma}(r) &= -\mathrm{Res}_{z=-\frac{d+2\gamma}{2\beta}}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right] - \mathrm{Res}_{z=-1}\left[\mathcal{H}^{(q)}_{\sigma,\gamma}(z)r^{-z}\right] + O(r^2)\\ &= -\left(-\frac{d+2\gamma}{2\beta}\right)^q\cdot(\hat{\kappa}_1\ln r + \kappa_1)\,r^2 - \kappa_2 r + O(r^2) \qquad (4.8)\\ &= -\left(-\frac{d+2\gamma}{2\beta}\right)^q\cdot(\hat{\kappa}_1\ln r + \kappa_1)\,r^2 + O(r)\end{aligned}$$

as $r \to 0$. Additionally, if $\sigma + \alpha \in \mathbb{N}$, then $\kappa_2 = 0$ in (4.8) hence we obtain the desired result. The lemma is proved. □

5. Upper estimates of $p(t,x)$ and its derivatives

Take the function $\mathbb{H}_{\sigma,\gamma}(r)$ from (4.3), and define

$$p_{\sigma,\gamma}(t,x) := \frac{2^{2\gamma}}{\pi^{d/2}}|x|^{-d-2\gamma}t^{-\sigma}\mathbb{H}_{\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right), \quad (t,x)\in(0,\infty)\times\mathbb{R}^d_0.$$

Write

$$p(t,x) := p_{0,0}(t,x) = \frac{|x|^{-d}}{\pi^{d/2}}\mathbb{H}_{0,0}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right). \tag{5.1}$$

Note that $\mathbb{H}_{\sigma,\gamma}(r)$ is a bounded function of $r\in(0,\infty)$. As a corollary of Lemmas 4.2 and 4.3, we obtain the following upper estimates of $p_{\sigma,\gamma}(t,x)$ for $\gamma\in[0,\infty)$ and $\sigma\in\mathbb{R}$.

Theorem 5.1. *Let $\alpha\in(0,2)$, $\beta\in(0,\infty)$, $\gamma\in[0,\infty)$, and $\sigma\in\mathbb{R}$. Then for $|x|^{2\beta}t^{-\alpha}\geq 1$*

$$|p_{\sigma,\gamma}(t,x)| \lesssim |x|^{-d-2\gamma}t^{-\sigma}$$

and for $|x|^{2\beta}t^{-\alpha}\leq 1$

$$|p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma<\beta-\frac{d}{2} \\ |x|^{-d-2\gamma+2\beta}t^{-\sigma-\alpha}\left(1+|\ln\left(|x|^{2\beta}t^{-\alpha}\right)|\right) & : \gamma=\beta-\frac{d}{2} \\ |x|^{-d-2\gamma+2\beta}t^{-\sigma-\alpha} & : \gamma>\beta-\frac{d}{2}. \end{cases}$$

Furthermore,

(i) *If $\sigma+\alpha\in\mathbb{N}$, then for $|x|^{2\beta}t^{-\alpha}\leq 1$*

$$|p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma<2\beta-\frac{d}{2} \\ |x|^{-d-2\gamma+4\beta}t^{-\sigma-2\alpha}\left(1+\left|\ln\left(|x|^{2\beta}t^{-\alpha}\right)\right|\right) & : \gamma=2\beta-\frac{d}{2} \\ |x|^{-d-2\gamma+4\beta}t^{-\sigma-2\alpha} & : \gamma>2\beta-\frac{d}{2}. \end{cases}$$

(ii) *If $\alpha=1$ and $\sigma=0$, then for $|x|^{2\beta}t^{-\alpha}\leq 1$*

$$|p_{0,\gamma}(t,x)| \lesssim t^{-\frac{d+2\gamma}{2\beta}}.$$

(iii) *If $\gamma\in\mathbb{Z}_+$, then for $|x|^{2\beta}t^{-\alpha}\geq 1$*

$$|p_{\sigma,\gamma}(t,x)| \lesssim |x|^{-d-2\gamma-2\beta}t^{-\sigma+\alpha}.$$

(iv) *If $\beta\in\mathbb{N}$ and $\gamma=0$, then there exists a constant $c=c(\alpha,\beta,\sigma)>0$ such that for $|x|^{2\beta}t^{-\alpha}\geq 1$,*

$$|p_{\sigma,0}(t,x)| \lesssim |x|^{-d}t^{-\sigma}\exp\left\{-c(t^{-\alpha}|x|^{2\beta})^{\frac{1}{2\beta-\alpha}}\right\}.$$

(v) *If $\gamma=\beta$, then for $|x|^{2\beta}t^{-\alpha}\geq 1$*

$$|p_{\sigma,\beta}(t,x)| \lesssim \begin{cases} |x|^{-d-4\beta}t^{-\sigma+\alpha} & : \sigma\in\mathbb{N} \\ |x|^{-d-2\beta}t^{-\sigma} & : \sigma\in\mathbb{R}\setminus\mathbb{N} \end{cases}$$

and for $|x|^{2\beta}t^{-\alpha} \leq 1$

$$|p_{\sigma,\beta}(t,x)| \lesssim \begin{cases} t^{-\sigma-\alpha-\frac{\alpha d}{2\beta}} & : \frac{d}{2} < \beta \\ |x|^{-d+2\beta}t^{-\sigma-2\alpha}\left(1+\left|\ln\left(|x|^{2\beta}t^{-\alpha}\right)\right|\right) & : \frac{d}{2} = \beta \\ |x|^{-d+2\beta}t^{-\sigma-2\alpha} & : \frac{d}{2} > \beta. \end{cases}$$

Remark 5.2. (i) By Theorem 5.1, $p_{\sigma,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ for all $\gamma \in [0,\beta]$ and $\sigma \in \mathbb{R}$. Furthermore, if $\alpha = 1$ and $\sigma = 0$, then $p_{0,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ for all $\gamma \in [0,\infty)$.

(ii) Observe that

$$p_{\sigma,\gamma}(t,x) = t^{-\sigma-\frac{\alpha(d+2\gamma)}{2\beta}} p_{\sigma,\gamma}(1, t^{-\frac{\alpha}{2\beta}}x) \tag{5.2}$$

which implies

$$\int_0^T \int_{\mathbb{R}^d} |p_{\sigma,\gamma}(t,x)|\, dxdt = \int_0^T t^{-\sigma-\frac{\alpha\gamma}{\beta}} \left(\int_{\mathbb{R}^d} |p_{\sigma,\gamma}(1,x)|\, dx\right) dt < \infty$$

if $\sigma + \frac{\alpha\gamma}{\beta} < 1$. Thus, under this condition, one can consider Riemann-Liouville fractional integral and the Fourier-Laplace transform of $p_{\sigma,\gamma}(t,x)$. However we do not use such transforms in this article.

The following theorem handles the interchangeability of Δ^γ and $\mathbb{D}_t^\sigma$.

Theorem 5.3. *Let* $\gamma \in [0,\infty)$. *For any* $\sigma \in \mathbb{R}$, $m \in \mathbb{N}$, *and* $\eta \in (-\infty, 1)$,

$$D_t^m p_{\sigma,\gamma}(t,x) = p_{\sigma+m,\gamma}(t,x),$$

and

$$\mathbb{D}_t^\sigma p_{\eta,\gamma}(t,x) = p_{\sigma+\eta,\gamma}(t,x).$$

Proof. First we show

$$\frac{\partial}{\partial t} p_{\sigma,\gamma}(t,x) = p_{\sigma+1,\gamma}(t,x) \tag{5.3}$$

for any $\sigma \in \mathbb{R}$. By (3.1),

$$\frac{\alpha z}{\Gamma(1-\sigma+\alpha z)} = \frac{1}{\Gamma(-\sigma+\alpha z)} + \frac{\sigma}{\Gamma(1-\sigma+\alpha z)}$$

which implies

$$\mathcal{H}_{\sigma,\gamma}(z)\alpha z = \mathcal{H}_{\sigma+1,\gamma}(z) + \sigma\mathcal{H}_{\sigma,\gamma}(z).$$

Then by Proposition 3.4,

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{H}_{\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right) &= \frac{t^{-1}}{2\pi\mathrm{i}}\int_L \mathcal{H}_{\sigma,\gamma}(z)\alpha z\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)^{-z} dz \\ &= \frac{t^{-1}}{2\pi\mathrm{i}}\int_L \left(\mathcal{H}_{1+\sigma,\gamma}(z) + \sigma\mathcal{H}_{\sigma,\gamma}(z)\right)\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)^{-z} dz \\ &= t^{-1}\left[\mathbb{H}_{\sigma+1,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right) + \sigma\mathbb{H}_{\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)\right]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial t}p_{\sigma,\gamma}(t,x)\\
&=\frac{2^{2\gamma}}{\pi^{d/2}}|x|^{-d-2\gamma}t^{-\sigma}\left(\frac{\partial}{\partial t}\mathbb{H}_{\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)-\sigma t^{-1}\mathbb{H}_{\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)\right)\\
&=\frac{2^{2\gamma}}{\pi^{d/2}}|x|^{-d-2\gamma}t^{-\sigma-1}\mathbb{H}_{\sigma+1,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)\\
&=p_{\sigma+1,\gamma}(t,x),
\end{aligned}
$$

and (5.3) is proved. Thus to complete the proof of the theorem, due to (2.2) and (5.3), it is sufficient to show that for $\sigma<0$ and $\eta<1$

$$
I_t^{|\sigma|}p_{\eta,\gamma}(t,x)=p_{\eta+\sigma,\gamma}(t,x).
$$

Take $\ell_0>-\frac{1-\eta}{\alpha}$ satisfying (3.15). For $\Re[z]>-\frac{1-\eta}{\alpha}$, it holds that

$$
\frac{1}{\Gamma(-\sigma)}\int_0^t(t-s)^{-\sigma-1}s^{-\eta+\alpha z}ds=\frac{\Gamma(1-\eta+\alpha z)}{\Gamma(1-\sigma-\eta+\alpha z)}t^{-\sigma-\eta+\alpha z}.
$$

Observe that

$$
\mathcal{H}_{\eta,\gamma}(z)\frac{\Gamma(1-\eta+\alpha z)}{\Gamma(1-\sigma-\eta+\alpha z)}=\mathcal{H}_{\eta+\sigma,\gamma}(z).
$$

By (3.13) and the Fubini theorem,

$$
\begin{aligned}
&I_t^{|\sigma|}p_{\eta,\gamma}(t,x)\\
&=\int_0^t\frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)}p_{\eta,\gamma}(s,x)ds\\
&=\frac{2^{2\gamma}\pi^{-\frac{d}{2}}|x|^{-d-2\gamma}}{2\pi\mathrm{i}}\int_0^t\int_L\frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)}\mathcal{H}_{\eta,\gamma}(z)s^{-\eta}\left(\frac{|x|^{2\beta}s^{-\alpha}}{2^{2\beta}}\right)^{-z}dzds\\
&=\frac{2^{2\gamma}\pi^{-\frac{d}{2}}|x|^{-d-2\gamma}}{2\pi\mathrm{i}}\int_L\mathcal{H}_{\eta,\gamma}(z)\left(\frac{|x|^{2\beta}}{2^{2\beta}}\right)^{-z}\left[\int_0^t\frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)}s^{-\eta+\alpha z}ds\right]dz\\
&=\frac{2^{2\gamma}\pi^{-\frac{d}{2}}|x|^{-d-2\gamma}}{2\pi\mathrm{i}}t^{-\sigma}\int_L\mathcal{H}_{\eta+\sigma,\gamma}(z)\left(\frac{|x|^{2\beta}t^{-\alpha}}{2^{2\beta}}\right)^{-z}dz\\
&=2^{2\gamma}\pi^{-\frac{d}{2}}|x|^{-d-2\gamma}t^{-\eta-\sigma}\mathbb{H}_{\eta+\sigma,\gamma}\left(\frac{|x|^{2\beta}}{2^{2\beta}}t^{-\alpha}\right)\\
&=p_{\eta+\sigma,\gamma}(t,x).
\end{aligned}
$$

The theorem is proved. □

Remark 5.4. Let $0<\varepsilon<T$. Then by Theorems 5.1 and 5.3, $p(t,x)$ and $\frac{\partial p}{\partial t}(t,x)$ are integrable in $x\in\mathbb{R}^d$ uniformly in $t\in[\varepsilon,T]$. Thus

$$
\frac{\partial}{\partial t}\int_{\mathbb{R}^d}e^{-\mathrm{i}(x,\xi)}p(t,x)dx=\int_{\mathbb{R}^d}e^{-\mathrm{i}(x,\xi)}\frac{\partial p}{\partial t}(t,x)dx.
$$

To estimate the classical derivatives of $p(t,x)$, we need the following theorem.

Theorem 5.5. *Let $\alpha,\beta,\gamma,\sigma$ be given as in Theorem 5.1 and $n\in\mathbb{N}$. Then for $|x|^{2\beta}t^{-\alpha}\geq 1$*

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} |x|^{-d-n}t^{-\sigma}\exp\left\{-(|x|^{2\beta}t^{-\alpha})^{\frac{1}{2\beta-\alpha}}\right\} & : \beta\in\mathbb{N},\gamma=0 \\ |x|^{-d-2\gamma-2\beta-n}t^{-\sigma+\alpha} & : \gamma\in\mathbb{Z}_+ \textit{ or } \sigma\in\mathbb{N} \\ |x|^{-d-2\gamma-n}t^{-\sigma} & : \textit{otherwise}, \end{cases}$$

and for $|x|^{2\beta}t^{-\alpha}\leq 1$

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} |x|^{2-n}t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma<\beta-\frac{d}{2}-1 \\ |x|^{2-n}t^{-\sigma-\alpha}(1+|\ln|x|^{2\beta}t^{-\alpha}|) & : \gamma=\beta-\frac{d}{2}-1 \\ |x|^{-d-2\gamma+2\beta-n}t^{-\sigma-\alpha} & : \gamma>\beta-\frac{d}{2}-1. \end{cases}$$

Additionally, for $|x|^{2\beta}t^{-\alpha}\leq 1$, the followings hold:

(i) *If $\alpha=1$ and $\sigma=0$,*

$$|D^n_x p_{0,\gamma}(t,x)| \lesssim |x|^{2-n}t^{-\frac{d+2\gamma+2}{2\beta}}.$$

(ii) *If $\sigma+\alpha\in\mathbb{N}$,*

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} |x|^{2-n}t^{-\sigma-\frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma<2\beta-\frac{d}{2}-1 \\ |x|^{2-n}t^{-\sigma-2\alpha}(1+\left|\ln|x|^{2\beta}t^{-\alpha}\right|) & : \gamma=2\beta-\frac{d}{2}-1 \\ |x|^{-d-2\gamma+4\beta-n}t^{-\sigma-2\alpha} & : \gamma>2\beta-\frac{d}{2}-1. \end{cases}$$

(iii) *If $\gamma=\beta$,*

$$|D^n_x p_{\sigma,\beta}(t,x)| \lesssim \begin{cases} |x|^{2-n}t^{-\sigma-\alpha-\frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2}+1<\beta \\ |x|^{2-n}t^{-\sigma-2\alpha}(1+\left|\ln|x|^{2\beta}t^{-\alpha}\right|) & : \frac{d}{2}+1=\beta \\ |x|^{-d+2\beta-n}t^{-\sigma-2\alpha} & : \frac{d}{2}+1>\beta. \end{cases}$$

Proof. Write $R=R(t,x):=2^{-2\beta}|x|^{2\beta}t^{-\alpha}$ and recall $\mathbb{H}^{(q)}_{\sigma,\gamma}$ from (4.5). By Proposition 3.4 and (3.1),

$$D_{x^i}\mathbb{H}^{(q)}_{\sigma,\gamma}(R) = -2\beta\frac{x^i}{|x|^2}\mathbb{H}^{(q+1)}_{\sigma,\gamma}(R)$$

for each $q\in\mathbb{Z}_+$ and $i=1,\ldots,d$. Hence

$$\begin{aligned} &|D_{x^i}p_{\sigma,\gamma}(t,x)| \\ &= \left|-\frac{2^{2\gamma}}{\pi^{d/2}}x^i|x|^{-d-2\gamma-2}t^{-\sigma}\left((d+2\gamma)\mathbb{H}_{\sigma,\gamma}(R)+2\beta\mathbb{H}^{(1)}_{\sigma,\gamma}(R)\right)\right| \\ &\leq C|x|^{-d-2\gamma-1}t^{-\sigma}\left|(d+2\gamma)\mathbb{H}_{\sigma,\gamma}(R)+2\beta\mathbb{H}^{(1)}_{\sigma,\gamma}(R)\right| \end{aligned}$$

and

$$|D_{x^j}D_{x^i}p_{\sigma,\gamma}(t,x)|$$

$$= \left| D_{x^j} \left\{ -\frac{2^{2\gamma}}{\pi^{d/2}} x^i |x|^{-d-2\gamma-2} t^{-\sigma} \left((d+2\gamma)\mathbb{H}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(1)}_{\sigma,\gamma}(R) \right) \right\} \right|$$

$$= \frac{2^{2\gamma}}{\pi^{d/2}} t^{-\sigma} \Bigg| - \delta_{ij} |x|^{-d-2\gamma-2} \left\{ (d+2\gamma)\mathbb{H}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(1)}_{\sigma,\gamma}(R) \right\}$$

$$+ (d+2\gamma+2) x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma)\mathbb{H}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(1)}_{\sigma,\gamma}(R) \right\}$$

$$+ 2\beta x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma)\mathbb{H}^{(1)}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(2)}_{\sigma,\gamma}(R) \right\} \Bigg|$$

$$\leq C|x|^{-d-2\gamma-2} t^{-\sigma} \sum_{q=1}^{2} \left| (d+2\gamma)\mathbb{H}^{(q-1)}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(q)}_{\sigma,\gamma}(R) \right|.$$

Inductively, for any $n \in \mathbb{N}$,

$$|D^n_x p_{\sigma,\gamma}(t,x)| \leq C|x|^{-d-2\gamma-n} t^{-\sigma} \sum_{q=1}^{n} \left| (d+2\gamma)\mathbb{H}^{(q-1)}_{\sigma,\gamma}(R) + 2\beta \mathbb{H}^{(q)}_{\sigma,\gamma}(R) \right|. \tag{5.4}$$

By Lemmas 4.2, 4.4, and (5.4),

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} t^{\frac{\alpha(d+n)}{2\beta} - \sigma} \exp\left\{ -c(|x|^{2\beta} t^{-\alpha})^{\frac{1}{2\beta-\alpha}} \right\} & : \gamma = 0, \beta \in \mathbb{N} \\ |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha} & : \gamma \in \mathbb{Z}_+ \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\gamma-n} t^{-\sigma} & : \text{otherwise} \end{cases}$$

for $|x|^{2\beta} t^{-\alpha} \geq 1$. To estimate the upper bounds for $|x|^{2\beta} t^{-\alpha} \leq 1$, observe that

$$(d+2\gamma)\left(-\frac{d+2\gamma}{2\beta} \right)^{q-1} + 2\beta \left(-\frac{d+2\gamma}{2\beta} \right)^{q} = 0.$$

Thus, by Lemma 4.5 and (5.4),

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} |x|^{2-|n|} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\ |x|^{2-|n|} t^{-\sigma-\alpha} (1 + |\ln |x|^{2\beta} t^{-\alpha}|) & : \gamma = \beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+2\beta-|n|} t^{-\sigma-\alpha} & : \gamma > \beta - \frac{d}{2} - 1 \end{cases}$$

for $R \leq 1$.

Additionally, if $\sigma + \alpha \in \mathbb{N}$, by Lemma 4.5,

$$|D^n_x p_{\sigma,\gamma}(t,x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma - \frac{\alpha(d+2\gamma+2)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{2-|n|} t^{-\sigma-\alpha} (1 + \left|\ln |x|^{2\beta} t^{-\alpha}\right|) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma+4\beta-|n|} t^{-\sigma-2\alpha} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases}$$

for $R \leq 1$. Also, if $\gamma = \beta$,

$$|D^n_x p_{\sigma,\beta}(t,x)| \lesssim \begin{cases} |x|^{2-n} t^{-\sigma-\alpha-\frac{\alpha(d+2)}{2\beta}} & : \frac{d}{2} + 1 < \beta \\ |x|^{2-|n|} t^{-\sigma-2\alpha} (1 + \left|\ln |x|^{2\beta} t^{-\alpha}\right|) & : \frac{d}{2} + 1 = \beta \\ |x|^{-d+2\beta-|n|} t^{-\sigma-2\alpha} & : \frac{d}{2} + 1 > \beta \end{cases}$$

for $R \leq 1$. The theorem is proved. □

6. Proofs of Theorems 2.1, 2.3, 2.4, and 2.5

By Remark 5.2, $p_{\sigma,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ if $\gamma \in [0,\beta]$ and $\sigma \in \mathbb{R}$, and $p_{0,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ if $\alpha = 1$, $\gamma \in [0,\infty)$, and $\sigma = 0$. Due to (2.5), Lemmas 4.2, 4.3, Theorems 5.1, 5.3, and 5.5, to prove our desired results, it is enough to show

$$\mathcal{F}\{p_{\sigma,\gamma}(t,\cdot)\} = |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-t^\alpha |\xi|^{2\beta}), \tag{6.1}$$

which is equivalent to

$$p_{\sigma,\gamma}(t,x) = \Delta^\gamma p_{\sigma,0}(t,x) = \Delta^\gamma \mathbb{D}_t^\sigma p(t,x).$$

We divide the proof into the cases $d \geq 2$ and $d = 1$.

Case 1: $d \geq 2$

We choose $\ell_0 \in \mathbb{R}$ in (4.3) such that

$$\max(-2, -1 - \frac{d-1}{4\beta}) < \ell_0 < -1 \tag{6.2}$$

if $\gamma = \beta$,

$$\max(-1, -\frac{\gamma}{\beta} - \frac{d-1}{4\beta}) < \ell_0 < -\frac{\gamma}{\beta} \tag{6.3}$$

if $\gamma \in (0,\beta)$,

$$\max(-1, -\frac{1}{\alpha}, -\frac{d-1}{4\beta}) < \ell_0 < 0 \tag{6.4}$$

if $\gamma = 0$. If $\alpha = 1$, $\gamma \neq 0$, and $\sigma = 0$, then we take ℓ_0 such that

$$-\frac{\gamma}{\beta} - \frac{d-1}{4\beta} < \ell_0 < -\frac{\gamma}{\beta}. \tag{6.5}$$

Under the above restriction on ℓ_0, (4.3) is well-defined and the value of $p_{\sigma,\gamma}(t,x)$ is independent of the choice of ℓ_0.

Due to the Fourier transform for radial function (see [35, Theorem IV.3.3])

$$\mathcal{F}\{p_{\sigma,\gamma}(t,\cdot)\}(\xi) = \frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \mathbb{H}_{\sigma,\gamma}\left(\frac{\rho^{2\beta}}{2^{2\beta}} t^{-\alpha}\right) J_{\frac{d}{2}-1}(|\xi|\rho) d\rho,$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of the first kind of order $\frac{d}{2}-1$, i.e., for $r \in [0,\infty)$,

$$J_{\frac{d}{2}-1}(r) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\frac{d}{2})} \left(\frac{r}{2}\right)^{2k-1+\frac{d}{2}}.$$

It is well-known if $m > -1$, then

$$J_m(t) = \begin{cases} O(t^m) & : t \to 0+ \\ O(t^{-1/2}) & : t \to \infty. \end{cases}$$

Due to (6.2)-(6.5),

$$\begin{aligned}&\int_0^\infty \left|\rho^{-\frac{d}{2}-2\gamma-2\beta\ell_0} J_{\frac{d}{2}-1}(|\xi|\rho)\right| d\rho \\ &\leq \int_0^1 \rho^{-2\beta\ell_0-2\gamma-1} d\rho + \int_1^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta\ell_0-\frac{1}{2}} d\rho < \infty.\end{aligned}$$

Recall $\Re[z] = \ell_0$ along L. By the above and (3.13),

$$\int_0^\infty \int_L \left|\rho^{-\frac{d}{2}} \mathcal{H}_{\sigma,\gamma}(z) \left(\frac{\rho^{2\beta} t^{-\alpha}}{2^{2\beta}}\right)^{-z} J_{\frac{d}{2}-1}(|\xi|\rho)\right| |dz| d\rho < \infty.$$

Therefore, by the Fubini theorem

$$\begin{aligned}&\int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \mathbb{H}_{\sigma,\gamma}\left(\frac{\rho^{2\beta}}{2^{2\beta}} t^{-\alpha}\right) J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi\mathrm{i}} \int_0^\infty \rho^{-\frac{d}{2}-2\gamma} \left[\int_L \mathcal{H}_{\sigma,\gamma}(z) \left(\frac{\rho^{2\beta} t^{-\alpha}}{2^{2\beta}}\right)^{-z} dz\right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho \\ &= \frac{1}{2\pi\mathrm{i}} \int_L \left[\int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho\right] \mathcal{H}_{\sigma,\gamma}(z) \left(\frac{t^{-\alpha}}{2^{2\beta}}\right)^{-z} dz.\end{aligned}$$

By using the formula [1, (11.4.16)],

$$\int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho = 2^{-\frac{d}{2}-2\gamma-2\beta z} |\xi|^{\frac{d}{2}+2\gamma+2\beta z-1} \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)}$$

we have

$$\mathcal{H}_{\sigma,\gamma}(z) \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)} = \frac{\Gamma(z+1)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)}.$$

Hence

$$\begin{aligned}&\frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \cdot \frac{1}{2\pi\mathrm{i}} \int_L \left[\int_0^\infty \rho^{-\frac{d}{2}-2\gamma-2\beta z} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho\right] \mathcal{H}_{\sigma,\gamma}(z) \left(\frac{t^{-\alpha}}{2^{2\beta}}\right)^{-z} dz \\ &= \frac{2^{\frac{d}{2}+2\gamma}}{|\xi|^{\frac{d}{2}-1}} t^{-\sigma} \cdot \frac{2^{-\frac{d}{2}-2\gamma}}{2\pi\mathrm{i}} |\xi|^{\frac{d}{2}+2\gamma-1} \int_L \frac{\Gamma(z+1)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)} \left(|\xi|^{-2\beta} t^{-\alpha}\right)^{-z} dz \\ &= |\xi|^{2\gamma} t^{-\sigma} \cdot \frac{1}{2\pi\mathrm{i}} \int_{-L} \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(1-\sigma-\alpha z)} \left(|\xi|^{2\beta} t^{\alpha}\right)^{-z} dz \\ &= |\xi|^{2\gamma} t^{-\sigma} \mathrm{H}_{12}^{11}\left[|\xi|^{2\beta} t^{\alpha} \,\middle|\, \begin{matrix} (0,1) & \\ (0,1) & (\sigma,\alpha) \end{matrix}\right] \\ &= |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^{\alpha}),\end{aligned}$$

where the last equality is due to Remark 3.3. Therefore,

$$\mathcal{F}\left\{p_{\sigma,\gamma}(t,\cdot)\right\} = |\xi|^{2\gamma} t^{-\sigma} E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^{\alpha}).$$

Case 2: $d = 1, \gamma \in (0, \beta)$**.**

We choose ℓ_0 such that

$$\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_0 < 0.$$

Since $p_{\sigma,\gamma}(t,x)$ is an even function,

$$\begin{aligned}
&\mathcal{F}\left\{p_{\sigma,\gamma}(t,\cdot)\right\} \\
&= \int_{-\infty}^{\infty} e^{-\mathrm{i}\xi x} p_{\sigma,\gamma}(t,x)dx \\
&= 2\int_0^{\infty} p_{\sigma,\gamma}(t,x)\cos(\xi x)dx \\
&= 2\left(\int_0^{t^{\alpha/2\beta}} p_{\sigma,\gamma}(t,x)\cos(\xi x)dx + \int_{t^{\alpha/2\beta}}^{\infty} p_{\sigma,\gamma}(t,x)\cos(\xi x)dx\right) \\
&= 2t^{-\sigma-\frac{\alpha\gamma}{\beta}}\left(\int_0^1 p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx + \int_1^{\infty} p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx\right).
\end{aligned}$$

The last equality holds due to (5.2). Set Bromwich contours

$$L_{<c} := \left\{z \in \mathbb{C} : \Re[z] = \ell_{<c}\right\}, \quad L_{>c} := \left\{z \in \mathbb{C} : \Re[z] = \ell_{>c}\right\},$$

where

$$\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_{<c} < -\frac{\gamma}{\beta}, \quad -\frac{\gamma}{\beta} < \ell_{>c} < 0.$$

By (3.13),

$$\begin{aligned}
&\int_0^1 \int_{L_{<c}} \left|2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)x^{-2\gamma-2\beta z-1}\right| |dz|dx \\
&+ \int_1^{\infty}\int_{L_{>c}} \left|2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)x^{-2\gamma-2\beta z-1}\right| |dz|dx < \infty.
\end{aligned}$$

Therefore, by the Fubini theorem,

$$\begin{aligned}
&\int_0^1 p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx \\
&= \frac{2^{2\gamma}\pi^{-1/2}}{2\pi\mathrm{i}}\int_0^1\int_{L_{<c}} 2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)x^{-2\gamma-2\beta z-1}\cos(t^{\frac{\alpha}{2\beta}}\xi x)dzdx \\
&= \frac{2^{2\gamma}\pi^{-1/2}}{2\pi\mathrm{i}}\int_{L_{<c}} 2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)\left[\int_0^1 x^{-2\gamma-2\beta z-1}\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx\right]dz.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_1^{\infty} p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx \\
&= \frac{2^{2\gamma}\pi^{-1/2}}{2\pi\mathrm{i}}\int_{L_{>c}} 2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)\left[\int_1^{\infty} x^{-2\gamma-2\beta z-1}\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx\right]dz.
\end{aligned}$$

For $\eta \in \mathbb{R}$, it holds that (see [31, (1.8.1.1), (1.8.1.2)])

$$\int_0^1 x^{-2\lambda-1}\cos(\eta x)dx = -\frac{{}_1F_2\left(-\lambda;\frac{1}{2},1-\lambda;-\frac{\eta^2}{4}\right)}{2\lambda},\quad \Re[\lambda]<0$$

and

$$\begin{aligned}&\int_1^\infty x^{-2\lambda-1}\cos(\eta x)dx\\ &=\frac{\Gamma(-2\lambda)\cos\lambda\pi}{|\eta|^{-2\lambda}}+\frac{{}_1F_2\left(-\lambda;\frac{1}{2},1-\lambda;-\frac{\eta^2}{4}\right)}{2\lambda},\ \Re[\lambda]>-\frac{1}{2},\end{aligned}$$

where ${}_1F_2\left(-\lambda;\frac{1}{2},1-\lambda;-\frac{\eta^2}{4}\right)$ denotes the general hypergeometric function, i.e.,

$$\begin{aligned}&{}_1F_2(p;q,r;z)\\ &=\frac{\Gamma(q)\Gamma(r)}{\Gamma(p)}\sum_{k=0}^\infty\frac{\Gamma(p+k)}{\Gamma(q+k)\Gamma(r+k)}\frac{z^k}{k!},\ p\in\mathbb{C},\ q,r\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.\end{aligned}$$

Observe that for $\gamma+\beta z\in\mathbb{C}\setminus\mathbb{N}$,

$$\begin{aligned}&{}_1F_2\left(-\gamma-\beta z;\frac{1}{2},1-\gamma-\beta z;-\frac{t^{\frac{\alpha}{\beta}}|\xi|^2}{4}\right)\\
&=\frac{\sqrt{\pi}\Gamma(1-\gamma-\beta z)}{\Gamma(-\gamma-\beta z)}\sum_{k=0}^\infty\frac{\Gamma(-\gamma-\beta z+k)}{\Gamma(\frac{1}{2}+k)\Gamma(1-\gamma-\beta z+k)}\frac{\left(-t^{\frac{\alpha}{\beta}}|\xi|^2/4\right)^k}{k!}\\
&=-\sum_{k=0}^\infty\frac{\sqrt{\pi}(\gamma+\beta z)}{\Gamma(\frac{1}{2}+k)k!(-\gamma-\beta z+k)}\left(-\frac{t^{\frac{\alpha}{\beta}}|\xi|^2}{4}\right)^k\\
&=1-\sum_{k=1}^\infty\frac{\gamma+\beta z}{(2k-1)!2k(-\gamma-\beta z+k)}\left(-\frac{t^{\frac{\alpha}{\beta}}|\xi|^2}{4}\right)^k=:F_{t,\xi}(z).\end{aligned}$$

Since the series in $F_{t,\xi}(z)$ converges absolutely for fixed t,ξ, one can easily see that $F_{t,\xi}(z)$ is holomorphic in $\{z\in\mathbb{C}:-\frac{2\gamma-1}{2\beta}<\Re[z]<-\frac{2\gamma+1}{2\beta}\}$ by Morera's theorem. So we obtain

$$\int_0^1 p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx=-\frac{2^{2\gamma}\pi^{-1/2}}{2\pi\mathrm{i}}\int_{L_{<c}}2^{2\beta z}\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)}F_{t,\xi}(z)dz,$$

and

$$\begin{aligned}&\int_1^\infty p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx\\
&=\frac{2^{2\gamma}\pi^{-1/2}}{2\pi\mathrm{i}}\left(\int_{L_{>c}}2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)\Gamma(-2\gamma-2\beta z)\cos\left((\gamma+\beta z)\pi\right)\left|t^{\frac{\alpha}{2\beta}}\xi\right|^{2\gamma+2\beta z}dz\right.\end{aligned}$$

$$+\int_{L_{>c}} 2^{2\beta z}\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)}F_{t,\xi}(z)dz\Bigg).$$

If $\gamma \neq \beta$, by (3.1),

$$\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)} = -\frac{\Gamma(\frac{1}{2}+\gamma+\beta z)\Gamma(1+z)\Gamma(-z)}{2\Gamma(1-\gamma-\beta z)\Gamma(1-\sigma+\alpha z)}.$$

Thus $z=-\frac{\gamma}{\beta}$ is a removable singularity. These facts lead to

$$\operatorname{Res}_{z=-\frac{\gamma}{\beta}}\left[2^{2\beta z}\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)}F_{t,\xi}(z)\right]=0 \tag{6.6}$$

and

$$\frac{1}{2\pi \mathrm{i}}\int_{L_{>c}} 2^{2\beta z}\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)}F_{t,\xi}(z)dz = \frac{1}{2\pi \mathrm{i}}\int_{L_{<c}} 2^{2\beta z}\frac{\mathcal{H}_{\sigma,\gamma}(z)}{2(\gamma+\beta z)}F_{t,\xi}(z)dz.$$

Therefore,

$$\int_0^\infty p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx$$

$$= \frac{2^{2\gamma}\pi^{-1/2}t^{\frac{\alpha\gamma}{\beta}}|\xi|^{2\gamma}}{2\pi \mathrm{i}}\int_{L_{>c}} 2^{2\beta z}\mathcal{H}_{\sigma,\gamma}(z)\Gamma(-2\gamma-2\beta z)\cos\left((\gamma+\beta z)\pi\right)\left(t^\alpha|\xi|^{2\beta}\right)^z dz.$$

By (3.2),

$$\mathcal{H}_{\sigma,\gamma}(z)\Gamma(-2\gamma-2\beta z)\cos\left((\gamma+\beta z)\pi\right) = \pi^{1/2}2^{-2\gamma-2\beta z-1}\frac{\Gamma(1+z)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)}.$$

Hence

$$\begin{aligned}\mathcal{F}\left\{p_{\sigma,\gamma}(t,\cdot)\right\}(\xi) &= 2t^{-\sigma-\frac{\alpha\gamma}{\beta}}\int_0^\infty p_{\sigma,\gamma}(1,x)\cos(t^{\frac{\alpha}{2\beta}}\xi x)dx\\ &= \frac{|\xi|^{2\gamma}t^{-\sigma}}{2\pi \mathrm{i}}\int_L \mathcal{H}_{\sigma,\gamma}(z)\frac{\Gamma(1+z)\Gamma(-z)}{\Gamma(1-\sigma+\alpha z)}\left(t^\alpha|\xi|^{2\beta}\right)^z dz\\ &= |\xi|^{2\gamma}t^{-\sigma}E_{\alpha,1-\sigma}\left(-|\xi|^{2\beta}t^\alpha\right).\end{aligned}$$

Case 3: $d=1,\gamma=0$**.**

Let

$$\max(-1,-\frac{1}{\alpha},-\frac{1}{2\beta})<\ell_0<1.$$

Again, we follow the argument in Case 2. Note that

$$\frac{\mathcal{H}_{\sigma,0}(z)}{2\beta z} = -\frac{\Gamma(\frac{1}{2}+\beta z)\Gamma(1+z)\Gamma(-z)}{2\Gamma(1-\beta z)\Gamma(1-\sigma+\alpha z)}$$

has a removable singularity at $z=0$ if and only if $\sigma\in\mathbb{N}$. Thus (6.6) holds if $\sigma=1$ and we immediately obtain

$$\mathcal{F}\left\{p_{1,0}(t,\cdot)\right\} = t^{-1}E_{\alpha,0}(-|\xi|^{2\beta}t^\alpha) = \frac{\partial}{\partial t}E_\alpha(-|\xi|^{2\beta}t^\alpha).$$

By Remark 5.4 and Theorem 5.3,

$$\frac{\partial}{\partial t}\mathcal{F}\left\{p(t,\cdot)\right\} = \mathcal{F}\{\frac{\partial p}{\partial t}(t,\cdot)\} = \mathcal{F}\left\{p_{1,0}(t,\cdot)\right\}.$$

Thus,

$$\mathcal{F}\left\{p(t,\cdot)\right\}(\xi) = E_\alpha(-|\xi|^{2\beta}t^\alpha) + R(\xi).$$

By (5.2),

$$\begin{aligned}\mathcal{F}\left\{p(t,\cdot)\right\}(\xi) &= \int_{\mathbb{R}} e^{-\mathrm{i}x\xi}p(t,x)dx = \int_{\mathbb{R}} e^{-\mathrm{i}x\xi}t^{-\frac{\alpha}{2\beta}}p(1,t^{-\frac{\alpha}{2\beta}}x)dx \\ &= \int_{\mathbb{R}} e^{-\mathrm{i}t^{\alpha/2\beta}x\xi}p(1,x)dx = \mathcal{F}\left\{p(1,\cdot)\right\}(t^{\frac{\alpha}{2\beta}}\xi).\end{aligned}$$

Then by the Riemann-Lebesgue lemma, $\mathcal{F}\left\{p(t,\cdot)\right\}(\xi)$ converges to 0 as $t\to\infty$. This implies $R(\xi)\equiv 0$ and we obtain the desired result.

Case 4: $d=1$, $\gamma=\beta$**.**

Now we additionally assume $\sigma+\alpha\in\mathbb{N}$ and take ℓ_0 so that

$$\max(-2,-1-\frac{1}{2\beta}) < \ell_0 < 0.$$

Note that every argument in Case 2 holds except that

$$\frac{\mathcal{H}_{\sigma,\beta}(z)}{2\beta(1+z)} = -\frac{\Gamma(\frac{1}{2}+\beta+\beta z)\Gamma(2+z)\Gamma(-z)}{2\Gamma(1-\beta-\beta z)\Gamma(1-\sigma+\alpha z)(z+1)}$$

has a removable singularity at $z=-1$ if $\sigma+\alpha\in\mathbb{N}$. Thus (6.6) holds and we obtain (6.1). The theorems are proved.

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