BLOCH-TYPE SPACES ON THE UPPER HALF-PLANE

XI FU AND JUNDING ZHANG

ABSTRACT. We define Bloch-type spaces of $C^1(\mathbb{H})$ on the upper half plane $\mathbb{H}$ and characterize them in terms of weighted Lipschitz functions. We also discuss the boundedness of a composition operator $C_\phi$ acting between two Bloch spaces. These obtained results generalize the corresponding known ones to the setting of upper half plane.

1. Introduction

Let $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ be the upper half-plane, and $C^1(\mathbb{H})$ be the set of all complex-valued functions having continuous partial derivatives on $\mathbb{H}$. For $\alpha > 0$, the $\alpha$-Bloch space on $\mathbb{H}$, denoted by $B^\alpha$, is defined to be the space of all functions $f \in C^1(\mathbb{H})$ such that

$$\| f \|_\alpha = \sup_{z \in \mathbb{H}} (\text{Im} z)^\alpha (| f_z(z) | + | f_{\bar{z}}(z) |) < \infty,$$

It is easy to check that the space $B^\alpha$ is a Banach space with the norm

$$\| f \|_{B^\alpha} = | f(i) | + \| f \|_\alpha.$$

Let $\omega : [0, +\infty) \to [0, +\infty)$ be an increasing function with $\omega(0) = 0$, we say that $\omega$ is a majorant if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [6]). Following [2], given a majorant $\omega$ and $\alpha > 0$, the $\omega$-$\alpha$-Bloch space $B^\alpha_\omega$ consists of all functions $f \in C^1(\mathbb{H})$ such that

$$\| f \|_{B^\alpha_\omega} = \sup_{z \in \mathbb{H}} \omega((\text{Im} z)^\alpha) (| f_z(z) | + | f_{\bar{z}}(z) |) < \infty,$$

and the little $\omega$-$\alpha$-Bloch space $B^\alpha_{\omega,0}$ consists of the functions $f \in B^\alpha_\omega$ such that

$$\lim_{z \to \partial^\infty \mathbb{H}} \omega((\text{Im} z)^\alpha) (| f_z(z) | + | f_{\bar{z}}(z) |) = 0,$$

where $\partial^\infty \mathbb{H}$ denotes the union of $\partial \mathbb{H}$ and $\{ \infty \}$.

If we denote by $\tilde{B}^\alpha_\omega$ the set of all functions $f \in B^\alpha_\omega$ such that $\omega((\text{Im} z)^\alpha) (| f_z(z) | + | f_{\bar{z}}(z) |)$ vanishing at $\infty$, then the condition (2) is equivalent to the condition

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that the function $f \in \tilde{B}_s^\omega$ satisfies
\begin{equation}
\lim_{\text{Im} z \to 0} \omega((\text{Im} z)^\alpha) (|f_z(z)| + |f_{\bar{z}}(z)|) = 0.
\end{equation}
In particular, when $\omega(t) = t$, we remark that the space $\tilde{B}_s^\omega$ is the $\alpha$-Bloch space $B^\alpha$.

For $0 < \alpha \leq 1$, the weighted Poincaré metric $ds_\alpha$ of $\mathbb{H}$, introduced in [1] is defined as
\[ ds_\alpha^2 = \frac{|dz|^2}{(\text{Im} z)^{2\alpha}}. \]
Suppose that $\gamma(t)(0 \leq t \leq 1)$ is a continuous and piecewise smooth curve in $\mathbb{H}$. Then the length of $\gamma(t)$ with respect to the weighted Poincaré metric $ds_\alpha$ is equal to
\[ L_{p_\alpha}(\gamma) = \int_0^1 ds_\alpha = \int_0^1 \frac{|\gamma'(t)|}{|\text{Im} \gamma(t)|^\alpha} dt. \]
Consequently, the associated distance between $z$ and $w$ in $\mathbb{H}$ is
\[ p_\alpha(z, w) = \inf \{ L_{p_\alpha}(\gamma) : \gamma(0) = z, \gamma(1) = w \}, \]
where $\gamma$ is a continuous and piecewise smooth curve in $\mathbb{H}$. Note that $p_1 (\alpha = 1)$ is the classical Poincaré distance in $\mathbb{H}$.

Let $\mu, \nu \geq 0$ and $f$ be a continuous function in $\mathbb{H}$. If there exists a constant $C$ such that
\[ (\text{Im} z)^\mu (\text{Im} w)^\nu |f(z) - f(w)| \leq C|z - w| \quad (\text{resp.} \leq C p_\alpha(z, w)) \]
for any $z, w \in \mathbb{H}$, then we say that $f$ is a weighted Euclidian (resp. hyperbolic) Lipschitz function of indices $(\mu, \nu)$. In particular, when $\mu = \nu = 0$, we say that $f$ is a Euclidian (resp. hyperbolic) Lipschitz function (cf. [14]).

In the theory of function spaces, the relationship between Bloch spaces and (weighted) Lipschitz functions has attracted much attention. In 1986, Holland and Walsh ([8]) established a classical criterion for analytic Bloch space in the unit disc $\mathbb{D}$ in terms of weighted Euclidian Lipschitz functions of indices $(\frac{1}{2}, \frac{1}{4})$. Since then, a series of work has been carried out to characterize Bloch, $\alpha$-Bloch, little $\alpha$-Bloch and Besov spaces of holomorphic and harmonic functions along this line. For instance, Ren and Tu [15] extended Holland and Walsh’s criterion to the Bloch space in the unit ball of $\mathbb{C}^n$, Li and Wulan [9], Zhao [18] characterized holomorphic $\alpha$-Bloch space in terms of $(1 - |z|^2)^{\frac{\alpha}{2}} (1 - |w|^2)^{\alpha - \beta} |f(z) - f(w)|/|z - w|$. In [19, 20], Zhu investigated the relationship between Bloch spaces and hyperbolic Lipschitz functions and proved that a holomorphic function belongs to Bloch space if and only if it is hyperbolic Lipschitz. For the related results of harmonic functions, we refer to [2, 3, 4, 7, 14] and the references therein.

Motivated by the known results mentioned above, we consider the corresponding problems in the setting of $C^1(\mathbb{H})$ in this paper. In Section 2, we collect some known results that will be needed in the sequel. The main results and their proofs are presented in Sections 3 and 4.
Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other. The notation $A \simeq B$ means that there is a positive constant $C$ such that $B/C \leq A \leq CB$.

2. Several lemmas

In this section, we introduce some notations and collect some preliminary results that we need later.

Let $z, w \in \mathbb{H}$, the pseudo-hyperbolic distance $\rho$ is defined as

$$\rho(z, w) = \left| \frac{z - w}{z - \overline{w}} \right|.$$

It is easy to see that $\rho$ on $\mathbb{H}$ is a distance function and horizontal translation and dilatation invariant (cf. [11]). For $z \in \mathbb{H}$ and $r \in (0, 1)$, the pseudo-hyperbolic ball with center $z$ and radius $r$ is denoted by

$$E(z, r) = \{ w \in \mathbb{H} : \rho(z, w) < r \}.$$

A straightforward calculation shows that $E(z, r)$ is a Euclidean ball $B(x^*, r_0)$ where

$$x^* = (x, \frac{1 + r^2}{1 - r^2}y), \quad r_0 = \frac{2ry}{1 - r^2}, \quad \text{and} \quad z = x + yi.$$ 

The following lemma is proved in [11, Lemma 2.1].

**Lemma 2.1.** The inequality

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \left| \frac{z - w}{z - \overline{w}} \right| \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$$

holds for all $z, w, u \in \mathbb{H}$.

As an application of Lemma 2.1, we easily get the following (see [4]).

**Corollary 2.1.** Let $r \in (0, 1)$, $u, v \in E(z, r)$. Then we have

$$\text{Im } u \asymp \text{Im } v \asymp |z - \overline{w}| \asymp |E(z, r)|^\frac{1}{2},$$

where $|E(z, r)|$ denotes the area of $E(z, r)$.

We end this section with two inequalities which will be used in the sequel.

**Lemma 2.2 ([2, Lemma 6]).** Let $\omega(t)$ be a majorant and $u \in (0, 1), v \in (1, \infty)$. Then for $t \in (0, \infty)$,

$$\omega(ut) \geq u\omega(t), \quad \omega(vt) \leq v\omega(t).$$

**Lemma 2.3.** Let $a, b > 0$, $0 < s < 1$. Then

$$sa + (1 - s)b \geq a^s b^{1-s}.$$
3. Bloch spaces

Let \( f \) be a harmonic Bloch mapping in the unit disc \( D \). In [5], Colonna proved that the Bloch constant \( B_f \) of \( f \) is equals to its Bloch semi-norm, i.e.,

\[
B_f = \sup_{z, w \in D, z \neq w} \frac{|f(z) - f(w)|}{h(z, w)} = \sup_{z \in D} (1 - |z|^2)(|f_z(z)| + |f_w(z)|),
\]
where \( h \) is the hyperbolic distance in \( \mathbb{H} \).

Firstly, we characterize the space \( B^\alpha \) in terms of hyperbolic Lipschitz functions and generalize Colonna’s result to the setting of \( \mathcal{C}^1(\mathbb{H}) \).

**Theorem 3.1.** Let \( f \in \mathcal{C}^1(\mathbb{H}) \) and \( 0 < \alpha \leq 1 \). Then \( f \in B^\alpha \) if and only if there is a constant \( C > 0 \) such that

\[
|f(z) - f(w)| \leq C p_\alpha(z, w), \quad z, w \in \mathbb{H}.
\]
Moreover, we have

\[
\|f\|_\alpha = \sup_{z, w \in \mathbb{H}, z \neq w} \frac{|f(z) - f(w)|}{p_\alpha(z, w)}
\]
for all \( f \in B^\alpha \).

**Proof.** We first prove the sufficiency. For any \( z, w \in \mathbb{H} \), by the definition of \( p_\alpha(z, w) \), we assume that \( \gamma(s) \) is the geodesic between \( z \) and \( w \) (parametrized by arc-length) with respect to \( p_\alpha \). Since \( p_\alpha(\gamma(0), \gamma(s)) = s \), we have

\[
|f(z) - f(w)| \leq C s.
\]
Dividing both sides by \( s \) and then letting \( s \to 0 \) in the above inequality gives

\[
(\|f_z(z)\| + \|f_w(z)\|) \gamma'(0) \leq C.
\]
From the minimal length property of geodesics,

\[
p_\alpha(\gamma(0), \gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{|\text{Im} \gamma(t)|^{\alpha}} dt = s, \quad 0 < s < \epsilon,
\]
we obtain that

\[
\lim_{s \to 0} \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{|\text{Im} \gamma(t)|^{\alpha}} dt = \frac{|\gamma'(0)|}{|\text{Im} z|^{\alpha}} = 1.
\]
It follows that \( (\text{Im} z)^{\alpha}(\|f_z(z)\| + \|f_w(z)\|) \leq C \) and hence \( f \in B^\alpha \) with

\[
(\text{Im} z)^{\alpha}(\|f_z(z)\| + \|f_w(z)\|) : z \in \mathbb{H} \leq \sup \{ \frac{|f(z) - f(w)|}{p_\alpha(z, w)} : z \neq w \}.
\]
Conversely, we assume that \( f \in B^\alpha \). Let \( z, w \in \mathbb{H} \) and \( \gamma(t)(0 \leq t \leq 1) \) be a smooth curve from \( z \) to \( w \). Then we have

\[
|f(z) - f(w)| = | \int_0^1 \frac{df}{dt}(\gamma(t)) dt |
\]
\[
\leq \int_0^1 (|f_z(\gamma(t))| + |f_w(\gamma(t))|)|\gamma'(t)| dt
\]
\[ \leq \|f\|_\alpha \int_0^1 \frac{|\gamma'(t)|}{|\text{Im}\gamma(t)|^\alpha} \, dt \]
\[ \leq \|f\|_{\alpha P_\alpha}(\gamma(t)). \]

Taking the infimum over all piecewise continuous curves connecting \( z \) and \( w \), we conclude that
\[ |f(z) - f(w)| \leq \|f\|_{\alpha P_\alpha}(z, w) \]
for all \( z, w \in \mathbb{H} \). This completes the proof. \( \square \)

In the following, we characterize the spaces \( B_\omega^\alpha, B_0^\alpha \) in terms of Euclidean weighted Lipschitz functions.

**Theorem 3.2.** Let \( r \in (0, 1) \), \( f \in C^1(\mathbb{H}) \), \( 0 < \beta \leq \alpha \). Then \( f \in B_\omega^\alpha \) if and only if
\[ K = \sup_{w \in E(z, r), z \neq w} \omega((\text{Im}z)^\beta(\text{Im}w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} < \infty. \]

**Proof.** We first prove the sufficiency. Let \( f \in C^1(\mathbb{H}) \). For each \( z \in \mathbb{H} \), we have
\[ K \sup_{w \in E(z, r), z \neq w} \left| \frac{f(z) - f(w)}{|z - w|} \right| \leq \frac{K}{\omega((\text{Im}z)^\beta(\text{Im}w)^{\alpha-\beta})}. \]
By letting \( w \to z \), we obtain that
\[ |f_z(z)| + |f_\bar{z}(z)| \leq \frac{K}{\omega((\text{Im}z)^\alpha)}, \]
from which we conclude that \( f \in B_\omega^\alpha \).

Conversely, let \( f \in B_\omega^\alpha \) and for any \( w \in E(z, r), z \neq w \),
\[ |f(z) - f(w)| = \left| \int_0^1 \frac{df}{ds}(sz + (1 - s)w)ds \right| \]
\[ \leq |z - w| \int_0^1 \left( |\frac{\partial f}{\partial s}(sz + (1 - s)w)| + |\frac{\partial f}{\partial \bar{z}}(sz + (1 - s)w)| \right) ds \]
\[ \leq C|z - w| \|f\|_{\omega, \alpha} \int_0^1 \frac{ds}{\omega((\text{Im}(sz + (1 - s)w))^{\alpha})} \]
\[ \leq C|z - w| \int_0^1 \frac{ds}{\omega((\text{Im}z)^{\alpha\lambda}(\text{Im}w)^{\alpha-\alpha\lambda})}, \]
where the last inequality follows from Lemma 2.3.

Since for each \( w \in E(z, r), z \neq w \), \( \text{Im}z \lesssim \text{Im}w \), we can find a \( \lambda \in (0, 1) \) such that \( \text{Im}z \geq \lambda(\text{Im}w) \) and \( \text{Im}w \geq \lambda(\text{Im}z) \). Then we infer that
\[ \frac{|f(z) - f(w)|}{|z - w|} \leq C \int_0^1 \frac{ds}{\omega((\text{Im}z)^{\alpha\lambda}(\text{Im}w)^{\alpha-\alpha\lambda})} \]
\[ \leq C \int_0^1 \frac{ds}{\omega((\text{Im}z)^{\alpha\lambda\lambda}(\text{Im}w)^{\alpha-\alpha\lambda})}. \]
Theorem 3.3. Let
\[ \lim_{t \to 0} \frac{ds}{\lambda^{a-\alpha}} \]
whenever \( 0 < \omega \leq 1342 \) and
\[ C \omega \leq \omega. \]

By Lemma 2.2 again, we deduce that
\[ \omega((\text{Im}z)^\alpha) \geq \lambda^{a-\beta} \omega((\text{Im}w)^{\alpha-\beta}), \]

from which we see that \( K < \infty \). The proof of Theorem 3.2 is completed. \( \square \)

**Theorem 3.3.** Let \( r \in (0, 1) \), \( f \in \mathcal{B}_{\omega}^0 \), \( 0 < \beta \leq \alpha \). Then \( f \in \mathcal{B}_{\omega,0}^0 \) if and only if
\[ \lim_{\text{Im}z \to 0} \sup_{w \in E(z, r), z \neq w} \frac{\omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta}) |f(z) - f(w)|}{|z - w|} = 0. \]

**Proof.** Sufficiency. Assume that (4) holds. Then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[ \sup_{w \in E(z, r), z \neq w} \frac{\omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta}) |f(z) - f(w)|}{|z - w|} < \epsilon \]
whenever \( 0 < \text{Im}z < \delta \). It follows by an argument similar to that in the proof of Theorem 3.2, we have
\[ \omega((\text{Im}z)^\alpha)(|f_z(z)| + |f_{\bar{z}}(z)|) < C \omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} < C \epsilon, \]
whenever \( 0 < \text{Im}z < \delta, w \in E(z, r) \). Hence
\[ \lim_{\text{Im}z \to 0} \frac{\omega((\text{Im}z)^\alpha)(|f_z(z)| + |f_{\bar{z}}(z)|)}{\omega((\text{Im}w)^{\alpha-\beta})} = 0. \]

Necessity. Now we assume that \( f \in \mathcal{B}_{\omega,0}^0 \). For \( t \in (0, +\infty) \), let \( f_t(z) = f(z + ti) \). By the proof of Theorem 3.2, for \( w \in E(z, r) \), we have
\[ \omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta}) \frac{|f_t(z) - f_t(w)|}{|z - w|} \leq C \| f - f_t \|_{\omega,0} \]
and
\[ \omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta}) \frac{|f_t(z) - f_t(w)|}{|z - w|} \]
\[ = \frac{C \omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta})}{\omega((\text{Im}z + t)^\beta (\text{Im}w + t)^{\alpha-\beta})} \frac{\omega((\text{Im}z + t)^\beta (\text{Im}w + t)^{\alpha-\beta}) |f(z + ti) - f(w + ti)|}{|z - w|} \leq \frac{C \omega((\text{Im}z)^\beta (\text{Im}w)^{\alpha-\beta})}{\omega((\text{Im}z + t)^\beta (\text{Im}w + t)^{\alpha-\beta})} \| f \|_{\omega,0}. \]
By the triangle inequality,
\[
\sup_{w \in E(z, r), z \neq w} \omega((\text{Im} z)^\beta(\text{Im} w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} \leq C\|f - f_t\|_{\omega, \alpha} + \frac{C\omega((\text{Im} z)^\beta(\text{Im} w)^{\alpha-\beta})}{\omega((\text{Im} z + t)^\beta(\text{Im} w + t)^{\alpha-\beta})} \|f\|_{\omega, \alpha}.
\]
In the above inequality, first by letting $\text{Im} z \to 0$ and then letting $t \to 0$, we obtain the desired result. □

**Remark 3.1.** When $\omega(t) = t$, Li and Wulan [9] obtained the analogues of Theorems 3.2 and 3.3 for holomorphic Bloch space on the unit ball of $\mathbb{C}^n$.

In the following, we remove the restriction $w \in E(z, r)$ in Theorems 3.2 and 3.3 and obtain the following results which can be viewed as generalizations of [2, Theorems 3, 5] to the case of $C^1(\mathbb{H})$.

**Theorem 3.4.** Let $f \in C^1(\mathbb{H})$, $0 < \beta < 1$, $\beta \leq \alpha < 1 + \beta$. Then $f \in B_{\omega}^{\alpha}$ if and only if
\[
\sup_{z, w \in \mathbb{H}, z \neq w} \omega((\text{Im} z)^\beta(\text{Im} w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} < \infty.
\]

**Proof.** Assume that (5) holds. Fix $r \in (0, 1)$ and $x \in \mathbb{H}$, it follows from the proof of Theorem 3.2, we can easily prove that $f \in B_{\omega}^{\alpha}$. For the converse, we assume that $f \in B_{\omega}^{\alpha}$. Then for $z, w \in \mathbb{H}$,
\[
|f(z) - f(w)| \leq C|z - w| \int_0^1 \frac{ds}{\omega((\text{Im} (sz + (1-s)w))^\alpha)}.
\]
Since for $z, w \in \mathbb{H}$ and $s \in [0, 1]$,
\[
(s\text{Im} z + (1-s)\text{Im} w)^{\alpha} \geq (s\text{Im} z)^\beta((1-s)\text{Im} w)^{\alpha-\beta},
\]
we get that
\[
|f(z) - f(w)| \leq C|z - w| \int_0^1 \frac{ds}{\omega((s\text{Im} z + (1-s)\text{Im} w)^{\alpha})}
\leq C|z - w| \int_0^1 \frac{ds}{\omega(s^{\beta}(1-s)^{\alpha-\beta})(\text{Im} z)^{\beta}(\text{Im} w)^{\alpha-\beta})}
\leq C|z - w| \int_0^1 \frac{ds}{s^{\beta}(1-s)^{\alpha-\beta}}
\leq \frac{C|z - w|}{\omega((\text{Im} z)^\beta(\text{Im} w)^{\alpha-\beta})},
\]
where the last integral converges since $\alpha < 1 + \beta$. Thus
\[
\sup_{z, w \in \mathbb{H}, z \neq w} \omega((\text{Im} z)^\beta(\text{Im} w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} < \infty.
\]
This completes the proof of Theorem 3.4. □
Similarly, we can prove the following.

**Theorem 3.5.** Let \( f \in \tilde{B}_\alpha^\omega \), \( 0 < \beta < 1 \), \( \beta \leq \alpha < 1 + \beta \). Then \( f \in B_\alpha^\omega \) if and only if

\[
\lim_{\operatorname{Im} z \to 0} \sup_{z, w \in \mathbb{H}, z \neq w} \omega((\operatorname{Im} z)^\beta (\operatorname{Im} w)^{\alpha - \beta}) \frac{|f(z) - f(w)|}{|z - w|} = 0.
\]

4. Composition operators

Let \( \phi \) be a holomorphic self-mapping of \( \mathbb{H} \). The composition operator \( C_\phi \), induced by \( \phi \) is defined by \( C_\phi(f) = f \circ \phi \) for \( f \in C^1(\mathbb{H}) \). During the past few years, composition operators have been studied extensively on spaces of holomorphic functions on various domains in \( \mathbb{C} \) and \( \mathbb{C}^n \), see e.g., [13, 10, 16, 21]. In this section, we discuss the boundedness of composition operators between Bloch spaces on the upper half plane \( \mathbb{H} \).

**Theorem 4.1.** Let \( \alpha, \beta > 0 \) and \( \phi \) be a holomorphic self-mapping of \( \mathbb{H} \). Then \( C_\phi : B^\alpha \to B^\beta \) is bounded if and only if

\[
\sup_{z \in \mathbb{H}} \frac{(\operatorname{Im} z)^\beta |\phi'(z)|}{(\operatorname{Im} \phi(z))^\alpha} < \infty.
\]

**Proof.** First suppose that

\[
M = \sup_{z \in \mathbb{H}} \frac{(\operatorname{Im} z)^\beta |\phi'(z)|}{(\operatorname{Im} \phi(z))^\alpha} < \infty.
\]

For \( f \in B^\alpha \) and \( z \in \mathbb{H} \), we have

\[
(\operatorname{Im} z)^\beta (|C_\phi(f)z| + |C_\phi(f)\overline{z}|) = (\operatorname{Im} z)^\beta ((f \circ \phi)_z(z) + (f \circ \phi)_{\overline{z}}(z))
\]

\[
= (\operatorname{Im} z)^\beta |\phi'(z)|(f_z(\phi(z)) + f_{\overline{z}}(\phi(z)))
\]

\[
\leq M(\operatorname{Im} \phi(z))^{\alpha} (|f_z(\phi(z))| + |f_{\overline{z}}(\phi(z))|)
\]

\[
\leq C\|f\|_{\alpha}.
\]

and

\[
|f(\phi(i))| \leq C\|f\|_{\alpha}.
\]

Hence \( C_\phi : B^\alpha \to B^\beta \) is bounded.

For the converse, assume that \( C_\phi : B^\alpha \to B^\beta \) is a bounded operator with

\[
\|C_\phi(f)\|_{\beta} \leq C\|f\|_{\alpha}
\]

for all \( f \in B^\alpha \). Fix a point \( z_0 \in \mathbb{H} \) and let \( w = \phi(z_0) \). If \( \alpha \neq 1 \), consider the function \( f_w(z) = (z - \overline{w})^{1-\alpha} \). Then it is easy to check that \( f_w \in B^\alpha \). The boundedness of \( C_\phi \) implies that

\[
\frac{(\operatorname{Im} z)^\beta |\phi'(z)|}{|\phi(z) - \overline{w}|^\alpha} \leq C.
\]
In particular, take \( z = z_0 \), we get
\[
\frac{\left| \text{Im} z_0 \right|^\beta |\phi'(z_0)|}{\left| \text{Im} \phi(z_0) \right|^{\alpha}} \leq C.
\]
Since \( z_0 \) is arbitrary, the result follows.

If \( \alpha = 1 \), we only need to consider the function \( f_w(z) = \ln(z-w) \). Following a discussion similar to the above, it can be proved that (6) holds. The proof of Theorem 4.1 is completed.

Recall that the classical Schwarz-Pick Lemma in the upper half-plane gives that for a holomorphic self-mapping \( \phi \) of \( \mathbb{H} \), \( (\text{Im} z) |\phi'(z)| \leq \text{Im} \phi(z) \) holds for all \( z \in \mathbb{H} \). As an application of this result, it is easy to derive the following corollary.

**Corollary 4.1.** Let \( \phi \) be a holomorphic self-mapping of \( \mathbb{H} \). Then \( C_\phi : B^1 \to B^1 \) is bounded.

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**References**


Xi Fu
Department of Mathematics
Shaoxing University
Shaoxing 312000, Zhejiang Province, P. R. China
E-mail address: fuxi1984@hotmail.com

Junding Zhang
Department of Mathematics
Shanghai University
Shangda Road No. 99, 200444, Shanghai, P. R. China
E-mail address: zjday521@i.shu.edu.cn