CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we obtain initial coefficient bounds for an unified subclass of analytic functions by using the Chebyshev polynomials. Furthermore, we find the Fekete-Szegö result for this class. All results are sharp. Consequences of the results are also discussed.

1. Introduction

Let \( A \) denote the class of all analytic functions in the unit disk \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \). Each function \( f \in A \) with the conditions \( f(0) = 0 \) and \( f'(0) = 1 \), has the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta).
\]

We shall denote by \( S \) the class of functions in \( A \) that are also univalent in \( \Delta \). We say that the function \( f \) is subordinate to \( g \) in \( \Delta \) and written as \( f \prec g \) if and only if \( f(z) = g(w(z)) \) for some analytic function \( w \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), for all \( z \in \Delta \).

A number of vital and well explored subclasses of class \( A \) are the class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) in \( \Delta \) and the class \( K(\alpha) \) of convex functions of order \( \alpha \) in \( \Delta \). By definition, we have

\[
S^*(\alpha) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha ; \ z \in \Delta, \ 0 \leq \alpha < 1 \right\}
\]

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\[ K(\alpha) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha ; \, z \in \Delta, \, 0 \leq \alpha < 1 \right\}, \]

respectively. In particular, we set \( S^*(0) = S^* \) and \( K(0) = K \).

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev. For a brief history of Chebyshev polynomials of first kind \( T_n(t) = \cos n\alpha \) \((-1 < t < 1)\), the second kind \( U_n(t) = \frac{\sin(n+1)\alpha}{\sin \alpha} \) \((-1 < t < 1)\) of the polynomials degree \( n \) and their applications one can refer \([2, 3, 4]\).

**Definition 1.1.** A function \( f \in A \) is said to be in the class \( G^\delta_{\lambda}(t) \), if the following subordination holds for all \( z \in \Delta(2) = (1 - \delta) \frac{zF_1'(z)}{F_\lambda(z)} + \delta \left( 1 + \frac{zF_n'(z)}{F_\lambda(z)} \right) \times H(z, t) = \frac{1}{1 - 2tz + z^2}, \]

where \( \delta \geq 0, \, 0 \leq \lambda \leq 1, \, t \in (1/2, 1] \) and

\[ F_\lambda(z) = (1 - \lambda)f(z) + \lambda zf'(z). \]

Further, we see, if \( t = \cos \alpha \), where \( \alpha \in (-\pi/3, \pi/3) \), then

\[ H(z, t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=0}^{\infty} \frac{\sin(n+1)\alpha}{\sin \alpha} z^n \quad (z \in \Delta). \]

Thus

\[ H(z, t) = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha) z^2 + \cdots \quad (z \in \Delta). \]

From \([5]\), we can write

\[ H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in \Delta, \, t \in (-1, 1)) \]

where \( U_{n-1} = \frac{\sin(n, \arccos t)}{\sqrt{1-t^2}} \) (\( n \in \mathbb{N} \)) are the Chebyshev polynomials of the second kind and we have

\[ U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \]

and

\[ U_1(t) = 2t, \, U_2(t) = 4t^2 - 1, \, U_3(t) = 8t^3 - 4t, \, \cdots. \]

The generating function of the first kind of Chebyshev polynomial \( T_n(t) \), \( t \in [-1, 1] \), is given by

\[ \sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in \Delta). \]
The first kind of Chebyshev polynomial $T_n(t)$ and second kind of Chebyshev polynomial $U_n(t)$ are connected by:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

**Remark 1.** It is interesting to note that for restricted values of the parameters involved in the class $G^\delta_\lambda(t)$ gives the following special sub-classes:

(i) A function $f \in A$ is said to be in the class $G^\delta_0(t) := K(\delta, t)$, $\delta \geq 0$ and $t \in (1/2, 1]$, if the following subordination holds:

$$(1 - \delta) \frac{zf'(z)}{f(z)} + \delta \left(1 + \frac{zf''(z)}{f'(z)}\right) < H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta).$$

This class was introduced and studied by Altınkaya and Yalçın [1].

(ii) A function $f \in A$ is said to be in the class $G^1_0(t) := H(t)$, $t \in (1/2, 1]$, if the following subordination holds:

$$1 + \frac{zf''(z)}{f'(z)} < H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta).$$

This class was introduced and studied by Dziok et al. [3].

(iii) A function $f \in A$ is said to be in the class $G^0_\lambda(t) := M(\lambda, t)$, $0 \leq \lambda \leq 1$ and $t \in (1/2, 1]$, if the following subordination holds:

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} < H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta).$$

(iv) A function $f \in A$ is said to be in the class $G^0_0(t) := M(t)$ and $t \in (1/2, 1]$, if the following subordination holds:

$$\frac{zf'(z)}{f(z)} < H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta).$$

(v) A function $f \in A$ is said to be in the class $G^1_\lambda(t) := N(\lambda, t)$, $0 \leq \lambda \leq 1$ and $t \in (1/2, 1]$, if the following subordination holds:

$$\frac{zf'(z) + (1 + 2\lambda) z^2 f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} < H(z,t) = \frac{1}{1 - 2tz + z^2},$$

where $z \in \Delta$.

In this present paper, motivated by the earlier works of Dziok et al. [3] and Altınkaya and Yalçın [1], we use the Chebyshev polynomials expansions to provide estimates for the initial coefficients of analytic functions in $G^\delta_\lambda(t)$. We also obtain Fekete-Szegö bounds.
2. MAIN RESULTS FOR THE FUNCTION CLASS $G_\delta^\lambda(t)$

Theorem 2.1. Let the function $f(z)$ given by (1) be in the class $G_\delta^\lambda(t)$. Then

\begin{equation}
|a_2| \leq \frac{2t}{(1+\delta)(1+\lambda)},
\end{equation}

\begin{equation}
|a_3| \leq \begin{cases}
\frac{t}{(1+2\delta)(1+2\lambda)}, & \frac{1}{2} < t \leq \frac{(1+\delta)^2+(1+\delta)\sqrt{55+22\delta+9}}{4(\delta^2+5\delta+2)} \\
\frac{8(\delta^2+5\delta+2)^2-(1+\delta)^2}{2(1+\delta)^2(1+2\delta)(1+2\lambda)}, & (1+\delta)^2+(1+\delta)\sqrt{55+22\delta+9} < t \leq 1
\end{cases}
\end{equation}

and

\begin{equation}
|a_3 - \eta a_2^2| \leq \begin{cases}
\frac{t}{(1+2\delta)(1+2\lambda)}, & \eta \in [\eta_1, \eta_2] \\
\frac{t}{(1+2\delta)(1+2\lambda)} \left[ \frac{4t^2-1}{2t^2} + \frac{2(1+2\delta)t}{(1+\delta)^2} - \eta \frac{4(1+2\delta)(1+2\lambda)t}{(1+\delta)^2(1+2\lambda)^2} \right], & \eta \notin [\eta_1, \eta_2]
\end{cases}
\end{equation}

where

\begin{equation}
\eta_1 = \frac{(1+\delta)^2[4(\delta^2+5\delta+2)^2-(1+\delta)^2(1+2\delta)]}{8(1+2\delta)(1+2\lambda)t^2}
\end{equation}

and

\begin{equation}
\eta_2 = \frac{(1+\delta)^2[4(\delta^2+5\delta+2)^2-(1+\delta)^2(1-2\delta)]}{8(1+2\delta)(1+2\lambda)t^2}.
\end{equation}

All of the inequalities are sharp.

Proof. Let the function $f(z)$ given by (1) be in the class $G_\delta^\lambda(t)$. From (2), we have,

\begin{equation}
(1-\delta) \frac{zF_\lambda''(z)}{F_\lambda(z)} + \delta \left(1 + \frac{zF_\lambda''(z)}{F_\lambda'(z)}\right) = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \cdots
\end{equation}

for some analytic function

\begin{equation}
w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots, \quad (z \in \Delta),
\end{equation}

such that $w(0) = 0$ and $|w(z)| < 1$. It is well-known that if $|w(z)| < 1$, for all $z \in \Delta$, then

\begin{equation}
|c_j| \leq 1, \quad \text{for all } j \in \mathbb{N}
\end{equation}

and

\begin{equation}
|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \quad \text{for all } \mu \in \mathbb{R}.
\end{equation}
From (9) and (10), we have

\begin{equation}
(1 - \delta) \frac{zF'_{\lambda}(z)}{F_{\lambda}(z)} + \delta \left( 1 + \frac{zF''_{\lambda}(z)}{F_{\lambda}(z)} \right) = 1 + U_1(t)c_1z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots.
\end{equation}

Equating the coefficients in (13), we get

\begin{equation}
(1 + \delta)(1 + \lambda)a_2 = U_1(t)c_1
\end{equation}

and

\begin{equation}
[2(1 + 2\delta)(1 + 2\lambda)a_3 - (1 + 3\delta)(1 + \lambda)^2a_2^2] = U_1(t)c_2 + U_2(t)c_1^2.
\end{equation}

Then, by using (3) in (14), we get

\[ a_2 = \frac{2c_1}{(1 + \delta)(1 + \lambda)}t. \]

Further, applying (11) in the above equation, we get desired bound \(|a_2|\) as given in (4).

By using (14) and (15) for some \( \eta \in \mathbb{R} \), we have

\[ |a_3 - \eta a_2^2| \leq \frac{U_1(t)}{2(1 + 2\delta)(1 + 2\lambda)} \left| c_2 - \left\{ \frac{-U_2(t)}{U_1(t)} - \frac{(1 + 3\delta)}{(1 + \delta)^2}U_1(t) + 2\eta \frac{(1 + 2\delta)(1 + 2\lambda)}{(1 + \delta)^2(1 + \lambda)^2}U_1(t) \right\} c_1^2 \right|. \]

From (12), it follows that

\[ |a_3 - \eta a_2^2| \leq \frac{U_1(t)}{2(1 + 2\delta)(1 + 2\lambda)} \max \left\{ 1, \left| \frac{U_2(t)}{U_1(t)} + \frac{(1 + 3\delta)}{(1 + \delta)^2}U_1(t) - 2\eta \frac{(1 + 2\delta)(1 + 2\lambda)}{(1 + \delta)^2(1 + \lambda)^2}U_1(t) \right| \right\}. \]

Next, using (3) in the above equation, we have,

\[ |a_3 - \eta a_2^2| \leq \frac{t}{(1 + 2\delta)(1 + 2\lambda)} \max \left\{ 1, \left| \frac{4t^2 - 1}{2t} + \frac{2(1 + 3\delta)t}{(1 + \delta)^2} - 4\eta \frac{(1 + 2\delta)(1 + 2\lambda)t}{(1 + \delta)^2(1 + \lambda)^2} \right| \right\}. \]

Since \( t > 0 \), we get

\begin{equation}
\left| \frac{4t^2 - 1}{2t} + \frac{2(1 + 3\delta)t}{(1 + \delta)^2} - 4\eta \frac{(1 + 2\delta)(1 + 2\lambda)t}{(1 + \delta)^2(1 + \lambda)^2} \right| \leq 1,
\end{equation}

if and only if \( \eta_1 \leq \eta \leq \eta_2 \) where \( \eta_1 \) and \( \eta_2 \) are given in (7) and (8). So we obtain (6). If we take \( \eta = 0 \), then we obtain the inequality (5).
The equality (9) with \( w(z) = z \) generate the function \( \hat{f} \in G^\delta_\lambda(t) \) such that
\[
\hat{f}(z) = z + \frac{2t}{(1 + \delta)(1 + \lambda)} z^2 + \frac{4(\delta^2 + 5\delta + 2)t^2 - (1 + \delta)^2}{2(1 + \delta)^2(1 + 2\delta)(1 + 2\lambda)} z^3 + \cdots,
\]
which shows that the inequalities (4) and (5) for
\[
\frac{(1 + \delta)^2 + (1 + \delta)\sqrt{5\delta^2 + 22\delta + 9}}{4(\delta^2 + 5\delta + 2)} \leq t \leq 1
\]
are sharp. Also, in this case
\[
|a_3 - \eta a_2^2| \leq \left\{
\begin{array}{ll}
\frac{1}{2} & < t \leq \frac{(1 + \delta)^2 + (1 + \delta)\sqrt{5\delta^2 + 22\delta + 9}}{4(\delta^2 + 5\delta + 2)}, \\
\frac{t}{1 + 2\delta}, & 0 < t \leq \frac{1}{2}.
\end{array}
\right.
\]

and
\[
|a_3| \leq \left\{
\begin{array}{ll}
\frac{t}{1 + 2\delta}, & 0 < t \leq \frac{1}{2}.
\end{array}
\right.
\]

which shows the sharpness of (5) for \( \frac{1}{2} < t \leq \frac{(1 + \delta)^2 + (1 + \delta)\sqrt{5\delta^2 + 22\delta + 9}}{4(\delta^2 + 5\delta + 2)} \), and (6) for \( \eta \in [\eta_1, \eta_2] \). This completes the proof of Theorem 2.1.

\[\square\]

3. Corollaries and Consequences

Taking different restricted values to parameters which are involved in
Theorem 2.1, we obtain the following corollaries.

Corollary 3.1. Let the function \( f(z) \) given by (1) be in the class \( K(\delta, t) \). Then
\[
|a_2| \leq \frac{2t}{1 + \delta},
\]
\[
|a_3| \leq \left\{ \begin{array}{ll} \frac{t}{1 + 2\delta}, & 0 < t \leq \frac{(1 + \delta)^2 + (1 + \delta)\sqrt{5\delta^2 + 22\delta + 9}}{4(\delta^2 + 5\delta + 2)}, \\
\frac{4((\delta^2 + 5\delta + 2)t^2 - (1 + \delta)^2)}{2(1 + \delta)^2(1 + 2\delta)(1 + 2\lambda)}, & \frac{(1 + \delta)^2 + (1 + \delta)\sqrt{5\delta^2 + 22\delta + 9}}{4(\delta^2 + 5\delta + 2)} < t \leq 1 \end{array} \right.
\]
and
\[
|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll} \frac{t}{1 + 2\delta}, & \eta \in [\eta_1, \eta_2] \\
\frac{4t^2 - 1}{2t} + \frac{2(1 + 3\delta)t}{(1 + \delta)^2} - \eta \frac{4(1 + 2\delta)t}{(1 + \delta)^2}, & \eta \notin [\eta_1, \eta_2] \end{array} \right.
\]
where
\[
\eta_1 = \frac{4((\delta^2 + 5\delta + 2)t^2 - (1 + \delta)^2(1 + 2t))}{8(1 + 2\delta)t^2}.
\]
and
\[ \eta_2 = \frac{4(\delta^2 + 5\delta + 2)t^2 - (1 + \delta)^2(1 - 2t)}{8(1 + 2\delta)t^2}. \]

All of the inequalities are sharp.

**Remark 2.** Note that Corollary 3.1 is an improvement of the results given by Altunkaya and Yalçın [1].

**Corollary 3.2 ([3]).** Let the function \( f(z) \) given by (1) be in the class \( \mathcal{H}(t) \). Then
\[
|a_2| \leq t, \\
|a_3| \leq \frac{4t^2}{3} - \frac{1}{6}
\]
and
\[
|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{t}{3}, & \eta \in [\eta_1, \eta_2] \\
\frac{8t^2 - 1 - 6\eta^2}{6}, & \eta \notin [\eta_1, \eta_2]
\end{array} \right.
\]
where
\[ \eta_1 = \frac{8t^2 - 2t - 1}{6t^2} \quad \text{and} \quad \eta_2 = \frac{8t^2 + 2t - 1}{6t^2}. \]
All of the inequalities are sharp.

**Corollary 3.3.** Let the function \( f(z) \) given by (1) be in the class \( \mathcal{M}(\lambda, t) \). Then
\[
|a_2| \leq \frac{2t}{1 + \lambda}, \\
|a_3| \leq \frac{8t^2 - 1}{2(1 + 2\lambda)}
\]
and
\[
|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{t}{1 + 2\lambda}, & \eta \in [\eta_1, \eta_2] \\
\frac{8t^2 - 1 - 2t}{2t} + 2t - \frac{4\eta(1 + 2\lambda)t}{(1 + \lambda)^2}, & \eta \notin [\eta_1, \eta_2]
\end{array} \right.
\]
where
\[ \eta_1 = \frac{(1 + \lambda)^2(8t^2 - 2t - 1)}{8(1 + 2\lambda)t^2} \quad \text{and} \quad \eta_2 = \frac{(1 + \lambda)^2(8t^2 + 2t - 1)}{8(1 + 2\lambda)t^2}. \]
All of the inequalities are sharp.

**Corollary 3.4.** Let the function \( f(z) \) given by (1) be in the class \( \mathcal{M}(t) \). Then
\[
|a_2| \leq 2t, \\
|a_3| \leq 4t^2 - \frac{1}{2}
\]
and
\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
  t, & \eta \in [\eta_1, \eta_2] \\
  4(1-\eta)t^2 - \frac{1}{2}, & \eta \notin [\eta_1, \eta_2]
\end{cases} \]
where
\[ \eta_1 = \frac{8t^2 - 2t - 1}{8t^2} \quad \text{and} \quad \eta_2 = \frac{8t^2 + 2t - 1}{8t^2}. \]

All of the inequalities are sharp.

**Corollary 3.5.** Let the function \( f(z) \) given by (1) be in the class \( \mathcal{N}(\lambda, t) \). Then
\[ |a_2| \leq \frac{t}{1 + \lambda}, \]
\[ |a_3| \leq \frac{8t^2 - 1}{6(1 + 2\lambda)} \]
and
\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
  \frac{t}{3(1+2\lambda)}, & \eta \in [\eta_1, \eta_2] \\
  \frac{t}{3(1+2\lambda)} \left[ \frac{8t^2 - 1}{2t} + 2t - \eta \frac{3(1+2\lambda)t}{(1+\lambda)^2} \right], & \eta \notin [\eta_1, \eta_2]
\end{cases} \]
where
\[ \eta_1 = \frac{(1+\lambda)^2 (8t^2 - 2t - 1)}{6(1+2\lambda)t^2} \quad \text{and} \quad \eta_2 = \frac{(1+\lambda)^2 (8t^2 + 2t - 1)}{6(1+2\lambda)t^2}. \]

All of the inequalities are sharp.

**References**


On the Chebyshev Polynomial Bounds

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