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TWO DESCRIPTIONS OF RELATIVE DERIVED CATEGORIES

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Abstract. In this paper, we provide two different descriptions for a relative derived category with respect to a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$. First, we construct an exact model structure on certain exact category which has as its homotopy category the relative derived category of $\mathcal{A}$. We also show that a relative derived category is equivalent to homotopy category of certain complexes. Moreover, we investigate the existence of certain recollements in such categories.

1. Introduction

Let $\mathcal{A}$ be an abelian category. The derived category of $\mathcal{A}$, denoted $\mathbb{D}(\mathcal{A})$, as introduced by Grothendieck and Verdier, is an important and well-established construction in Homological Algebra. It is widely used throughout mathematics and an essential topic of modern research. Formally, the derived category can be defined as the Verdier quotient of the homotopy category of chain complexes by its subcategory of acyclic complexes. Moreover, Gao and Zhang [16] used Gorenstein homological algebra to study Gorenstein derived categories. The Gorenstein derived category of $\mathcal{A}$, denoted $\mathbb{D}_{\mathcal{GP}}(\mathcal{A})$, is by definition the Verdier quotient of the homotopy category of $\mathcal{A}$, $\mathbb{K}(\mathcal{A})$, modulo the thick subcategory $\mathbb{K}_{\mathcal{GP}\text{-ac}}(\mathcal{A})$ of $\mathcal{GP}$-acyclic complexes, where $\mathcal{GP}$ denotes the class of Gorenstein projective objects. Some of the basic properties of $\mathbb{D}_{\mathcal{GP}}(\mathcal{A})$ are investigated in [16].

Let $\mathcal{X}$ be a subcategory of an abelian category $\mathcal{A}$. In a similar way, one can define the $\mathcal{X}$-relative derived category of $\mathcal{A}$, denoted by $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$. In this relative construction, acyclic chain complexes are replaced by $\mathcal{X}$-acyclic ones (i.e., complexes that become acyclic after applying the functor $\mathrm{Hom}_{\mathcal{A}}(X,-)$ for all $X \in \mathcal{X}$). In general, not much seems to be known about relative derived

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categories. Recently, the authors of [2] studied some aspects of this relative derived category, with special attention to a relative version of Rickard's theorem as well as invariants under Gorenstein derived equivalence.

Still there are several open questions transferred naturally from derived setting to relative setting, e.g. when is $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ compactly generated, or when is $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ closed under taking (co)products? See [10, Appendix C] for a (not complete) list of open problems.

Let us explain the structure of the paper. In Section 2 we recall some generalities on model structures and provide any background information needed through this paper. In Section 3 we will give two different descriptions of relative derived categories. First, we construct an exact model structure on a certain exact category and show that its homotopy category is equivalent to the relative derived category. Let $\mathcal{W}_{\mathcal{X}}$ be the class of all $\mathcal{X}$-acyclic complexes in $\mathbb{C}(\mathcal{A})$. A complex $\mathbf{X} \in \mathbb{C}(\mathcal{X})$ is called DG-$\mathcal{X}$-Prj if $\mathrm{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{W})$ is acyclic for all $\mathbf{W} \in \mathcal{W}_{\mathcal{X}}$. As another description, it will be shown that $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ is equivalent to the homotopy category of all DG-$\mathcal{X}$-Prj complexes. As an immediate consequence, a relative version of the Spaltenstein's result [30, Theorem C], will be obtained.

In the following we represent another description of relative derived category such that introduced in [1]. Indeed if Λ is an artin algebra, then $\mathbb{D}_{\mathcal{X}}(\text{Mod-}\Lambda)$, for a certain subcategory $\mathcal{X}$ of Mod-Λ, has a description as the (absolute) derived category of a functor category. This has some consequences, among them we get that $\mathbb{D}_{\mathrm{pur}}(\text{Mod-}\Lambda) \simeq \mathbb{D}(\mathrm{Mod}(\text{mod-}\Lambda))$, where the pure derived category $\mathbb{D}_{\mathrm{pur}}(\text{Mod-}\Lambda)$, is the derived category with respect to the pure exact structure, that is studied in a paper of Krause [24].

Recollements are important in algebraic geometry and representation theory, see for example [4, 23]. They were introduced by Beilinson et al. [4] with the idea that one triangulated category can be considered as being 'glued together' from two others. Many authors study the existence of recollements as well as the link between recollements and other concepts in representation theory, see for instance [11, 21, 23, 28]. Gao [15, Theorem 2.5] proved that if R is an n-Gorenstein ring, then there is a colocalization sequence of the homotopy category $\mathbb{K}^{\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}R)$. In Section 4 we study recollements of triangulated categories in which relative derived categories appear.

2. Preliminaries

We assume that Λ is an arbitrary ring, associative with the identity element and denote the category of all, resp. all finitely presented, Λ-modules by Mod-Λ, resp. mod-Λ.

Let $\mathcal{C}$ be an additive category. We denote by $\mathbb{C}(\mathcal{C})$ the category of complexes in $\mathcal{C}$; the objects are complexes and morphisms are genuine chain maps. We grade the complexes cohomologically, so an object of $\mathbb{C}(\mathcal{C})$ is of the following form

$$\cdots \to C^{n-1} \overset{\partial^{n-1}}{\to} C^n \overset{\partial^n}{\to} C^{n+1} \to \cdots .$$

It is known that in case $\mathcal{C}$ is additive, resp. abelian, then so is $\mathbb{C}(\mathcal{C})$. In particular, $\mathbb{C}(\text{Mod-}\Lambda)$ is an abelian category.

We write $\mathbb{K}(\mathcal{C})$ for the homotopy category. $\mathbb{K}^-(\mathcal{C})$, resp. $\mathbb{K}^+(\mathcal{C})$, $\mathbb{K}^{\mathrm{b}}(\mathcal{C})$, denotes the full subcategory of $\mathbb{K}(\mathcal{C})$ consisting of all bounded above, resp. bounded below, bounded, complexes. Also, let $\mathbb{K}^{-,\mathrm{b}}(\mathcal{C})$ denote the full subcategory of $\mathbb{K}^-(\mathcal{C})$ consisting of complexes with only finitely many nonzero cohomologies.

Definition 2.1. Let $\mathcal{X}$ be an abelian category. A complex $\mathbf{X} \in \mathbb{C}(\mathcal{X})$ is called acyclic if $\mathrm{H}^n(\mathbf{X}) = 0$ for all $n \in \mathbb{Z}$. $\mathbf{X}$ is called $\mathcal{X}$-totally acyclic if the induced complexes $\mathcal{X}(\mathbf{X}, X)$ and $\mathcal{X}(X, \mathbf{X})$ of abelian groups are acyclic for all $X \in \mathcal{X}$. We denote by $\mathbb{C}_{\mathrm{ac}}(\mathcal{X})$, resp. $\mathbb{C}_{\mathrm{tac}}(\mathcal{X})$, the full subcategory of $\mathbb{C}(\mathcal{X})$ consisting of acyclic, resp. totally acyclic, complexes. Moreover, the triangulated subcategory of $\mathbb{K}(\mathcal{X})$ consisting of acyclic, resp. totally acyclic, complexes will be denoted by $\mathbb{K}_{\mathrm{ac}}(\mathcal{X})$, resp. $\mathbb{K}_{\mathrm{tac}}(\mathcal{X})$.

Let $\mathcal{A}$ be an abelian category. If $\mathcal{X} = \text{Prj-}A$, resp. $\mathcal{Y} = \text{Inj-}A$, is the class of projectives, resp. injectives, object in $\mathcal{A}$, then an object $G \in \mathcal{A}$ is called Gorenstein projective, resp. Gorenstein injective, if it is isomorphic to a syzygy of an object $\mathbf{P}$, resp. $\mathbf{I}$, in $\mathbb{K}_{\mathrm{tac}}(\mathcal{X})$, resp. $\mathbb{K}_{\mathrm{tac}}(\mathcal{Y})$. We let $\mathcal{GP}\text{-}\mathcal{A}$, resp. $\mathcal{GI}\text{-}\mathcal{A}$, denote the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective, resp. Gorenstein injective, objects. In case $\mathcal{A} = \text{Mod-}\Lambda$, we abbreviate the notions to $\mathcal{GP}\text{-}\Lambda$ and $\mathcal{GI}\text{-}\Lambda$. We set $\mathcal{G}p\text{-}\Lambda = \mathcal{GP}\text{-}\Lambda \cap \text{mod-}\Lambda$ and $\mathcal{G}i\text{-}\Lambda = \mathcal{GI}\text{-}\Lambda \cap \text{mod-}\Lambda$.

Definition 2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{X} \subseteq \mathcal{A}$ be a full additive subcategory which is closed under taking direct summands. Let M be an object of $\mathcal{A}$. A right $\mathcal{X}$-approximation (an $\mathcal{X}$-precover) of M is a morphism $\varphi : X \to M$ with $X \in \mathcal{X}$ such that any morphism from an object $\mathcal{X}$ to M factors through φ. $\mathcal{X}$ is called contravariantly finite (precovering) if any object in $\mathcal{A}$ admits a right $\mathcal{X}$-approximation. One can define left $\mathcal{X}$-approximations ($\mathcal{X}$-preenvelopes) and covariantly finite (preenveloping) classes dually. A subcategory $\mathcal{X}$ is called functorially finite if $\mathcal{X}$ is both covariantly and contravariantly finite.

Let $\mathcal{M}$ be a class of objects in $\mathcal{A}$. We let Add-$\mathcal{M}$, resp. add-$\mathcal{M}$, denote the class of all objects of $\mathcal{A}$ that are isomorphic to a direct summand of a direct sum, resp. a finite direct sum, of objects in $\mathcal{M}$. In case $\mathcal{M} = \{M\}$, contains a single object, we write Add-M, resp. add-M, instead of Add-$\{M\}$, resp. add-$\{M\}$.

Recall that a pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects of $\mathcal{A}$ is said to be a cotorsion pair if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$, where the right and left orthogonal are defined as follows

$${}^{\perp}\mathcal{Y} := \{A \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(A, Y) = 0 \text{ for all } Y \in \mathcal{Y}\}$$

and

$$\mathcal{X}^{\perp} := \{B \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, B) = 0 \text{ for all } X \in \mathcal{X}\}.$$

The cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called complete if for every $A \in \mathcal{A}$ there exist exact sequences

$$0 \to Y \to X \to A \to 0 \quad \text{and} \quad 0 \to A \to Y' \to X' \to 0,$$

where $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

Definition 2.3. Let Λ be an artin algebra. Λ is called *Gorenstein* if $\mathrm{id}_\Lambda \Lambda < \infty$ and $\mathrm{id}\Lambda_\Lambda < \infty$. It is known that in this case $\mathrm{id}_\Lambda \Lambda = \mathrm{id}\Lambda_\Lambda$. Λ is called *virtually Gorenstein* algebra if $(\mathcal{GP}\text{-}\Lambda)^\perp = {}^\perp(\mathcal{GI}\text{-}\Lambda)$.

Virtually Gorenstein algebras was first introduced by Beligiannis and Reiten [7] as a common generalisation of Gorenstein algebras and algebras of finite representation type. See [6, Remark 4.6], for a list of properties. It is proved in [5] that $\mathcal{GP}\text{-}\Lambda$ is contravariantly finite provided Λ is an artin algebra. Moreover, Beligiannis proved that if Λ is a virtually Gorenstein algebra, then $\mathcal{G}p\text{-}\Lambda$ is a contravariantly finite subcategory of mod-Λ, [6, Proposition 4.7].

Definition 2.4. An artin algebra Λ is called of finite Cohen-Macaulay type (finite CM-type, for short) provided that there are only finitely many indecomposable finitely generated Gorenstein projective Λ-modules, up to isomorphism.

Suppose that Λ is an artin algebra of finite CM-type and $\{M_1, \dots, M_n\}$ is the set of all non-isomorphic finitely generated indecomposable Gorenstein projective Λ-modules. It is easy to check that $\mathcal{G}p\text{-}\Lambda = \text{add-}M$, where $M = M_1 \oplus \cdots \oplus M_n$.

2.1. Relative derived categories

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ be a subcategory of $\mathcal{A}$. A complex $\mathbf{A}$ in $\mathcal{A}$ is called $\mathcal{X}$-*acyclic* (sometimes $\mathcal{X}$-proper exact) if the induced complex $\mathrm{Hom}_{\mathcal{A}}(X, \mathbf{A})$ is acyclic for each $X \in \mathcal{X}$. A chain map $f : \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathbb{C}(\mathcal{A})$ is called an $\mathcal{X}$-quasi-isomorphism if for any $X \in \mathcal{X}$, the induced chain map $\mathcal{A}(X, f)$ is a quasi-isomorphism.

The relative derived category $\mathbb{D}^*_{\mathcal{X}}(\mathcal{A})$ with $* \in \{\text{blank}, -, \text{b}\}$ is defined as the Verdier quotient of the homotopy category $\mathbb{K}^*(\mathcal{A})$ with respect to the thick triangulated subcategory $\mathbb{K}^*_{\mathcal{X}\text{-ac}}(\mathcal{A})$ of $\mathcal{X}$-acyclic complexes. Note that if $\mathcal{X} = \text{Prj-}\mathcal{A}$, we get back the classical derived category, see [16, §2].

Let $\mathcal{A}$ have enough projective objects and $\mathcal{X} = \mathcal{GP}\text{-}\mathcal{A}$ be the class of all Gorenstein projective objects of $\mathcal{A}$. $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ is known as the Gorenstein derived category of $\mathcal{A}$, denoted by $\mathbb{D}_{\mathcal{GP}}(\mathcal{A})$, which is introduced and studied by Gao and Zhang [16]. In case $\mathcal{X} = \mathcal{GP}\text{-}\Lambda$, where Λ is a ring, we write $\mathbb{K}_{\mathcal{GP}\text{-ac}}(\text{Mod-}\Lambda)$ for $\mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{A})$ and $\mathbb{D}_{\mathcal{GP}}(\text{Mod-}\Lambda)$ for $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$.

Observe that one can define dual concepts for a subcategory $\mathcal{Y}$ of $\mathcal{A}$; a $\mathcal{Y}$-coacyclic complex, i.e., a complex $\mathbf{A}$ such that the induced complex $\mathrm{Hom}_{\mathcal{A}}(\mathbf{A}, Y)$ is acyclic for all $Y \in \mathcal{Y}$; a $\mathcal{Y}$-quasi-isomorphism, i.e., a chain map $f : \mathbf{A} \longrightarrow \mathbf{B}$ such that $\mathrm{Hom}_{\mathcal{A}}(f, Y)$ is a quasi-isomorphism for all $Y \in \mathcal{Y}$; and so the relative derived category $\mathbb{D}_{\mathcal{Y}}(\mathcal{A})$, defines as $\mathbb{D}_{\mathcal{Y}}(\mathcal{A}) := \mathbb{K}(\mathcal{A})/\mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{A})$, where $\mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{A})$ denotes the homotopy category of $\mathcal{Y}$-coacyclic complexes. Assume that $\mathcal{A}$ has enough injective objects and $\mathcal{Y} = \mathcal{GI}\text{-}\mathcal{A}$ is the class of all Gorenstein injective objects of $\mathcal{A}$. Then the relative derived category $\mathbb{D}_{\mathcal{GI}}(\mathcal{A})$ is called Gorenstein (injective) derived category of $\mathcal{A}$. In case $\mathcal{A} = \text{Mod-}\Lambda$ and $\mathcal{Y} = \mathcal{GI}\text{-}\Lambda$, we write $\mathbb{K}_{\mathcal{GI}\text{-ac}}(\text{Mod-}\Lambda)$ for $\mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{A})$ and $\mathbb{D}_{\mathcal{GI}}(\text{Mod-}\Lambda)$ for $\mathbb{D}_{\mathcal{Y}}(\mathcal{A})$.

2.2. Model structures and Hovey pairs

Model categories were first introduced by Quillen [29]. Let $\mathcal{C}$ be a category. A model structure on $\mathcal{C}$ is a triple $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ of classes of morphisms, called *cofibrations*, *weak equivalences* and *fibrations*, respectively, such that satisfying certain axioms. Morphisms in $\mathrm{Cof} \cap \mathrm{W}$ are called *trivial cofibrations* and morphisms in $\mathrm{W} \cap \mathrm{Fib}$ are *trivial fibrations*.

A model category is a category with model structure such that $\mathcal{C}$ has an initial object $\emptyset$, a terminal object $*$, all pushouts of trivial cofibrations along trivial cofibrations exist, and dually all pullbacks of trivial fibrations along trivial fibrations exist. We refer the reader to [12] for a readable introduction to model categories and to [19] for a more in-depth presentation.

The definition of model structure then was modified by some authors. The one that is commonly used nowadays is due to Hovey [20]. Hovey discovered that the existence of a model structure on any abelian category $\mathcal{A}$ is equivalent to the existence of two complete cotorsion pairs in $\mathcal{A}$ which are compatible in a precise way. The advantage of the Hovey's theorem is that we can construct a model structure on abelian category $\mathcal{A}$ determined by three class of objects, called *cofibrant*, *trivial* and *fibrant* objects.

Gillespie followed [20] and focused on exact categories with model structure compatible with the exact structure. He defined cotorsion pairs in exact categories and saw that Hovey's correspondence between abelian model structures and cotorsion pairs naturally carries over to a correspondence between exact model structures and cotorsion pairs. Let us provide some context on exact categories and exact model structures.

2.2.1. *Exact categories.* An exact category is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is a distinguished class of diagrams of the form $X \xrightarrow{i} Y \xrightarrow{d} Z$ called *conflation*, satisfying certain axioms which make conflations behave similar to short exact sequences in an abelian category. A map such as i in the language of exact categories is called an *inflation* (denoted by $\rightarrowtail$) while d is called a *deflation* (denoted by $\twoheadrightarrow$).

For the results presented later to work, we need to impose extra conditions on our exact categories. So we recall the notion of weakly idempotent complete exact categories and efficient exact categories.

Definition 2.5. An additive category $\mathcal{A}$ is called *weakly idempotent complete* if every split monomorphism has a cokernel and every split epimorphism has a kernel. An exact category $(\mathcal{A}, \mathcal{E})$ is called weakly idempotent complete if the additive category $\mathcal{A}$ is so.

Definition 2.6. An exact category $(\mathcal{A}, \mathcal{E})$ is called *efficient* if

- $\mathcal{A}$ is weakly idempotent complete.
- Arbitrary transfinite compositions of inflations exist and are themselves inflations.
- Every object of $\mathcal{A}$ is small relative to the class of all inflations.
- $\mathcal{A}$ admits a generator. That is, there is an object $G \in \mathcal{A}$ such that every $X \in \mathcal{A}$ admits a deflation $\coprod_I G \twoheadrightarrow X$.

We refer the reader to [31] for more details about efficient exact category.

Now let $(\mathcal{A}, \mathcal{E})$ be an exact category. The axioms of exact category allow us to define Yoneda Ext groups with usual properties. The abelian group $\mathrm{Ext}^1_{\mathcal{E}}(X, Y)$ is the group of equivalence classes of short exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$. In particular, $\mathrm{Ext}^1_{\mathcal{E}}(X, Y) = 0$ if and only if every short exact sequence $Y \rightarrowtail Z \twoheadrightarrow X$ is isomorphic to the split exact sequence $Y \rightarrowtail Y \oplus X \twoheadrightarrow X$.

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A pair $(\mathcal{F}, \mathcal{D})$ of full subcategories of $\mathcal{A}$ is called a cotorsion pair provided that

$$\mathcal{F} = {}^{\perp}\mathcal{D} \qquad \text{and} \qquad \mathcal{F}^{\perp} = \mathcal{D},$$

where ${}^{\perp}$ is taken with respect to the functor $\mathrm{Ext}^1_{\mathcal{E}}$. The cotorsion pair $(\mathcal{F}, \mathcal{D})$ is said to have enough projectives if for every $X \in \mathcal{A}$ there is a short exact sequence $D \rightarrowtail F \twoheadrightarrow X$ with $D \in \mathcal{D}$ and $F \in \mathcal{F}$. We say that it has enough injectives if it satisfies the dual statement. If both of these hold we say the cotorsion pair is complete.

2.2.2. *Exact model structures.* As we said before Hovey defined in [20], Definition 2.1, the notion of an abelian model structure. In order to define exact model structure on exact category $(\mathcal{A}, \mathcal{E})$ we need to specify cofibrant, fibrant and trivial object in this category. Note that for any $X \in \mathcal{A}$, $0 \rightarrowtail X$ is an inflation and $X \twoheadrightarrow 0$ is a deflation. Now suppose $(\mathcal{A}, \mathcal{E})$ has a model structure as defined in Definition 1.1.3 of [19]. An object $W \in \mathcal{A}$ is said to be a trivial if $0 \rightarrowtail W$ is a weak equivalence. An object $A \in \mathcal{A}$ is said to be a cofibrant (resp. trivially cofibrant) if $0 \rightarrowtail A$ is a cofibration (resp. trivially cofibration). Dually $B \in \mathcal{A}$ is fibrant (resp. trivially fibrant) if $B \twoheadrightarrow 0$ is fibration (resp. trivial fibration).

Definition 2.7. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. An exact model structure on $(\mathcal{A}, \mathcal{E})$ is a model structure in the sense of Definition 1.1.3 of [19] in which each of the following hold.

(1) A map is a (trivial) cofibration if and only if it is an inflation with a (trivially) cofibrant cokernel.
(2) A map is a (trivial) fibration if and only if it is a deflation with a (trivially) fibrant kernel.

The next theorem is a result due to Hovey [20] which is described by Gillespie in the sense of exact category, see [17]. We just recall that a class of objects $\mathcal{W} \in \mathcal{A}$ is a *thick subcategory* of $\mathcal{A}$ if it is closed under direct summands and if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

Theorem 2.8 ([17, Theorem 3.3])**.** *Let $(\mathcal{A}, \mathcal{E})$ be an exact category with an exact model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ be the class of fibrant objects and $\mathcal{W}$ be the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathcal{A}$ and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. If we further assume that $(\mathcal{A}, \mathcal{E})$ is weakly idempotent complete, then the converse holds. That is, given two compatible cotorsion pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, each complete and with $\mathcal{W}$ a thick subcategory, then there is an exact model structure on $\mathcal{A}$ where $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects.*

2.3. Homotopy category of an exact model structure

Model categories are used to give an effective construction of the localization of categories, where the problem is to convert the class of weak-equivalence into isomorphisms. Suppose $\mathcal{C}$ is a category with a subcategory of $\mathcal{W}$. The localized category that denoted by $\mathcal{C}[\mathcal{W}^{-1}]$ is defined in classical algebra. In case $\mathcal{C}$ is a model category with weak equivalence $\mathcal{W}$, define $\mathcal{C}[\mathcal{W}^{-1}]$ as the *Homotopy category* associated to $\mathcal{C}$ and denote by $\mathrm{Ho}\mathcal{C}$. Our reason for not adopting the right notation is that in this case, we have an identity between the morphisms of localized category and homotopy class of morphisms under a certain *homotopy relation* which is determined by the model structure. The abstract notion of homotopy relation can be found in any references on model category such as [19], but whenever $(\mathcal{A}, \mathcal{E})$ is an exact model category we can determine a homotopy relation by the following lemma:

Lemma 2.9. *Let $(\mathcal{A}, \mathcal{E})$ be an exact model category and $f, g : X \to Y$ be two morphisms. If X is cofibrant and Y is fibrant, then f and g are homotopic (we denote by $f \sim g$) if and only if $f - g$ factor through a trivially fibrant and cofibrant object.*

Proof. We refer to [17, Proposition 4.4]. □

Finally, we introduce the fundamental theorem about model categories.

Theorem 2.10. *Let $\mathcal{C}$ be a model category, let $\gamma : \mathcal{C} \to \mathrm{Ho}\mathcal{C}$ be the canonical localisation functor and denote by $\mathcal{C}_{cf}$ the full subcategory of $\mathcal{C}$ given by the objects which are cofibrant and fibrant. Then the composition*

$$\mathcal{C}_{cf} \xrightarrow{\subseteq} \mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}\mathcal{C}$$

induces a category equivalence $(\mathcal{C}_{cf}/\sim) \to \mathrm{Ho}\mathcal{C}$, where $\mathcal{C}_{cf}/\sim$ is defined by

$$(\mathcal{C}_{cf}/\sim)(X, Y) = \mathcal{C}_{cf}(X, Y)/\sim .$$

3. Model structures and relative derived categories

In this section we describe relative derived categories using different tools: as the homotopy category of a special exact model structure, as the homotopy category of certain complexes, called DG-$\mathcal{X}$ complexes.

Setup. Throughout the section $\mathcal{A}$ is an abelian category with set-indexed coproducts and a projective generator, and $\mathcal{X} \subseteq \mathcal{A}$ is a full subcategory containing projective generator such that $\mathcal{X} = \text{Add-}\mathcal{S}$ for some set $\mathcal{S} \subseteq \mathcal{X}$.

Denote by $\mathcal{E}_{\mathcal{X}}$ the class of all short exact sequences $0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$ in $\mathbb{C}(\mathcal{A})$ such that for each $i \in \mathbb{Z}$, the sequence $0 \to X^i \to Y^i \to Z^i \to 0$ belongs to $\mathfrak{E}_{\mathcal{X}}$, where $\mathfrak{E}_{\mathcal{X}}$ is the class of all short exact sequences in $\mathcal{A}$ that are $\mathcal{X}$-acyclic. One can check easily that the pair $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ satisfies Quillen's axioms and so forms an exact category. Clearly $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ is weakly idempotent complete. Moreover $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ is an efficient exact category. Indeed it is enough to show that $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ admits a generator. A complex $\mathbf{X}$ in the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ is called generator if for every complex $\mathbf{A} \in \mathbb{C}(\mathcal{A})$ there is an epic $\mathcal{X}$-approximation $\coprod_I \mathbf{X} \to \mathbf{A}$. Let $\bar{S}$ be the complex

$$\cdots \to 0 \to \bigoplus_{S \in \mathcal{S}} S \to \bigoplus_{S \in \mathcal{S}} S \to 0 \to \cdots .$$

It can be easily seen that $\bigoplus_{i \in \mathbb{Z}} \bar{S}[i]$ is a generator for the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$.

Remark 3.1. Note that $(\mathcal{A}, \mathfrak{E}_{\mathcal{X}})$ is also an exact category. In case $\mathcal{X}$ is admissible, i.e., every right $\mathcal{X}$-approximation is epic, $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ is just Neeman's derived category of the exact category $(\mathcal{A}, \mathfrak{E}_{\mathcal{X}})$, see [27, Construction 1.5].

Our aim is to construct an exact model structure on $\mathbb{C}(\mathcal{A})$ in such a way that its homotopy category is the relative derived category $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$. First we need the following definition:

Definition 3.2. Let $\mathcal{W}_{\mathcal{X}}$ be the class of all $\mathcal{X}$-acyclic complexes in $\mathbb{C}(\mathcal{A})$. A complex $\mathbf{X} \in \mathbb{C}(\mathcal{X})$ is called DG-$\mathcal{X}$-Prj if $\text{Hom}_{\mathcal{A}}(\mathbf{X}, \mathbf{W})$ is acyclic for all $\mathbf{W} \in \mathcal{W}_{\mathcal{X}}$. We denote the class of all DG-$\mathcal{X}$-Prj complexes by DG-$(\mathcal{X})$. Note that when $\mathcal{X} = \text{Prj-}R$ for a ring R, then DG-$(\mathcal{X})$ is just the class of all DG-projective complexes.

So in view of Theorem 2.8, in order to have an exact model structure, it is enough to present two compatible cotorsion pairs. We claim that the pairs $(\text{DG-}(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}, \mathbb{C}(\mathcal{A}))$ and $(\text{DG-}(\mathcal{X}), \mathcal{W}_{\mathcal{X}})$ are the desired ones.

Proposition 3.3. $(\text{DG-}(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}, \mathbb{C}(\mathcal{A}))$ *is a complete cotorsion pair in* $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$.

Proof. Let $\mathbf{X} \in \text{DG-}(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}$. So by definition we can say that $\text{Hom}(\mathbf{X}, \mathbf{X})$ is an exact complex. Hence, $\text{H}^i\text{Hom}(\mathbf{X}, \mathbf{X}) = 0$ for all $i \in \mathbb{Z}$. Since we have the following famous isomorphism

$$\text{Hom}_{\mathbb{K}(\mathcal{A})}(\mathbf{X}, \mathbf{X}) \cong \text{H}^i\text{Hom}(\mathbf{X}, \mathbf{X})$$

then we can say that $1_{\mathbf{X}} \sim 0$, so the complex $\mathbf{X}$ is contractible. Therefore $\mathbf{X}$ is an $\mathcal{E}_{\mathcal{X}}$-projective objects in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$. Also clearly every $\mathcal{E}_{\mathcal{X}}$-projective objects in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ is belong to DG-$(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}$. So (DG-$(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}, \mathbb{C}(\mathcal{A}))$ is a complete cotorsion pair in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$. □

Now we prove that (DG-$(\mathcal{X}), \mathcal{W}_{\mathcal{X}})$ is also a complete cotorsion pair in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$. To this end, we need some preparations.

Proposition 3.4. *${}^{\perp}\mathcal{W}_{\mathcal{X}}$ is exactly the class of all* DG-$\mathcal{X}$-Prj *complexes.*

Proof. Let $\mathbf{X} \in {}^{\perp}\mathcal{W}_{\mathcal{X}}$ and $W \in \mathcal{S}^{\perp}$. We have

$$\mathrm{Ext}^1_{\mathfrak{E}_{\mathcal{X}}}(X^i, W) \cong \mathrm{Ext}^1_{\mathcal{E}_{\mathcal{X}}}(\mathbf{X}, e^i_{\rho}(W)),$$

where $e^i_{\rho}(W)$ is the complex

$$\cdots \to 0 \to W \xrightarrow{\mathrm{id}} W \to 0 \to \cdots$$

with the W on the right hand side sits on the i-th position.

It can be easily checked that $e^i_{\rho}(W) \in \mathcal{W}_{\mathcal{X}}$. Hence $\mathrm{Ext}^1_{\mathcal{E}_{\mathcal{X}}}(\mathbf{X}, e^i_{\rho}(W)) = 0$. This implies that $\mathrm{Ext}^1_{\mathfrak{E}_{\mathcal{X}}}(X^i, W) = 0$. So $X^i \in {}^{\perp}(\mathcal{S}^{\perp})$. By [31, Theorem 5.16] we can say that X^i is an $\mathcal{S}$-filtered object. Hence there is an $\mathcal{S}$-filtration $(X_{\alpha} | \alpha \leq \lambda)$ such that $X_0 = 0$ and $X_{\lambda} = X^i$. Now consider the conflation $0 \to X_1 \to X_2 \to S_1 \to 0$. Since $S_1 \in \mathcal{S}$, therefore $X_2 = X_1 \oplus S_1$. Following this procedure we can say that, for each $\alpha < \lambda$, there is a conflation $0 \to X_{\alpha} \to X_{\alpha+1} \to S_{\alpha} \to 0$ in $\mathfrak{E}_{\mathcal{X}}$ with $S_{\alpha} \in \mathcal{S}$ and $X_{\alpha+1} = X_{\alpha} \oplus S_{\alpha}$. For limit ordinal μ it is easy to see that $X_{\mu} = \sum_{\alpha<\mu} S_{\alpha}$ such that S_{α}s are pairwise disjoint elements of $\mathcal{S}$. Hence we can say that $X^i \in$ Add-$\mathcal{S}$.

Now, by definition, it remains to show that $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ is an exact complex for any $\mathbf{Y} \in \mathcal{W}_{\mathcal{X}}$. We know that $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ is an exact complex if and only if for all $i \in \mathbb{Z}$, $\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(\mathbf{X}, \mathbf{Y}[i]) = 0$. By [8, lemma 2.1], $f : \mathbf{X} \to \mathbf{Y}[i]$ is null-homotopic if and only if the exact sequence $0 \to \mathbf{Y}[i] \to \mathrm{cone}(f) \to \mathbf{X}[1] \to 0$ associated to the mapping cone $\mathrm{cone}(f)$ splits in $\mathbb{C}(\mathcal{A})$. So it is enough to show that $\mathrm{Ext}^1_{\mathcal{E}_{\mathcal{X}}}(\mathbf{X}[1], \mathbf{Y}[i]) = 0$. This follows from the fact that $\mathcal{W}_{\mathcal{X}}$ is closed under shifting.

On the other hand, let $\mathbf{X} \in$ DG-$(\mathcal{X})$ and $0 \to \mathbf{Y} \to \mathbf{A} \to \mathbf{X} \to 0$ be a short exact sequence in $\mathcal{E}_{\mathcal{X}}$. Since $X^i \in \mathcal{X}$, this short exact sequence splits in each degree. Therefore, there is a morphism $f : \mathbf{X} \to \mathbf{Y}[1]$ such that the above sequence is isomorphic to the short exact sequence $0 \to \mathbf{Y} \to \mathrm{cone}(f) \to \mathbf{X} \to 0$. Now, this sequence is split if and only if the complex $\mathrm{Hom}(\mathbf{X}, \mathbf{Y}[1])$ is acyclic and so the proof is complete. □

The idea of the proof of the following proposition is taken from Lemma 5.1 of [8].

Proposition 3.5. *Let $\mathbf{A} \in \mathbb{C}(\mathcal{A})$. Then for all $X \in \mathcal{X}$,*

$$\mathrm{Ext}^1_{\mathcal{E}_{\mathcal{X}}}(X[i], \mathbf{A}) \cong \mathrm{H}^{i-1}\mathrm{Hom}(X, \mathbf{A}).$$

Proof. Given an object X in $\mathcal{X}$, let $\mathbf{X}$ be the complex $\mathbf{X}^0 = \mathbf{X}^{-1} = X$ with the identity map and zero elsewhere, and let $\underline{\mathbf{X}}$ to be the stalk complex with X in degree zero and zero elsewhere. Thus $\underline{\mathbf{X}}$ is a subcomplex of $\mathbf{X}$ and $\mathbf{X}/\underline{\mathbf{X}} = \underline{\mathbf{X}}[1]$. So there is a short exact sequence $X^\bullet : 0 \to \underline{\mathbf{X}} \to \mathbf{X} \to \underline{\mathbf{X}}[1] \to 0$. Applying $\mathrm{Hom}(-, \mathbf{A})$ to $X^\bullet$, we have the following exact sequence

$$\mathrm{Hom}(\mathbf{X}, \mathbf{A}) \xrightarrow{\varphi} \mathrm{Hom}(\underline{\mathbf{X}}, \mathbf{A}) \xrightarrow{\psi} \mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\underline{\mathbf{X}}[1], \mathbf{A}) \longrightarrow \mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\mathbf{X}, \mathbf{A}).$$

Observe that every $\mathcal{X}$-acyclic sequence ending in $\mathbf{X}$ is split. Therefore, $\mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\mathbf{X}, \mathbf{A}) = 0$ and so there is an isomorphism

$$\mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\underline{\mathbf{X}}[1], \mathbf{A}) \cong \frac{\mathrm{Hom}(\underline{\mathbf{X}}, \mathbf{A})}{\mathrm{Ker}\psi}.$$

Note that $\mathrm{Ker}\psi = \mathrm{Im}\varphi$.

Now consider the complex

$$\cdots \longrightarrow \mathrm{Hom}(X, A^{-1}) \xrightarrow{d_*^{-1}} \mathrm{Hom}(X, A^0) \xrightarrow{d_*^0} \mathrm{Hom}(X, A^1) \longrightarrow \cdots.$$

We have an epimorphism

$$\eta : \mathrm{Hom}(\underline{\mathbf{X}}, \mathbf{A}) \longrightarrow \frac{\mathrm{Ker}d_*^0}{\mathrm{Im}d_*^{-1}}$$

defined by $\eta(f) = f_0$ for every $f = (f^n)_{n\in\mathbb{Z}} \in \mathrm{Hom}(\underline{\mathbf{X}}, \mathbf{A})$. It is easy to check that $\mathrm{Ker}\eta = \mathrm{Im}\varphi$. So $\mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\underline{\mathbf{X}}[1], \mathbf{A}) \cong \mathrm{H}^0(\mathrm{Hom}(X, \mathbf{A}))$.

Similar argument shows that, for each $i \in \mathbb{Z}$, we have the isomorphism

$$\mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\underline{\mathbf{X}}[i], \mathbf{A}) \cong \mathrm{H}^{i-1}(\mathrm{Hom}(X, \mathbf{A})). \qquad \square$$

As a direct consequence of the above proposition we have the following corollary.

Corollary 3.6. $({}^\perp\mathcal{W}_\mathcal{X})^\perp = \mathcal{W}_\mathcal{X}$.

Proof. Clearly $\mathcal{W}_\mathcal{X} \subseteq ({}^\perp\mathcal{W}_\mathcal{X})^\perp$. Now let $\mathbf{W} \in ({}^\perp\mathcal{W}_\mathcal{X})^\perp$ and $X \in \mathcal{X}$. Since $\underline{\mathbf{X}}$ belongs to DG-$(\mathcal{X})$ and ${}^\perp\mathcal{W}_\mathcal{X} =$ DG-$(\mathcal{X})$, then $\mathrm{Ext}^1_{\mathcal{E}_\mathcal{X}}(\underline{\mathbf{X}}, \mathbf{W}) = 0$, hence by previous proposition we can say that $\mathbf{W}$ is an $\mathcal{X}$-acyclic complexes. $\square$

Proposition 3.7. (DG-$(\mathcal{X}), \mathcal{W}_\mathcal{X}$) *is a complete cotorsion pair.*

Proof. As we mentioned before, the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\mathcal{X})$ is efficient with a generator $\bigoplus_{i\in\mathbb{Z}} \bar{S}[i]$. Now, consider the set $T = \{S[i] \mid S \in \mathcal{S}, i \in \mathbb{Z}\} \bigcup \{\bar{S}[i] \mid S \in \mathcal{S}, i \in \mathbb{Z}\}$. Theorem 5.16 of [31] implies that $({}^\perp(T^\perp), T^\perp)$ is a complete cotorsion pair. By Proposition 3.5, $T^\perp = \mathcal{W}_\mathcal{X}$ and then Proposition 3.4 completes the proof. $\square$

Corollary 3.8. *There is an exact model structure on the exact category* $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\mathcal{X})$. *In this model structure,* DG-$(\mathcal{X})$ *is the class of cofibrant objects,* $\mathbb{C}(\mathcal{A})$ *is the class of fibrant objects and* $\mathcal{W}_\mathcal{X}$ *is the class of trivial objects. As usual, we denote this model structure by the triple* (DG-$(\mathcal{X}), \mathcal{W}_\mathcal{X}, \mathbb{C}(\mathcal{A})$).

Proof. This follows from the above proposition in conjunction with Theorem 2.8. □

Now we are ready to prove the main result of this section.

Theorem 3.9. *Let $\mathcal{A}$ and $\mathcal{X} = \text{Add-}\mathcal{S}$ be as our setup. Consider the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$ with the exact model structure $(\text{DG-}(\mathcal{X}), \mathcal{W}_{\mathcal{X}}, \mathbb{C}(\mathcal{A}))$. The homotopy category $\text{Ho}\mathbb{C}(\mathcal{A})$ is equivalent to $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ as triangulated categories.*

Proof. By construction, a map f in $\mathbb{C}(\mathcal{A})$ is a weak equivalence if and only if f is an $\mathcal{X}$-quasi-isomorphism. Now, the universal property in the theory of localisations of categories implies that $\text{Ho}\mathbb{C}(\mathcal{A})$ is equivalent to $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$. □

Theorem 3.10. *As in our setup, let $\mathcal{X}$ be a full subcategory of an abelian category $\mathcal{A}$ with $\mathcal{X} = \text{Add-}\mathcal{S}$ for some set $\mathcal{S} \subseteq \mathcal{X}$. Then there is the following triangle equivalence*

$$\mathbb{D}_{\mathcal{X}}(\mathcal{A}) \simeq \mathbb{K}(\text{DG-}(\mathcal{X})).$$

Proof. By Corollary 3.8, there is a model structure $(\text{DG-}(\mathcal{X}), \mathcal{W}_{\mathcal{X}}, \mathbb{C}(\mathcal{A}))$ on the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_{\mathcal{X}})$. By Lemma 2.9, a map f in $\mathbb{C}(\mathcal{A})_{cf}$ is homotopic to zero if and only if f factors through an object of $\text{DG-}(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}$. But, $\text{DG-}(\mathcal{X}) \bigcap \mathcal{W}_{\mathcal{X}}$ consists of all contractible complexes of $\mathcal{X}$. So the homotopy relation in this model category coincides with the usual notion of chain homotopy equivalences. This implies that $\mathbb{C}(\mathcal{A})_{cf}/\sim \simeq \mathbb{K}(\text{DG-}(\mathcal{X}))$.

Theorem 2.10 implies that $\text{Ho}\mathbb{C}(\mathcal{A}) \simeq \mathbb{C}(\mathcal{A})_{cf}/\sim$. Therefore $\text{Ho}\mathbb{C}(\mathcal{A}) \simeq \mathbb{K}(\text{DG-}(\mathcal{X}))$. On the other hand, by Theorem 3.9, $\text{Ho}\mathbb{C}(\mathcal{A}) \simeq \mathbb{D}_{\mathcal{X}}(\mathcal{A})$. Consequently, we have the desired equivalence

$$\mathbb{D}_{\mathcal{X}}(\mathcal{A}) \simeq \mathbb{K}(\text{DG-}(\mathcal{X})).$$ □

We say that a complex $\mathbf{A} \in \mathbb{C}(\mathcal{A})$ has a DG-$\mathcal{X}$-Prj resolution if there exists an $\mathcal{X}$-quasi-isomorphism $\mathbf{X} \longrightarrow \mathbf{A}$ with $\mathbf{X}$ a DG-$\mathcal{X}$-Prj complex.

Remark 3.11. Let Λ be a ring. Set $\mathcal{A} = \text{Mod-}\Lambda$ and $\mathcal{X} = \text{Prj-}\Lambda$. A famous result of Spaltenstein [30] states that every complex in $\mathbb{C}(\text{Mod-}R)$ admits a DG-projective resolution. The theorem above, in particular, implies that every complex in $\mathbb{C}(\mathcal{A})$ admits a DG-$\mathcal{X}$-Prj-resolution and so it can be considered as a relative version of Spaltenstein's result [30, Theorem C].

In the following we need another description of relative derived category which is proved in [1]. They show that when $\mathcal{A} = \text{Mod-}\Lambda$, $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$, for certain subcategories $\mathcal{X}$ of $\mathcal{A}$, have a nice interpretation as the (absolute) derived category of a functor category.

Let $\mathcal{C}$ be a skeletally small additive category and consider the functor category $(\mathcal{C}^{\text{op}}, \mathcal{A}b)$ of abelian group valued functors. Following Auslander [3], we denote this category by $\text{Mod}(\mathcal{C})$, and call it the category of modules on $\mathcal{C}$.

It is proved in Proposition 3.1 of [1] that if Λ is an artin algebra and $\mathcal{S}$ is a set contained in mod-Λ, then there is an equivalence $\mathbb{D}_{\mathcal{X}}(\text{Mod-}\Lambda) \simeq \mathbb{D}(\text{Mod}(\mathcal{S}))$

of triangulated categories where $\mathcal{X} := \text{Add-}\mathcal{S}$. The proof presented in this proposition also work when Λ is an arbitrary ring, associative with the identity element. So we have the following proposition.

Proposition 3.12. *Let Λ be a ring and $\mathcal{S}$ be a set contained in* mod-Λ. *Set $\mathcal{X} = \text{Add-}\mathcal{S}$. Then there is an equivalence*

$$\mathbb{D}_{\mathcal{X}}(\text{Mod-}\Lambda) \simeq \mathbb{D}(\text{Mod}(\mathcal{S}))$$

of triangulated categories.

Proof. We refer to the proof of [1, Proposition 3.1]. □

Example 3.13. Here we list some interesting examples.

(i) Let Λ be an artin algebra. Let $\mathcal{A} = \text{Mod-}\Lambda$ and $\mathcal{S} = \mathcal{G}p\text{-}\Lambda$. Then we can construct an exact model structure, given by the triple (DG-(Add-$\mathcal{G}p$-Λ), $\mathcal{W}_{\mathcal{G}p}, \mathbb{C}(\text{Mod-}\Lambda)$) on the exact category $(\mathbb{C}(\text{Mod-}\Lambda), \mathcal{E}_{\mathcal{G}p})$. In particular, we have an equivalence

$$\mathbb{D}_{\mathcal{G}p}(\text{Mod-}\Lambda) \simeq \mathbb{K}(\text{DG-}(\text{Add-}\mathcal{G}p\text{-}\Lambda))$$

of triangulated categories.

(ii) Let Λ be a Gorenstein ring. It is proved in [13], that there are complete cotorsion pairs $(\mathbb{C}(\mathcal{GP}\text{-}\Lambda), \mathcal{F})$ and $(\mathcal{L}, \mathbb{C}(\mathcal{GI}\text{-}\Lambda))$ in $\mathbb{C}(\text{Mod-}\Lambda)$, where $\mathcal{F}$, resp. $\mathcal{L}$, is the full subcategory of $\mathbb{C}(\text{Mod-}\Lambda)$ consisting of all complexes of finite projective, resp. injective, dimension. This, in particular, implies that for any complex $\mathbf{A} \in \mathbb{C}(\text{Mod-}\Lambda)$, there exist $\mathbf{G} \in \mathbb{C}(\mathcal{GP}\text{-}\Lambda)$ and a Gorenstein projective precover $\mathbf{G} \longrightarrow \mathbf{A}$ which is a $\mathcal{GP}$-quasi-isomorphism. But, the same argument as in [9, Proposition 3.5] applies to show that $\mathbb{K}(\mathcal{GP}\text{-}\Lambda) = \mathbb{K}(\text{DG-}(\mathcal{GP}\text{-}\Lambda))$. This means that we have the following equivalence of triangulated categories

$$\mathbb{D}_{\mathcal{GP}}(\text{Mod-}\Lambda) \simeq \mathbb{K}(\text{DG-}(\mathcal{GP}\text{-}\Lambda)).$$

Note that in general, $\mathcal{GP}\text{-}\Lambda \neq \text{Add-}\mathcal{G}p\text{-}\Lambda$, so compare this argument to (i).

(iii) The result stated in (ii) can be stated in a dual manner to prove that over a Gorenstein ring Λ, $\mathbb{K}(\text{DG-}(\mathcal{GI}\text{-}\Lambda)) \simeq \mathbb{D}_{\mathcal{GI}}(\text{Mod-}\Lambda)$.

Let Λ be a ring. Consider the pure exact structure on the category $\mathcal{A} = \text{Mod-}\Lambda$, that is, a sequence $X \to Y \to Z$ of morphisms in $\mathcal{A}$ is pure exact if the induced sequence $0 \to \text{Hom}_{\mathcal{A}}(C, X) \to \text{Hom}_{\mathcal{A}}(C, Y) \to \text{Hom}_{\mathcal{A}}(C, Z) \to 0$ is exact for each finitely presented object C. The projective objects with respect to this exact structure are called pure projective; they are precisely the direct summands of coproducts of finitely presented objects. The derived category with respect to this exact structure is by definition the pure derived category that we denote by $\mathbb{D}_{\text{pur}}(\mathcal{A})$.

In [24], Krause studied this category and proved that $\mathbb{D}_{\text{pur}}(\text{Mod-}\Lambda)$ is compactly generated and $\mathbb{D}_{\text{pur}}(\text{Mod-}\Lambda)^{\text{c}} \simeq \mathbb{K}^{\text{b}}(\text{mod-}\Lambda)$. In the following example, we see that our result can be specialized to get similar equivalences as above for $\mathbb{D}_{\text{pur}}(\text{Mod-}\Lambda)$.

Example 3.14. Let Λ be a ring. Set $\mathcal{A} = \text{Mod-}\Lambda$ and $\mathcal{S} = \text{mod-}\Lambda$. The relative derived category $\mathbb{D}_{\mathcal{X}}(\mathcal{A})$ is the pure derived category, since $\mathcal{X} = \text{Add-mod-}\Lambda$. So our result implies the following triangulated equivalence

$$\mathbb{D}_{\text{pur}}(\text{Mod-}\Lambda) \simeq \mathbb{K}(\text{DG-}(\text{Add-mod-}\Lambda)).$$

Furthermore, Proposition 3.12 implies that

$$\mathbb{D}_{\text{pur}}(\text{Mod-}\Lambda) \simeq \mathbb{D}(\text{Mod}(\text{mod-}\Lambda)).$$

Note that the second equivalence above have been proved in [24, Corollary 4.8].

4. Recollements involving relative derived categories

In this section, we give recollements as well as (co)localisation sequences of homotopy categories of Gorenstein projectives. Let us first recall some definitions.

A sequence

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{4.1}$$

of exact functors between triangulated categories is called a localisation sequence if

(1) The functor F is fully faithful and has a right adjoint.
(2) The functor G has a fully faithful right adjoint.
(3) There is an equality of triangulated subcategories $\text{Im}(F) = \text{Ker}(G)$.

We say that (4.1) is a colocalisation sequence if the pair $(F^{\text{op}}, G^{\text{op}})$ of opposite functors is a localisation sequence. A sequence of functors is called a recollement if it is both a localisation and a colocalisation sequence. We usually denote the functors and their adjoints involving in a recollement as follows:

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "F"] & \mathcal{T} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "G"] & \mathcal{T}'' \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proposition 4.1. *Let* $\mathcal{A} = \text{Mod-}\Lambda$*, where* Λ *is an arbitrary ring, and* $\mathcal{X}$ *be a full subcategory of* $\mathcal{A}$ *such that* $\mathcal{X} = \text{Add-}\mathcal{S}$ *for some set* $\mathcal{S} \subset \text{mod-}\Lambda$. *Then there is the following recollement*:

$$\begin{tikzcd} \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X}) \arrow[r] & \mathbb{K}(\mathcal{X}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}_{\mathcal{X}}(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. In view of [18, Corollary 6.7], there is the following recollement:

$$\begin{tikzcd} \mathbb{K}_{\text{ac}}(\text{Prj-Mod}(\mathcal{S})) \arrow[r] & \mathbb{K}(\text{Prj-Mod}(\mathcal{S})) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}(\text{Mod}(\mathcal{S})) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Now, it follows from the proof of Proposition 3.1 of [1] we can say that

$$\mathbb{K}(\text{Prj-Mod}(\mathcal{S})) \simeq \mathbb{K}(\mathcal{X}) \quad \text{and} \quad \mathbb{K}_{\text{ac}}(\text{Prj-Mod}(\mathcal{S})) \simeq \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X})$$

and also by Proposition 3.12 $\mathbb{D}_{\mathcal{X}}(\text{Mod-}\Lambda) \simeq \mathbb{D}(\text{Mod}(\mathcal{S}))$ so we get the result. □

Corollary 4.2. *Let Λ be an arbitrary ring. Then there exist the following recollements*:

$$\begin{tikzcd} \mathbb{K}_{\mathcal{G}p\text{-ac}}(\mathrm{Add}\text{-}\mathcal{G}p\text{-}\Lambda) \arrow[r] & \mathbb{K}(\mathrm{Add}\text{-}\mathcal{G}p\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}_{\mathcal{G}p}(\mathrm{Mod}\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

$$\begin{tikzcd} \mathbb{K}_{(\mathrm{mod}\text{-}\Lambda)\text{-ac}}(\mathrm{mod}\text{-}\Lambda) \arrow[r] & \mathbb{K}(\mathrm{Add}\text{-}\mathrm{mod}\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}_{\mathrm{pur}}(\mathrm{Mod}\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

In particular, if Λ is a virtually Gorenstein artin algebra of finite CM-type, then there is a recollement:

$$\begin{tikzcd} \mathbb{K}_{\mathcal{GP}\text{-ac}}(\mathcal{GP}\text{-}\Lambda) \arrow[r] & \mathbb{K}(\mathcal{GP}\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbb{D}_{\mathcal{GP}}(\mathrm{Mod}\text{-}\Lambda) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

Proof. The first two recollements follow directly from Proposition 4.1. Moreover, by [6, Theorem 4.10], if Λ is virtually Gorenstein of finite CM-type, then $\mathcal{GP}\text{-}\Lambda = \mathrm{Add}\text{-}\mathcal{G}p\text{-}\Lambda$. This fact together with Proposition 4.1 imply the existence of the last recollement. $\square$

In the following we intend to prove the existence of localisation sequences of the homotopy category of Gorenstein projective modules. The key of the proof is the existence of an stable t-structure. Let us recall the definition of a stable t-structure.

Definition 4.3. Let $\mathcal{T}$ be a triangulated category. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{T}$ is called an stable t-structure in $\mathcal{T}$ if the following conditions are satisfied.

(i) $\mathcal{U} = \Sigma\mathcal{U}$ and $\mathcal{V} = \Sigma\mathcal{V}$.
(ii) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.
(iii) For every $X \in \mathcal{T}$, there is a triangle $U \to X \to V \rightsquigarrow$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Miyachi [25] studied the relationship between (co)localisation sequences, recollements, and stable t-structures. Indeed, he showed that a (co)localisation sequence induces a stable t-structure, and vice versa.

Proposition 4.4 (Miyachi,[25])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a stable t-structure in $\mathcal{T}$. The following statements hold true.*

(i) *There is a localisation sequence*

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{V} \arrow[l, bend left, "j_*"] \end{tikzcd},$$

where $j_* : \mathcal{V} \longrightarrow \mathcal{T}$ is the canonical embedding.

(ii) *There is a localisation sequence*

$$\begin{tikzcd} \mathcal{U} \arrow[r, "j_!"] & \mathcal{T} \arrow[l, bend left, "j^*"] \arrow[r, "i^*"] & \mathcal{T}' \arrow[l, bend left, "i_*"] \end{tikzcd},$$

where $j_! : \mathcal{U} \longrightarrow \mathcal{T}$ is the canonical embedding.

As we saw in Remark 3.11, in our setup, every complex $\mathbf{A}$ admits a DG-$\mathcal{X}$-Prj resolution $\mathbf{X}$. Here we show that for a different condition on $\mathcal{X}$, if $\mathbf{A}$ is bounded above complex in $\mathcal{A} = \text{Mod-}\Lambda$, then it has a DG-$\mathcal{X}$-Prj resolution $\mathbf{X} \to \mathbf{A}$. Moreover $\mathbf{X}$ is bonded above as we expected. We denote by $\mathbb{K}^{\mathcal{X}\mathrm{b}}(\mathcal{X})$, resp. $\mathbb{K}^{-,\mathcal{X}\mathrm{b}}(\mathcal{X})$, the full subcategory of $\mathbb{K}(\mathcal{X})$, resp. $\mathbb{K}^{-}(\mathcal{X})$, consisting of all complexes $\mathbf{X}$ in which there is an integer $n = n(\mathbf{X})$ such that $\mathrm{H}^i(\mathrm{Hom}_\Lambda(X, \mathbf{X}))$ $= 0$ for all $i > |n|$, resp. all $i \leq n$, and all $X \in \mathcal{X}$.

Lemma 4.5. *Let Λ be a ring and $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{A} = \text{Mod-}\Lambda$ that is closed under products. Then every bounded above complex $\mathbf{A}$ has a* DG-$\mathcal{X}$-Prj *resolution $\mathbf{X} \longrightarrow \mathbf{A}$ with $\mathbf{X} \in \mathbb{K}^{-}(\mathcal{X})$.*

Proof. First observe that if $\mathbf{A}$ is a bounded complex, then by using the same argument as in the proof of [16, Proposition 3.4] we can obtain a DG-$\mathcal{X}$-Prj resolution $\mathbf{X} \longrightarrow \mathbf{A}$ with $\mathbf{X} \in \mathbb{K}^{-,\mathcal{X}b}(\mathcal{X})$. Now let $\mathbf{A} = (A^i, \partial^i)$ be a bounded above complex in $\mathcal{A}$. Without loss of generality, we may assume that $A^i = 0$ for all $i > 0$. For any $i \leq 0$, set

$$\mathbf{A}^{\geq i} : 0 \longrightarrow \mathrm{Ker}\partial^i \longrightarrow A^i \xrightarrow{\partial^i} A^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

and consider the inverse system $\{\varphi^i : \mathbf{A}^{\geq i-1} \longrightarrow \mathbf{A}^{\geq i}\}$. Since $\mathbf{A}^{\geq i}$ is bounded, there is $\mathbf{X}^i \in \mathbb{K}^{-,\mathcal{X}b}(\mathcal{X})$ with a $\mathcal{X}$-quasi-isomorphism $\mathbf{X}^i \longrightarrow \mathbf{A}^{\geq i}$. By the axioms of a triangulated category, there exists a morphism $\underleftarrow{\mathrm{hlim}}\mathbf{X}^i \longrightarrow \underleftarrow{\mathrm{hlim}}\mathbf{A}^{\geq i}$ making the following diagram commutative:

$$\begin{tikzcd}
\Sigma^{-1}\prod \mathbf{X}^i \arrow[r] \arrow[d] & \underleftarrow{\mathrm{hlim}}\mathbf{X}^i \arrow[r] \arrow[d, dotted] & \prod \mathbf{X}^i \arrow[r, "1-\text{shift}"] \arrow[d] & \prod \mathbf{X}^i \arrow[d] \\
\Sigma^{-1}\prod \mathbf{A}^{\geq i} \arrow[r] & \underleftarrow{\mathrm{hlim}}\mathbf{A}^{\geq i} \arrow[r] & \prod \mathbf{A}^{\geq i} \arrow[r, "1-\text{shift}"] & \prod \mathbf{A}^{\geq i}
\end{tikzcd}$$

Since $\mathcal{X}$ is closed under products, $\prod \mathbf{X}^i \longrightarrow \prod \mathbf{A}^{\geq i}$ is a $\mathcal{X}$-quasi-isomorphism with $\prod \mathbf{X}^i \in \mathbb{K}^{-}(\mathcal{X})$. Now, for any X in $\mathcal{X}$, apply the homological functor $\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, \Sigma^i(-))$ to the above diagram, to deduce that $\underleftarrow{\mathrm{hlim}}\mathbf{X}_i \longrightarrow \underleftarrow{\mathrm{hlim}}\mathbf{A}^{\geq i}$ is a $\mathcal{X}$-quasi-isomorphism. Also, by construction, $\underleftarrow{\mathrm{hlim}}\mathbf{X}_i$ is a bounded above complex in $\mathbb{C}(\mathcal{X})$. On the other hand, by [26, Proposition 11.7] $\underleftarrow{\mathrm{hlim}}\mathbf{A}^{\geq i} = \underleftarrow{\lim}\mathbf{A}^{\geq i} = \mathbf{A}$. Note that every complex in $\mathbb{K}^{-}(\mathcal{X})$ is DG-$\mathcal{X}$-Prj. Hence the proof is complete. $\square$

For any covariantly finite subcategory $\mathcal{Y}$ of $\mathcal{A}$, DG-$\mathcal{Y}$-Inj complexes and DG-$\mathcal{Y}$-Inj coresolutions are defined dually. Moreover, one can show that every bounded below complex in $\mathbb{C}(\mathcal{Y})$ is a DG-$\mathcal{Y}$-Inj complex. We have a dual version of Lemma 4.5 that is stated as follows. Since the argument of the proof is similar to the proof of Lemma 4.5, we skip the proof.

Lemma 4.6. *Let $\mathcal{Y}$ be covariantly finite subcategory of $\mathcal{A} = \text{Mod-}\Lambda$ that is closed under coproducts. Then every bounded below complex $\mathbf{A}$ has a* DG-$\mathcal{Y}$-Inj *coresolution $\mathbf{A} \longrightarrow \mathbf{Y}$ with $\mathbf{Y} \in \mathbb{K}^{+}(\mathcal{Y})$.*

It is known that over a noetherian ring Λ, $\mathcal{GI}$-Λ is closed under coproducts. Also, if Λ is an artin algebra, then $\mathcal{GP}$-Λ is closed under products [10, Proposition 2.2.12]. Moreover, note that by [5] when Λ is an artin algebra, $\mathcal{GP}$-Λ is a contravariantly finite subcategory of Mod-Λ. When Λ is noetherian a result of Enochs and López-Ramos [14, Corollary 2.7], implies that $\mathcal{GI}$-Λ is a covariantly finite subcategory of Mod-Λ. Using these facts, we have the following corollary.

Corollary 4.7. (i) *Let Λ be an artin algebra. Then every bounded above complex has a DG-Gorenstein projective resolution. In particular, there is a triangulated equivalence $\mathbb{D}^{-}_{\mathcal{GP}}(\text{Mod-}\Lambda) \simeq \mathbb{K}^{-}(\mathcal{GP}\text{-}\Lambda)$.*

(ii) *Let Λ be a noetherian ring. Then every bounded below complex has a DG-Gorenstein injective resolution. In particular, there is a triangulated equivalence $\mathbb{D}^{+}_{\mathcal{GI}}(\text{Mod-}\Lambda) \simeq \mathbb{K}^{+}(\mathcal{GI}\text{-}\Lambda)$.*

Proof. We just prove part (i). Proof of the part (ii) is similar. We have already know that every bounded above complex has a DG-Gorenstein projective resolution. Let $F : \mathbb{K}^{-}(\mathcal{GP}\text{-}\Lambda) \to \mathbb{D}^{-}_{\mathcal{GP}}(\text{Mod-}\Lambda)$ be the composition of the embedding $\mathbb{K}^{-}(\mathcal{GP}\text{-}\Lambda) \to \mathbb{K}^{-}(\text{Mod-}\Lambda)$ and the localisation functor $\mathbb{K}^{-}(\text{Mod-}\Lambda) \to \mathbb{D}^{-}_{\mathcal{GP}}(\text{Mod-}\Lambda)$. By Lemma 4.5 F is dense; and by [16, Proposition 2.8] F is fully faithful. Hence it is an equivalence. □

Now, we can state and prove our result. The proof is based on the proof of Proposition 2.2 of [22].

Proposition 4.8. *Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{A} = \text{Mod-}\Lambda$ that contains* Prj-Λ *and closed under products. Then the following statements hold true.*

(i) *The pair $(\mathbb{K}^{-,\mathcal{X}\text{b}}(\mathcal{X}), \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X}))$ is a stable t-structure in $\mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X})$.*

(ii) *For the canonical embedding functor $j_* : \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X}) \longrightarrow \mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X})$, there exists the following colocalisation sequence*

$$\begin{tikzcd} \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X}) \arrow[r, "j_*"'] & \mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X}) \arrow[l, bend right, "j^*"'] \arrow[r, "i^!"'] & \mathbb{D}^{\text{b}}_{\mathcal{X}}(\mathcal{A}). \arrow[l, bend right, "i_*"'] \end{tikzcd}$$

Consequently, $\mathbb{D}^{\text{b}}_{\mathcal{X}}(\mathcal{A}) \simeq \mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X})/\mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X})$.

Proof. (i) Clearly $\text{Hom}_{\mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X})}(\mathbb{K}^{-,\mathcal{X}\text{b}}(\mathcal{X}), \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X})) = 0$, since every complex in $\mathbb{K}^{-,\mathcal{X}\text{b}}(\mathcal{X})$ is DG-$\mathcal{X}$-Prj. Let $\mathbf{X} \in \mathbb{K}^{\mathcal{X}\text{b}}(\mathcal{X})$. Then $\mathbf{X}$ is $\mathcal{X}$-quasi-isomorphic to the complex

$$\mathbf{Y} : \cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} \text{Im} d^n \longrightarrow 0$$

for some integer n. In view of Lemma 4.5, $\mathbf{Y}$ admits a DG-$\mathcal{X}$ resolution $\mathbf{G} \longrightarrow \mathbf{Y}$, where $\mathbf{G} \in \mathbb{K}^{-,\mathcal{X}\text{b}}(\mathcal{X})$. So we get a DG-$\mathcal{X}$ resolution $\mathbf{G} \xrightarrow{\zeta_{\mathbf{X}}} \mathbf{X}$ of $\mathbf{X}$. Hence, we obtain a triangle

$$\mathbf{G} \xrightarrow{\zeta_{\mathbf{X}}} \mathbf{X} \longrightarrow \mathbf{U}_{\mathbf{X}} \rightsquigarrow,$$

where $\mathbf{U}_{\mathbf{X}} \in \mathbb{K}_{\mathcal{X}\text{-ac}}(\mathcal{X})$. This completes the proof of the first part.

(ii) Observe that by [2, Theorem 3.3], $\mathbb{D}^{\mathrm{b}}_{\mathcal{X}}(\mathcal{A}) \simeq \mathbb{K}^{-,\mathcal{X}\mathrm{b}}(\mathcal{X})$. Now the result follows directly from (i) and Proposition 4.4. □

Dual to Proposition 4.8 we have the following result. Since its proof is similar we skip it.

Proposition 4.9. *Let $\mathcal{Y}$ be a covariantly finite subcategory of $\mathcal{A} = \mathrm{Mod}\text{-}\Lambda$ that contains* Inj-Λ *and closed under coproducts. Then the following hold true.*

(i) *The pair $(\mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{Y}), \mathbb{K}^{+,\mathcal{Y}\mathrm{b}}(\mathcal{Y}))$ is a stable t-structure in $\mathbb{K}^{\mathcal{Y}\mathrm{b}}(\mathcal{Y})$.*

(ii) *For the canonical embedding functor $j_! : \mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{Y}) \longrightarrow \mathbb{K}^{\mathcal{Y}\mathrm{b}}(\mathcal{Y})$, there exists the following localisation sequence*

$$\begin{tikzcd} \mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{Y}) \arrow[r, shift left, "j_!"] & \mathbb{K}^{\mathcal{Y}\mathrm{b}}(\mathcal{Y}) \arrow[l, shift left, "j^*"] \arrow[r, shift left, "i^*"] & \mathbb{D}^{\mathrm{b}}_{\mathcal{Y}}(\mathcal{A}). \arrow[l, shift left, "i_*"] \end{tikzcd}$$

Consequently $\mathbb{D}^{\mathrm{b}}_{\mathcal{Y}}(\mathcal{A}) \simeq \mathbb{K}^{\mathcal{Y}\mathrm{b}}(\mathcal{Y})/\mathbb{K}_{\mathcal{Y}\text{-coac}}(\mathcal{Y})$.

The above propositions enable us to prove a version of [15, Theorem 2.5], for artin algebras.

Corollary 4.10. (i) *Let Λ be an artin algebra. Then there is the following colocalisation sequence*

$$\begin{tikzcd} \mathbb{K}_{\mathcal{GP}\text{-ac}}(\mathcal{GP}\text{-}\Lambda) \arrow[r, shift right, "j_*"'] & \mathbb{K}^{\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}\Lambda) \arrow[l, shift right, "j^*"'] \arrow[r, shift right, "i^!"'] & \mathbb{D}^{\mathrm{b}}_{\mathcal{GP}}(\mathrm{Mod}\text{-}\Lambda). \arrow[l, shift right, "i_*"'] \end{tikzcd}$$

Consequently, $\mathbb{D}^{\mathrm{b}}_{\mathcal{GP}}(\mathrm{Mod}\text{-}\Lambda) \simeq \mathbb{K}^{\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}\Lambda)/\mathbb{K}_{\mathcal{GP}\text{-ac}}(\mathcal{GP}\text{-}\Lambda)$.

(ii) *Let Λ be a noetherian ring. Then there is the following localisation sequence*

$$\begin{tikzcd} \mathbb{K}_{\mathcal{GI}\text{-ac}}(\mathcal{GI}\text{-}\Lambda) \arrow[r, shift left, "j_!"] & \mathbb{K}^{\mathcal{GI}\mathrm{b}}(\mathcal{GI}\text{-}\Lambda) \arrow[l, shift left, "j^*"] \arrow[r, shift left, "i^*"] & \mathbb{D}^{\mathrm{b}}_{\mathcal{GI}}(\mathrm{Mod}\text{-}\Lambda). \arrow[l, shift left, "i_*"] \end{tikzcd}$$

Consequently $\mathbb{D}^{\mathrm{b}}_{\mathcal{GI}}(\mathrm{Mod}\text{-}\Lambda) \simeq \mathbb{K}^{\mathcal{GI}\mathrm{b}}(\mathcal{GI}\text{-}\Lambda)/\mathbb{K}_{\mathcal{GI}\text{-ac}}(\mathcal{GI}\text{-}\Lambda)$.

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