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ON THE g-CIRCULANT MATRICES

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Abstract. In this paper, firstly we compute the spectral norm of g-circulant matrices $C_{n,g} = g\text{-Circ}(c_0, c_1, \ldots, c_{n-1})$, where $c_i \geq 0$ or $c_i \leq 0$ (equivalently $c_i \cdot c_j \geq 0$). After, we compute the spectral norms, determinants and inverses of the g-circulant matrices with the Fibonacci and Lucas numbers.

1. Introduction

An $n \times n$ matrix C is called a circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}$$

or an $n \times n$ matrix C is circulant if there exist $c_0, c_1, \ldots, c_{n-1}$ such that the i,j entry of C is $c_{i-j \bmod n}$, where the rows and columns are numbered from 0 to $n-1$ and $k \bmod n$ means the number between 0 to $n-1$ that is congruent to $k \bmod n$. Thus, we denote the circulant matrix C as $C = \text{Circ}(c_0, c_1, \ldots, c_{n-1})$.

An $n \times n$ matrix C_g is called a g-circulant matrix if it is of the form

$$C_g = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-g} & c_{n-g+1} & \cdots & c_{n-g-1} \\ c_{n-2g} & c_{n-2g+1} & \cdots & c_{n-2g-1} \\ \vdots & \vdots & & \vdots \\ c_g & c_{g+1} & \cdots & c_{g-1} \end{bmatrix}, \tag{1.1}$$

where g is a nonnegative integer. The entries of the matrix C_g are characterized by the rule $C_g = [c_{(i-jg) \bmod n}]_{i,j=0}^{n-1}$. Also, the matrix C_g is determined by its first row elements and the parameter g, that is, its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions. Thus, we denote the

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g-circulant matrix C_g as $C_g = g\text{-Circ}(c_0, c_1, \dots, c_{n-1})$. When we take $g = 1$, the matrix $C_g = \text{Circ}(c_0, c_1, \dots, c_{n-1})$ is an ordinary circulant matrix (briefly, circulant matrix). Any circulant matrix and g-circulant matrix has many interesting properties. Some of them are [8]:

1. Let A be $n \times n$. Then A is a circulant if and only if

$$A\pi = \pi A,$$

where the matrix $\pi = \text{Circ}(0, 1, 0, \dots, 0)$.

2. $\text{Circ}(c_0, c_1, \dots, c_{n-1}) = c_0 I + c_1 \pi + \cdots + c_{n-1}\pi^{n-1}$.

3. All circulants of the same order commute. If C is a circulant so is C^*. Hence C and C^* commute and therefore all circulants are normal matrices, where C^* is conjugate transpose of C.

4. A is a g-circulant matrix if and only if $A\pi = \pi A^g$.

5. Let A be a nonsingular g-circulant matrix. Then A^{-1} is a g^{-1}-circulant.

6. If the $n \times n$ matrices A, C and Q are of the forms $A = g\text{-Circ}(a_0, a_1, \dots, a_{n-1})$, $C = \text{Circ}(c_0, c_1, \dots, c_{n-1})$ and $Q = g\text{-Circ}\,(1, 0, 0, \dots, 0)$, then

$$AA^* \text{ is a circulant matrix,} \tag{1.2}$$

$$A = QC, \tag{1.3}$$

and for $(n, g) = 1$,

$$Q^{-1} = Q^* \text{ (Q is unitary).} \tag{1.4}$$

For more introduction and algebraic properties of circulant (or g-circulant) matrices, please refer to the classical book by Davis [8].

The g-circulant matrices play important roles in physics, signal and image processing, statistics, coding theory and so on. There are lots of articles concerning the determinants, inverses, spectral norms and many applications of circulant (or g-circulant) matrices [2, 4–7, 9, 10, 13, 14, 16–18, 22–24]. Solak [17, 18] has presented some bounds for the spectral and Euclidean norms of circulant matrices with the Fibonacci and Lucas numbers. Shen et al. [16] have computed the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. Similarly, Bozkurt and Tam [7] have computed the determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal–Lucas Numbers. Ngongiep [14] has showed the singular values of g-circulants. Zhou and Jiang [24] have derived explicit expressions of spectral norms for certain of g-circulant matrices with classical Fibonacci and Lucas numbers entries when $(n, g) = 1$. In this paper, we study some properties of g-circulant matrices with the Fibonacci and Lucas numbers when $(n, g) = 1$ and $(n, g) \neq 1$.

The main contents of this paper are organized as follows: In Section 2, we give some preliminaries, definitions and lemmas related to our study. In Section 3, firstly we compute the spectral norm of g-circulant matrix with all entries are nonnegative or nonpositive. Secondly, we give some special cases of our results including Fibonacci and Lucas numbers. In Sections 4 and 5, we compute determinants and inverses of the g-circulant matrices with the Fibonacci and Lucas numbers by using results of the paper [16].

2. Preliminaries

The sequences of the Fibonacci numbers are one of the most well-known sequences, and it has many applications to different fields such as mathematics, statistics and physics. The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$, $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ $(n \geq 1)$, $L_0 = 2$ and $L_1 = 1$. Let α and β be the roots of the characteristic equation $x^2 - x - 1 = 0$, then the Binet formulas of F_n and L_n are [16]:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \;\; L_n = \alpha^n + \beta^n.$$

The Fibonacci and Lucas numbers and their generalized forms have many applications in matrices and have many interesting identities [1, 3, 7, 9, 11, 12, 15–21]. Two of them are:

$$\sum_{s=0}^{n-1} F_s = F_{n+1} - 1, \tag{2.1}$$

and

$$\sum_{s=0}^{n-1} L_s = L_{n+1} - 1. \tag{2.2}$$

Definition 1. Let $A = (a_{ij})$ be any $m \times n$ matrix. The *spectral norm* of A is

$$\|A\|_2 = \sqrt{\max_i \lambda_i \left(A^* A\right)},$$

where $\lambda_i \left(A^* A\right)$ are eigenvalues of $A^* A$ and A^* is conjugate transpose of A.

Lemma 1 ([16])**.** *Let $A_n = \mathrm{Circ}(F_1, F_2, \ldots, F_n)$ be a circulant matrix. Then we have*

$$\det A_n = (1 - F_{n+1})^{n-1} + F_n^{n-2} \sum_{k=1}^{n-1} F_k \left(\frac{1 - F_{n+1}}{F_n} \right)^{k-1}.$$

Lemma 2 ([16])**.** *Let $A_n = \mathrm{Circ}(F_1, F_2, \ldots, F_n)\, (n > 2)$ be a circulant matrix. Then we have*

$$A_n^{-1} = \frac{1}{f_n} \mathrm{Circ}(f_0, f_1, f_2, \ldots, f_{n-1}),$$

where

$$(2.3)\qquad f_s=\begin{cases} 1+\sum\limits_{i=1}^{n-2}\frac{F_{n-i}F_n^{i-1}}{(F_1-F_{n+1})^i}, & \text{if } s=0,\\ -1+\sum\limits_{i=1}^{n-2}\frac{F_{n-1-i}F_n^{i-1}}{(F_1-F_{n+1})^i}, & \text{if } s=1,\\ -\frac{F_n^{s-2}}{(F_1-F_{n+1})^{s-1}}, & \text{if } 2\le s\le n-1,\\ F_1-F_n+\sum\limits_{i=1}^{n-2}F_i\left(\frac{F_n}{F_1-F_{n+1}}\right)^{n-i-1}, & \text{if } s=n. \end{cases}$$

Lemma 3 ([16]). *Let $B_n=\mathrm{Circ}(L_1,L_2,\ldots,L_n)$ be a circulant matrix. Then we have*

$$\det B_n=(1-L_{n+1})^{n-1}+(L_n-2)^{n-2}\sum_{k=1}^{n-1}(L_{k+2}-3L_{k+1})\left(\frac{1-L_{n+1}}{L_n-2}\right)^{k-1}.$$

Lemma 4 ([16]). *Let $B_n=\mathrm{Circ}(L_1,L_2,\ldots,L_n)$ be a circulant matrix. Then we have*

$$(2.4)\qquad B_n^{-1}=\frac{1}{l_n}\mathrm{Circ}(l_0,l_1,l_2,\ldots,l_{n-1}),$$

where

$$l_s=\begin{cases} 1+\sum\limits_{i=1}^{n-2}\frac{(L_{n+2-i}-3L_{n+1-i})(L_n-2)^{i-1}}{(L_1-L_{n+1})^i}, & \text{if } s=0,\\ -3+\sum\limits_{i=1}^{n-2}\frac{(L_{n+1-i}-3L_{n-i})(L_n-2)^{i-1}}{(L_1-L_{n+1})^i}, & \text{if } s=1,\\ -\frac{5(L_n-2)^{s-2}}{(L_1-L_{n+1})^{s-1}}, & \text{if } 2\le s\le n-1,\\ L_1-3L_n+\sum\limits_{i=1}^{n-2}(L_{i+2}-3L_{i+1})\left(\frac{L_n-2}{L_1-L_{n+1}}\right)^{n-i-1}, & \text{if } s=n. \end{cases}$$

Throughout this paper the $n\times n$ matrices $C_{n,g}$, $C_{n,1}$, $C_{n,g}(F)$, $C_{n,g}(L)$, $C_{n,1}(F), C_{n,1}(L)$ and $Q_{n,g}$ denote the following matrices:

$$\begin{aligned} C_{n,g}&=g\text{-}\mathrm{Circ}(c_0,c_1,\ldots,c_{n-1}),\\ C_{n,1}&=\mathrm{Circ}(c_0,c_1,\ldots,c_{n-1}),\\ C_{n,g}(F_s)&=g\text{-}\mathrm{Circ}(F_s,F_{s+1},\ldots,F_{s+n-1}),\\ C_{n,1}(F_s)&=\mathrm{Circ}(F_s,F_{s+1},\ldots,F_{s+n-1}),\\ C_{n,g}(L_s)&=g\text{-}\mathrm{Circ}(L_s,L_{s+1},\ldots,L_{s+n-1}),\\ C_{n,1}(L_s)&=\mathrm{Circ}(L_s,L_{s+1},\ldots,L_{s+n-1}) \end{aligned}$$

and

$$Q_{n,g}=g\text{-}\mathrm{Circ}\,(1,0,\ldots,0)\,,$$

where $c_i \geq 0$ or $c_i \leq 0$ (equivalently $c_i.c_j \geq 0$) $(i, j = 0, 1, \ldots, n-1)$, F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Also, $[n, g]$ and (n, g) denote the least common multiple of n, g and the greatest common divisor of n, g, respectively.

3. The spectral norm

Theorem 1. *The spectral norm of the matrix $C_{n,g}$ holds*

$$\left\| C_{n,g} \right\|_2 = \left[(n,g) \sum_{m=0}^{k-1} \sum_{s=0}^{n-1} c_s c_{s-mg} \right]^{\frac{1}{2}},$$

where $k = \frac{[n,g]}{g}$.

Proof. From (1.2), the matrix $C_{n,g}C_{n,g}^*$ is a circulant matrix. If the first row of $C_{n,g}C_{n,g}^*$ is $(a_0, a_1, \ldots, a_{n-1})$, then

$$C_{n,g}C_{n,g}^* = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-g} & c_{n-g+1} & \cdots & c_{n-g-1} \\ \vdots & \vdots & & \vdots \\ c_g & c_{g+1} & \cdots & c_{g-1} \end{bmatrix} \begin{bmatrix} c_0 & c_{n-g} & \cdots & c_g \\ c_1 & c_{n-g+1} & \cdots & c_{g+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-g-1} & \cdots & c_{g-1} \end{bmatrix}.$$

From matrix multiplication, we obtain

$$\underset{0 \leq i \leq n-1}{a_i} = \sum_{s=0}^{n-1} c_s c_{s-ig}. \tag{3.1}$$

Since $c_{[s-(pk+j)g]} = c_{(s-jg)}$ under mod n, we have $a_j = a_{pk+j}$, where $k = \frac{[n,g]}{g}$, $j = 0, 1, \ldots, k-1$, $p = 0, 1, \ldots, \frac{n}{k} - 1$. Then, for every $j = 0, 1, \ldots, k-1$, a_j(equivalently the block $(a_0, a_1, \ldots, a_{k-1})$) is repeated $\frac{n}{k}$ times in the row $(a_0, a_1, \ldots, a_{n-1})$. Thus, the first row of $C_{n,g}C_{n,g}^*$ is

$$\left(\underbrace{a_0, a_1, \ldots, a_{k-1}}_{1}, \underbrace{a_0, a_1, \ldots, a_{k-1}}_{2}, \ldots, \underbrace{a_0, a_1, \ldots, a_{k-1}}_{\frac{n}{k}} \right).$$

Since the circulant matrix $C_{n,g}C_{n,g}^*$ is normal, its spectral norm is equal to its spectral radius. Furthermore, by considering $C_{n,g}C_{n,g}^*$ is irreducible and its entries are nonnegative, we have that the spectral radius (or spectral norm) of the matrix $C_{n,g}C_{n,g}^*$ is equal to its Perron root. We select an n-dimensional column vector $v = (1, 1, \ldots, 1)^T$, then by (3.1)

$$\left[C_{n,g}C_{n,g}^*\right] v = \left(\sum_{i=0}^{n-1} a_i\right) v = \left(\frac{n}{k}\sum_{m=0}^{k-1} a_m\right) v = \left(\frac{n}{k}\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} c_s c_{s-mg}\right) v.$$

Obviously, $\frac{n}{k}\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} c_s c_{s-mg}$ is an eigenvalue of $C_{n,g}C_{n,g}^*$ associated with v and it is the Perron root of $C_{n,g}C_{n,g}^*$. Hence

$$\left\|C_{n,g}C_{n,g}^*\right\|_2 = \frac{n}{k}\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} c_s c_{s-mg}.$$

Finally, from the equalities $\|C_{n,g}\|_2^2 = \left\|C_{n,g}C_{n,g}^*\right\|_2$ and $\frac{n}{k} = (n, g)$, $\|C_{n,g}\|_2$ holds

$$\|C_{n,g}\|_2 = \left[(n, g)\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} c_s c_{s-mg}\right]^{\frac{1}{2}}.$$

□

Example 1. If we take firstly $c_i = F_i$ and secondly $c_i = L_i$ in Theorem 1, we have

$$\|C_{n,g}(F_0)\|_2 = \left[(n, g)\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} F_s F_{s-mg}\right]^{\frac{1}{2}}$$

and

$$\|C_{n,g}(L_0)\|_2 = \left[(n, g)\sum_{m=0}^{k-1}\sum_{s=0}^{n-1} L_s L_{s-mg}\right]^{\frac{1}{2}}.$$

Theorem 2. *The spectral norm of the matrix $C_{n,g}$ holds*

$$\|C_{n,g}\|_2 = \begin{cases} \sum_{s=0}^{n-1} c_s, & \text{if } c_s \geq 0, \quad 0 \leq s \leq n-1, \\ -\sum_{s=0}^{n-1} c_s, & \text{if } c_s \leq 0, \quad 0 \leq s \leq n-1, \end{cases}$$

where $(n, g) = 1$.

Proof. If we take $(n, g) = 1$ in Theorem 1, then $k = n$ and

$$\|C_{n,g}\|_2 = \left[\sum_{m=0}^{n-1}\sum_{s=0}^{n-1} c_s c_{s-mg}\right]^{\frac{1}{2}} = \left[\sum_{s=0}^{n-1} c_s \sum_{m=0}^{n-1} c_{s-mg}\right]^{\frac{1}{2}}.$$

Let $\underset{0\leq s\leq n-1}{H_s} = \{s - mg : 0 \leq m \leq n-1 \text{ and } (n, g) = 1\}$. Then

$$\underset{0\leq s\leq n-1}{H_s} = \{0, 1, \ldots, n-1\}$$

and

$$\sum_{m=0}^{n-1} c_{s-mg} = \sum_{t=0}^{n-1} c_t$$

under mod n. Thus

$$\|C_{n,g}\|_2 = \left[\sum_{s=0}^{n-1} c_s \sum_{t=0}^{n-1} c_t\right]^{\frac{1}{2}} = \left[\left(\sum_{s=0}^{n-1} c_s\right)^2\right]^{\frac{1}{2}} = \left|\sum_{s=0}^{n-1} c_s\right| = \sum_{s=0}^{n-1} |c_s| \,.$$

This completes the proof. □

Example 2. Let $(n,g)=1$. If we take firstly $c_i = F_i$ and secondly $c_i = L_i$ in Theorem 2, we have by (2.1)

$$\|C_{n,g}(F_0)\|_2 = F_{n+1} - 1$$

and by (2.2)

$$\|C_{n,g}(L_0)\|_2 = L_{n+1} - 1.$$

The our equalities in Example 2 have also been given as Theorem 3.1 and Theorem 3.2 in [24].

4. Determinants of $\mathbf{C}_{n,g}(F_1)$ and $\mathbf{C}_{n,g}(L_1)$

Theorem 3. *The determinant of $C_{n,g}(F_1)$ holds*

$$\det C_{n,g}(F_1) = \begin{cases} 0, & \textit{if} \ \ (n,g) \neq 1, \\ (1-F_{n+1})^{n-1} + F_n^{n-2} \sum\limits_{k=1}^{n-1} F_k \left(\frac{1-F_{n+1}}{F_n}\right)^{k-1}, & \textit{if} \ \ (n,g) = 1. \end{cases}$$

Proof. From (1.3), we write

$$C_{n,g}(F_1) = Q_{n,g} C_{n,1}(F_1).$$

Then, we have

$$\det C_{n,g}(F_1) = \det Q_{n,g} \det C_{n,1}(F_1),$$

where

$$\det Q_{n,g} = \begin{cases} 0, & \text{if} \ \ (n,g) \neq 1, \\ 1, & \text{if} \ \ (n,g) = 1. \end{cases}$$

By Lemma 1, the proof is completed. □

Theorem 4. *The determinant of $C_{n,g}(L_1)$ holds*

$$\det C_{n,g}(L_1) = \begin{cases} 0, \quad \textit{if} \ \ (n,g) \neq 1, \\ (1-L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum\limits_{k=1}^{n-1} (L_{k+2} - 3L_{k+1}) \left(\dfrac{1-L_{n+1}}{L_n - 2}\right)^{k-1}, \\ \quad \textit{if} \ \ (n,g) = 1. \end{cases}$$

Proof. By using Lemma 3 and the method of the proof of Theorem 3, the statement of theorem is proved easily. □

5. Inverses of $\mathbf{C}_{n,g}(F_1)$ and $\mathbf{C}_{n,g}(L_1)$

The matrices $C_{n,g}(F_1)$ and $C_{n,g}(L_1)$ are not invertible when $(n,g) \neq 1$, because their determinants are zero. Consequently, in this section we compute inverses of the matrices $C_{n,g}(F_1)$ and $C_{n,g}(L_1)$ under the condition $(n,g) = 1$.

Theorem 5. *Let f_i's be as in* (2.3) *and*

$$M_{n,g}(F) = \frac{1}{f_n}\left[g\text{-Circ}(f_0, f_{n-1}, f_{n-2}, \ldots, f_1)\right].$$

Then, for $n > 2$

$$[C_{n,g}(F_1)]^{-1} = (M_{n,g}(F))^T,$$

where $(M_{n,g}(F))^T$ is transpose of $M_{n,g}(F)$.

Proof. By (1.3) and (1.4), we have

$$C_{n,g}(F_1) = Q_{n,g}C_{n,1}(F_1)$$

and

$$(Q_{n,g})^{-1} = (Q_{n,g})^T.$$

Thus

$$[C_{n,g}(F_1)]^{-1} = [C_{n,1}(F_1)]^{-1}[Q_{n,g}]^T = \left[Q_{n,g}\left([C_{n,1}(F_1)]^{-1}\right)^T\right]^T. \tag{5.1}$$

From Lemma 2,

$$[C_{n,1}(F_1)]^{-1} = \frac{1}{f_n}\text{Circ}(f_0, f_1, \ldots, f_{n-1}).$$

Then

$$\left([C_{n,1}(F_1)]^{-1}\right)^T = \frac{1}{f_n}\text{Circ}(f_0, f_{n-1}, f_{n-2}, \ldots, f_1).$$

From (1.3), $Q_{n,g}\left([C_{n,1}(F_1)]^{-1}\right)^T$ is a g-circulant matrix and thus the first row of the matrix $Q_{n,g}\left([C_{n,1}(F_1)]^{-1}\right)^T$ is

$$\frac{1}{f_n}(f_0, f_{n-1}, f_{n-2}, \ldots, f_1).$$

Thus,

$$M_{n,g}(F) = Q_{n,g}\left([C_{n,1}(F_1)]^{-1}\right)^T. \tag{5.2}$$

By (5.1) and (5.2), we have

$$[C_{n,g}(F_1)]^{-1} = (M_{n,g}(F))^T.$$

□

Theorem 6. *Let l_i's be as in* (2.4) *and*

$$M_{n,g}(L) = \frac{1}{l_n}\left[g\text{-Circ}(l_0, l_{n-1}, l_{n-2}, \ldots, l_1)\right].$$

Then,

$$[C_{n,g}(L_1)]^{-1} = (M_{n,g}(L))^T,$$

where $(M_{n,g}(L))^T$ is transpose of $M_{n,g}(L)$.

Proof. The proof is completed easily by considering Lemma 4 and by using the method of the proof of Theorem 5. □

6. Conclusion

In this paper, we have dealt with the spectral norm of g-circulant matrix $C_{n,g} = g\text{-Circ}(c_0, c_1, \ldots, c_{n-1})$, where $c_i \geq 0$ or $c_i \leq 0$ (equivalently $c_i \cdot c_j \geq 0$). Also, we have computed the spectral norms, determinants and inverses of the g-circulant matrices with the Fibonacci and Lucas numbers by using results of the paper [16].

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