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SOME FINITE INTEGRALS INVOLVING THE PRODUCT OF BESSEL FUNCTION WITH JACOBI AND LAGUERRE POLYNOMIALS

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Abstract. The main object of this paper is to set up two (conceivably) valuable double integrals including the multiplication of Bessel function with Jacobi and Laguerre polynomials, which are given in terms of Srivastava and Daoust functions. By virtue of the most broad nature of the function included therein, our primary findings are equipped for yielding an extensive number of (presumably new) fascinating and helpful results involving orthogonal polynomials, Whittaker functions, sine and cosine functions.

1. Introduction

In recent years, various (potentially useful) integral operators including various kind of special functions have been considered through numerous authors (see, for example, [6], [7] and [1–5, 9–15]). Such integrals have numerous applications in varied field of designing and sciences. In particular, integrals associated with different kind of Bessel functions play an essential role in numerous field of material science, for example in the field of neutron physical science, plasma material science, radio material science and so forth. In a continuation of such sort of works specified above, in this paper, we establish two unified double integrals involving the multiplication of Bessel function with Jacobi and Laguerre polynomials, which are explicitly written in terms of Srivastava and Daoust functions.

$J_\mu(x)$ is the Bessel function (of the first kind) of order μ, (see [16], [18]) which is defined by

$$J_\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{\mu+2k}}{k!\Gamma(\mu+k+1)};\ \forall\ x \in C \setminus (-\infty, 0]; \tag{1}$$

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This is competently familiar that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \tag{2}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{3}$$

$P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial, defined by (see [16] and [18]):

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n, \quad 1+\alpha+\beta+n; & \frac{1-z}{2} \\ 1+\alpha; & \end{array} \right], \tag{4}$$

or equivalently

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k} (z-1)^k}{k!(n-k)!(1+\alpha)_k (1+\alpha+\beta)_n 2^k}, \tag{5}$$

is an important class of orthogonal polynomials which is the generalization of ultraspherical polynomial. From (4) and (5) we list below the number of important polynomials which can be expressed in terms of Jacobi polynomials for different values of α and β:

For $\beta = \alpha$,

$$P_n^{(\alpha,\alpha)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+2\alpha)_{n+k} (z-1)^k}{k!(n-k)!(1+\alpha)_k (1+2\alpha)_n 2^k}, \tag{6}$$

where $P_n^{(\alpha,\alpha)}(z)$ is the ultraspherical polynomial;

For $\alpha = \beta = \mu - \frac{1}{2}$,

$$P_n^{\left(\mu-\frac{1}{2},\mu-\frac{1}{2}\right)}(z) = \frac{\left(\mu+\frac{1}{2}\right)_n}{(2\mu)_n} C_n^{\mu}(z), \tag{7}$$

where $C_n^{\mu}(z)$ is the Gegenbauer polynomial;

For $\alpha = \beta = -\frac{1}{2}$,

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) = \frac{\left(\frac{1}{2}\right)_n}{(n)!} T_n(z), \tag{8}$$

where $T_n(z)$ is the Tchebycheff polynomial of first kind;

For $\alpha = \beta = \frac{1}{2}$,

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) = \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} U_n(z), \tag{9}$$

where $U_n(z)$ is the Tchebycheff polynomial of second kind;

For $\alpha = \beta = 0$,

$$P_n^{(0,0)}(z) = P_n(z), \tag{10}$$

where $P_n(z)$ is the Legendre polynomial.

The generalized Leguerre polynomial $L_n^\alpha(z)$ is defined by [16]:

$$L_n^\alpha(z) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n; 1+\alpha; z). \tag{11}$$

For $\alpha = 0$ the above polynomial is called the Laguerre or simple Laguerre polynomial.

From (11) it follows that

$$L_n^\alpha(z) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n z^k}{k!(n-k)!(1+\alpha)_k}. \tag{12}$$

Also, it is well known that

$$L_n^{-\frac{1}{2}}(z^2) = \frac{H_{2n}(z)}{(-1)^n 2^{2n} n!}; \tag{13}$$

$$L_n^{\frac{1}{2}}(z^2) = \frac{H_{2n+1}(z)}{(-1)^n 2^{2n+1} n! z}, \tag{14}$$

where $H_n(x)$ is the Hermite polynomial [16];

$$L_n^\alpha(z) = \frac{(1+\alpha)_n}{n!} z^{-\frac{1}{2}-\frac{\alpha}{2}} e^{\frac{z}{2}} M_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}(z); \tag{15}$$

$$L_n^\alpha(z) = \frac{(-1)^n}{n!} z^{-\frac{1}{2}-\frac{\alpha}{2}} e^{\frac{z}{2}} W_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}(z), \tag{16}$$

where $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are the Whittaker functions defined by Whittaker [19] (also see Whittaker and Watson [20]) in terms of confluent hypergeometric function ${}_1F_1$(or Kummer's functions) as follows:

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}}\,{}_1F_1\left(\frac{1}{2}+\mu-k, 2\mu+1; z\right),$$

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} U\left(\frac{1}{2}+\mu-k, 2\mu+1; z\right).$$

We retrace here the under mentioned intriguing and helpful result due to Edward [8], which is requisite in our current analysis

$$\int_0^1 \int_0^1 u^\mu (1-v)^{\mu-1} (1-u)^{\nu-1} (1-uv)^{1-\mu-\nu} dv du = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \tag{17}$$

provided $\Re(\mu) > 0$ and $\Re(\nu) > 0$.

2. Main results

Through this segment, we underlay two unified integrals involving the multiplication of Bessel function with Jacobi and Laguerre polynomials, which are explicitly written in terms of Srivastava and Daoust functions.

Theorem 2.1. *The under mentioned integral holds true, for* $\Re(\nu) > -1$, $\Re(\lambda+\nu) > 0$ *and* $\Re(\delta+\nu) > 0$,

$$\begin{aligned}
&\int_0^1\int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta}J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)\\
&\times P_n^{(\alpha,\beta)}\left(1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)dvdu = \frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}\\
&\times F^{4:0;0}_{5:0;1}\left[\begin{matrix}(\lambda+\nu:2,3), & (\delta+\nu:2,3), & (1+\alpha+\beta:1,2),\\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1+\alpha+\beta:1,1),\end{matrix}\right.\\
&\left.\begin{matrix}(1+\alpha:1,1): & _; & _; \\ (1:1,1): & _; & (1+\alpha,1);\end{matrix}\quad -1,1\right],
\end{aligned}\tag{18}$$

where $J_\nu(x)$ *and* $P_n^{(\alpha,\beta)}(z)$ *are the well known special functions defined by* (1) *and* (5)*, respectively,* $\Delta(m;l)$ *abbreviates the array of* m *parameters* $\frac{l}{m}, \frac{l+1}{m}, \ldots, \frac{l+m-1}{m}$, $m \geq 1$ *and* $F^{p:m_1;\cdots;m_s}_{\ell:q_1;\cdots;q_s}$ *is the Srivastava and Daoust multivariable hypergeometric function defined by* (*see,* [17, p. 454])

$$\begin{aligned}
&F^{p:m_1;\cdots;m_s}_{\ell:q_1;\cdots;q_s}\left[\begin{matrix}(a_j:\alpha_j^1,\ldots,\alpha_j^s)_{1,p}: & (c_j^1,r_j^1)_{1,q_1}; & \cdots; & (c_j^s,r_j^s)_{1,q_s};\\ (b_j:\beta_j^1,\ldots,\beta_j^s)_{1,\ell}: & (d_j^1,\delta_j^1)_{1,m_1}; & \cdots; & (d_j^s,\delta_j^s)_{1,m_s};\end{matrix}\ x_1,x_2,\ldots,x_s\right]\\
&= \sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha_j^1+\cdots+n_s\alpha_j^s}\prod_{j=1}^{q_1}(c_j^1)_{n_1r_j^1}\cdots\prod_{j=1}^{q_s}(c_j^s)_{n_sr_j^s}}{\prod_{j=1}^{l}(b_j)_{n_1\beta_j^1+\cdots+n_s\beta_j^s}\prod_{j=1}^{m_1}(d_j^1)_{n_1\delta_j^1}\cdots\prod_{j=1}^{m_s}(d_j^s)_{n_s\delta_j^s}}\frac{x_1^{n_1}}{(n_1)!}\cdots\frac{x_s^{n_s}}{(n_s)!}.
\end{aligned}\tag{19}$$

Proof. In the way to prove Theorem 2.1, we notify the left hand side of (2.1) by I, writing J_ν and $P_n^{(\alpha,\beta)}$ as a series with the help of (1) and (5), to get

$$\begin{aligned}
I &= \int_0^1\int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta}\sum_{n=0}^{\infty}\frac{(-1)^n[4u(1-v)(1-u)]^{\nu+2n}}{n!\Gamma(\nu+n+1)(1-uv)^{2\nu+4n}}\\
&\times\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n(1+\alpha+\beta)_{n+k}[4u(1-v)(1-u)]^k}{k!(n-k)!(1+\alpha)_k(1+\alpha+\beta)_n(1-uv)^{2k}}dvdu.
\end{aligned}\tag{20}$$

Using the following Lemma (see, [16]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),\tag{21}$$

in the aloft equation and then changing the order of integration and summation (which is true under the given conditions), we arrive at

(22)
$$I = 2^{2\nu} \sum_{n,k=0}^{\infty} \frac{(-1)^{n+2k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}2^{4n+6k}}{(n+k)!\Gamma(\nu+n+k+1)(1+\alpha)_k(1+\alpha+\beta)_{n+k}k!n!}$$
$$\times \int_0^1 \int_0^1 u^{\lambda+\nu+2n+3k}(1-v)^{\lambda+\nu+2n+3k-1}(1-u)^{\delta+\nu+2n+3k-1}(1-uv)^{1-\lambda-\delta-2\nu-4n-6k} dvdu.$$

Now using the result (17) and after a little simplification, we get

(23)
$$I = \frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+\nu)}$$
$$\times \sum_{n,k=0}^{\infty} \frac{(\lambda+\nu)_{2n+3k}(\delta+\nu)_{2n+3k}(1+\alpha+\beta)_{n+2k}(1+\alpha)_{n+k}(-1)^{n+2k}}{\left(\frac{\lambda+\delta+2\nu}{2}\right)_{2n+3k}\left(\frac{\lambda+\delta+2\nu+1}{2}\right)_{2n+3k}(\nu+1)_{n+k}(1+\alpha+\beta)_{n+k}(1)_{n+k}(1+\alpha)_k n!k!}.$$

In the end, summing up the series (23) by the assistance of (19), we readily attain our expected result. □

Theorem 2.2. *The under mentioned integral formula holds true, for* $\Re(\nu) > -1$, $\Re(\lambda+\nu) > 0$ *and* $\Re(\delta+\nu) > 0$,

$$\int_0^1 \int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$
$$\times L_n^\alpha\left(\tfrac{4u(1-v)(1-u)}{(1-uv)^2}\right) dvdu = \frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$
$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{llll} (\lambda+\nu:2,3), & (\delta+\nu:2,3), & (1+\alpha:1,1): & \_;\quad \_; \\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1:1,1): & \_;\quad (1+\alpha,1); \end{array} -1, 1\right], \tag{24}$$

where $L_n^\alpha(z)$ *is the generalized Laguerre polynomial.*

Proof. In the way to prove Theorem 2.2, we notify the left hand side of (24) by I', writing J_ν and L_n^α in their defining series, to get

$$I' = \int_0^1 \int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta} \sum_{n=0}^{\infty} \frac{(-1)^n[4u(1-v)(1-u)]^{\nu+2n}}{n!\Gamma(\nu+n+1)(1-uv)^{2\nu+4n}}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^k(1+\alpha)_n[4u(1-v)(1-u)]^k}{k!(n-k)!(1+\alpha)_k(1-uv)^{2k}} dvdu. \tag{25}$$

Now using the result (21) in (25) and then by changing the order of integration and summation (which is true under the given conditions), we arrive at

(26)
$$I' = 2^{2\nu} \sum_{n,k=0}^{\infty} \frac{(-1)^{n+2k}(1+\alpha)_{n+k} 2^{4n+6k}}{(n+k)!\Gamma(\nu+n+k+1)(1+\alpha)_k k! n!}$$
$$\times \int_0^1 \int_0^1 u^{\lambda+\nu+2n+3k}(1-v)^{\lambda+\nu+2n+3k-1}(1-u)^{\delta+\nu+2n+3k-1}(1-uv)^{1-\lambda-\delta-2\nu-4n-6k} dudv.$$

Further by using the result (17) in the above expression and after a little simplification, we have

(27)
$$I' = \frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+\nu)}$$
$$\times \sum_{n,k=0}^{\infty} \frac{(\lambda+\nu)_{2n+3k}(\delta+\nu)_{2n+3k}(1+\alpha)_{n+k}(-1)^{n+2k}}{\left(\frac{\lambda+\delta+2\nu}{2}\right)_{2n+3k}\left(\frac{\lambda+\delta+2\nu+1}{2}\right)_{2n+3k}(\nu+1)_{n+k}(1)_{n+k}(1+\alpha)_k n! k!}.$$

In the end, summing up the series (27) by the assistance of (19), we readily attain the right hand side of (24). This completes the proof of Theorem 2.2. □

3. Special cases

Corollary 3.1. *Applying* $\beta = \alpha$ *in* (18) *and then by assistance of* (6)*, we arrive at the under mentioned integral formula*

$$\int_0^1 \int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$
$$\times P_n^{(\alpha,\alpha)}\left(1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right) dvdu = \frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$
$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{l} (\lambda+\nu:2,3), \quad (\delta+\nu:2,3), \quad (1+2\alpha:1,2), \\ (\Delta(2,\lambda+\delta+2\nu):2,3), \quad (\nu+1:1,1), \quad (1+2\alpha:1,1), \end{array}\right.$$
$$\left.\begin{array}{l} (1+\alpha:1,1): \ \_; \quad \_; \\ (1:1,1): \ \_; \quad (1+\alpha,1); \end{array} -1,1\right], \tag{28}$$

where $\Re(\nu) > -1, \Re(\lambda+\nu) > 0$, $\Re(\delta+\nu) > 0$ *and* $P_n^{(\alpha,\alpha)}(z)$ *is the ultraspherical polynomial* [16].

Corollary 3.2. *Applying* $\beta = \alpha = \mu - \frac{1}{2}$ *in* (18) *and then by assistance of* (7)*, we obtain the under mentioned integral formula*

$$\int_0^1 \int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times C_n^{\mu}\left(1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)dvdu=\frac{2^{2\nu}(2\mu)_n\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{(\mu+\frac{1}{2})_n\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{c}(\lambda+\nu:2,3),\quad(\delta+\nu:2,3),\quad(2\mu:1,2),\\ \\ (\Delta(2,\lambda+\delta+2\nu):2,3),\quad(\nu+1:1,1),\quad(2\mu:1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(\frac{1}{2}+\mu:1,1):\ \_;\quad \_;\\ \\ (1:1,1):\ \_;\quad(\frac{1}{2}+\mu,1);\end{array}-1,1\right],\tag{29}$$

where $\Re(\nu)>-1,\Re(\lambda+\nu)>0,\Re(\delta+\nu)>0$ *and* $C_n^{\mu}(z)$ *is the Gegenbauer polynomial* [16].

Corollary 3.3. *If we place* $\beta=\alpha=\frac{1}{2}$ *in* (18)*, then by assistance of* (9)*, we get under mentioned integral formula*

$$\int_0^1\int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta}J_{\nu}\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times U_n\left(1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)dvdu=\frac{2^{2\nu}(n+1)!\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{(\frac{3}{2})_n\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{c}(\lambda+\nu:2,3),\quad(\delta+\nu:2,3),\quad(2:1,2),\\ \\ (\Delta(2,\lambda+\delta+2\nu):2,3),\quad(\nu+1:1,1),\quad(2:1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(\frac{3}{2}:1,1):\ \_;\quad \_;\\ \\ (1:1,1):\ \_;\quad(\frac{3}{2},1);\end{array}-1,1\right],\tag{30}$$

where $\Re(\nu)>-1,\Re(\lambda+\nu)>0,\Re(\delta+\nu)>0$ *and* $U_n(z)$ *is the Tchebycheff polynomial of second kind* [16].

Corollary 3.4. *If we place* $\beta=\alpha=0$ *in* (18)*, then by assistance of* (10)*, we get the under mentioned integral*

$$\int_0^1\int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta}J_{\nu}\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times P_n\left(1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)dvdu=\frac{2^{2\nu}\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{c}(\lambda+\nu:2,3),\quad(\delta+\nu:2,3),\quad(1:1,2):\\ \\ (\Delta(2,\lambda+\delta+2\nu):2,3),\quad(\nu+1:1,1),\quad(1:1,1):\end{array}\right.$$

$$\left.\begin{array}{c}\_;\quad \_;\\ \\ \_;\quad(1,1);\end{array}-1,1\right],\tag{31}$$

where $\Re(\nu) > -1, \Re(\lambda+\nu) > 0, \Re(\delta+\nu) > 0$ *and* $P_n(z)$ *is the Legendre polynomial* [16].

Corollary 3.5. *If we assign* $\nu = \frac{1}{2}$ *in* (18)*, then by assistance of* (2)*, we have*

$$\int_0^1 \int_0^1 u^{\lambda-\frac{1}{2}}(1-v)^{\lambda-\frac{3}{2}}(1-u)^{\delta-\frac{3}{2}}(1-uv)^{2-\lambda-\delta} \sin\left[\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right]$$

$$\times P_n^{(\alpha,\beta)}\left[1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right] dvdu = \frac{8\Gamma(\lambda+\frac{1}{2})\Gamma(\delta+\frac{1}{2})}{\Gamma(\lambda+\delta+1)}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{llll} (\lambda+\frac{1}{2}:2,3), & (\delta+\frac{1}{2}:2,3), & (1+\alpha+\beta:1,2), & \\ (\Delta(2,\lambda+\delta+1):2,3), & (\frac{3}{2}:1,1), & (1+\alpha+\beta:1,1), & \end{array}\right.$$

$$\left.\begin{array}{lll} (1+\alpha:1,1): & \_; & \_; \\ (1:1,1): & \_; & (1+\alpha,1); \end{array} -1,1\right], \tag{32}$$

where $\Re(\lambda) > -\frac{1}{2}$ *and* $\Re(\delta) > -\frac{1}{2}$.

Corollary 3.6. *Applying* $\nu = -\frac{1}{2}$ *in* (18) *and then by assistance of* (3)*, we get the under mentioned integral*

$$\int_0^1 \int_0^1 u^{\lambda-\frac{1}{2}}(1-v)^{\lambda-\frac{3}{2}}(1-u)^{\delta-\frac{3}{2}}(1-uv)^{2-\lambda-\delta} \cos\left[\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right]$$

$$\times P_n^{(\alpha,\beta)}\left[1-\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right] dvdu = \frac{\Gamma(\lambda-\frac{1}{2})\Gamma(\delta-\frac{1}{2})}{(\Gamma(\lambda+\delta-1)}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{llll} (\lambda-\frac{1}{2}:2,3), & (\delta-\frac{1}{2}:2,3), & (1+\alpha+\beta:1,2), & \\ (\Delta(2,\lambda+\delta+1):2,3), & (\frac{1}{2}:1,1), & (1+\alpha+\beta:1,1), & \end{array}\right.$$

$$\left.\begin{array}{lll} (1+\alpha:1,1): & \_; & \_; \\ (1:1,1): & \_; & (1+\alpha,1); \end{array} -1,1\right], \tag{33}$$

where $\Re(\lambda) > \frac{1}{2}$ *and* $\Re(\delta) > \frac{1}{2}$.

Corollary 3.7. *By assistance of* (15) *in* (24)*, we have the integral*

$$\int_0^1 \int_0^1 u^{\lambda-\frac{1}{2}-\frac{\alpha}{2}}(1-v)^{\lambda-\frac{3}{2}-\frac{\alpha}{2}}(1-u)^{\delta-\frac{3}{2}-\frac{\alpha}{2}}(1-uv)^{2-\lambda-\delta+\alpha} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times e^{\frac{2u(1-v)(1-u)}{(1-uv)^2}} M_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}\left(\tfrac{4u(1-v)(1-u)}{(1-uv)^2}\right) dvdu = \frac{2^{2\nu+\alpha+1} n!\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{(1+\alpha)_n\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{llll} (\lambda+\nu:2,3), & (\delta+\nu:2,3), & (1+\alpha:1,1): & \_; \\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1:1,1): & \_; \end{array}\right.$$

$$\left. \begin{matrix} \_; \\ (1+\alpha,1); \end{matrix} \; -1,1 \right], \tag{34}$$

where $\Re(\nu) > -1, \Re(\alpha) > -1, \Re(\lambda+\nu) > 0, \Re(\delta+\nu) > 0$ *and* $M_{k,\nu}(z)$ *is the Whittaker function of first kind* [20].

Corollary 3.8. *By assistance of* (16) *in* (24)*, we have the integral*

$$\int_0^1 \int_0^1 u^{\lambda-\frac{1}{2}-\frac{\alpha}{2}}(1-v)^{\lambda-\frac{3}{2}-\frac{\alpha}{2}}(1-v)^{\delta-\frac{3}{2}-\frac{\alpha}{2}}(1-uv)^{2-\lambda-\delta+\alpha} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times e^{\frac{2u(1-v)(1-u)}{(1-uv)^2}} W_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}\left(\tfrac{4u(1-v)(1-u)}{(1-uv)^2}\right) dvdu = \frac{2^{2\nu+\alpha+1} n!\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{(-1)^n\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{matrix} (\lambda+\nu:2,3), & (\delta+\nu:2,3), & (1+\alpha:1,1): & \_; \\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1:1,1): & \_; \end{matrix} \right.$$

$$\left. \begin{matrix} \_; \\ (1+\alpha,1); \end{matrix} \; -1,1 \right], \tag{35}$$

where $\Re(\nu) > -1, \Re(\alpha) > -1, \Re(\lambda+\nu) > 0, \Re(\delta+\nu) > 0$ *and* $W_{k,\nu}(z)$ *is the Whittaker function of second kind* [20].

Corollary 3.9. *Applying* $\alpha = -\frac{1}{2}$ *in* (24) *and then by assistance of* (13)*, we get the integral*

$$\int_0^1 \int_0^1 u^{\lambda}(1-v)^{\lambda-1}(1-u)^{\delta-1}(1-uv)^{1-\lambda-\delta} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times H_{2n}\left(\tfrac{2\sqrt{u(1-v)(1-u)}}{(1-uv)}\right) dvdu = \frac{2^{2\nu+2n}(-1)^n n!\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{matrix} (\lambda+\nu:2,3), & (\delta+\nu:2,3), & (\frac{1}{2}:1,1): \\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1:1,1): \end{matrix} \right.$$

$$\left. \begin{matrix} \_; & \_; \\ \_; & (\frac{1}{2},1); \end{matrix} \; -1,1 \right], \tag{36}$$

where $\Re(\nu) > -1, \Re(\lambda+\nu) > 0, \Re(\delta+\nu) > 0$ *and* $H_n(x)$ *is the Hermite polynomial* [16].

Corollary 3.10. *Applying* $\alpha = \frac{1}{2}$ *in* (24) *and then by assistance of* (14)*, we arrive at the integral*

$$\int_0^1 \int_0^1 u^{\lambda-\frac{1}{2}}(1-v)^{\lambda-\frac{3}{2}}(1-u)^{\delta-\frac{3}{2}}(1-uv)^{2-\lambda-\delta} J_\nu\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times H_{2n+1}\left(\tfrac{2\sqrt{u(1-v)(1-u)}}{(1-uv)}\right)dudv=\frac{2^{2\nu+2n+2}(-1)^n n!\Gamma(\lambda+\nu)\Gamma(\delta+\nu)}{\Gamma(\nu+1)\Gamma(\lambda+\delta+2\nu)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{cccccc}(\lambda+\nu:2,3), & (\delta+\nu:2,3), & (\tfrac{3}{2}:1,1): & \_; & \_; & \\ & & & & & -1,1\\ (\Delta(2,\lambda+\delta+2\nu):2,3), & (\nu+1:1,1), & (1:1,1): & \_; & (\tfrac{3}{2},1); & \end{array}\right], \tag{37}$$

where $\Re(\nu)>-1, \Re(\lambda+\nu)>0$ *and* $\Re(\delta+\nu)>0$.

Corollary 3.11. *If we place* $\nu=\frac{1}{2}$ *in* (24), *then by assistance of* (2), *we have the integral*

$$\int_0^1\int_0^1 u^{\lambda-\frac{1}{2}}(1-v)^{\lambda-\frac{3}{2}}(1-u)^{\delta-\frac{3}{2}}(1-uv)^{2-\lambda-\delta}\sin\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times L_n^{(\alpha)}\left(\tfrac{4u(1-v)(1-u)}{(1-uv)^2}\right)dvdu=\frac{8\Gamma(\lambda+\frac{1}{2})\Gamma(\delta+\frac{1}{2})}{\Gamma(\lambda+\delta+1)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{cccccc}(\lambda+\frac{1}{2}:2,3), & (\delta+\frac{1}{2}:2,3), & (1+\alpha:1,1): & \_; & \_; & \\ & & & & & -1,1\\ (\Delta(2,\lambda+\delta+1):2,3), & (\frac{3}{2}:1,1), & (1:1,1): & \_; & (1+\alpha,1); & \end{array}\right], \tag{38}$$

where $\Re(\lambda)>-\frac{1}{2}$ *and* $\Re(\delta)>-\frac{1}{2}$.

Corollary 3.12. *If we place* $\nu=-\frac{1}{2}$ *in* (24), *then by assistance of* (3), *we get the integral*

$$\int_0^1\int_0^1 u^{\lambda-\frac{1}{2}}(1-v)^{\lambda-\frac{3}{2}}(1-u)^{\delta-\frac{3}{2}}(1-uv)^{2-\lambda-\delta}\cos\left(\tfrac{8u(1-v)(1-u)}{(1-uv)^2}\right)$$

$$\times L_n^{(\alpha)}\left(\tfrac{4u(1-v)(1-u)}{(1-uv)^2}\right)dvdu=\frac{\Gamma(\lambda-\frac{1}{2})\Gamma(\delta-\frac{1}{2})}{\Gamma(\lambda+\delta-1)}$$

$$\times F^{3:0;0}_{4:0;1}\left[\begin{array}{cccccc}(\lambda-\frac{1}{2}:2,3), & (\delta-\frac{1}{2}:2,3), & (1+\alpha:1,1): & \_; & \_; & \\ & & & & & -1,1\\ (\Delta(2,\lambda+\delta-1):2,3), & (\frac{1}{2}:1,1), & (1:1,1): & \_; & (1+\alpha,1); & \end{array}\right], \tag{39}$$

where $\Re(\lambda)>\frac{1}{2}$ *and* $\Re(\delta)>\frac{1}{2}$.

Remark 3.13. With the help of Theorem 2.1 we can establish some other corollaries in which the double integral including the product of sine and cosine functions with ultraspherical polynomial, Gegenbauer polynomial, Tchebycheff polynomial of second kind and Legendre polynomial.

Remark 3.14. In identical way, with the help of Theorem 2.2 we can establish some other corollaries in which the double integral including the product of sine and cosine functions with Whittaker function of first kind, Whittaker function of second kind and Hermite polynomial.

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