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CERTAIN RESULTS ON CONTACT METRIC GENERALIZED (κ, μ)-SPACE FORMS

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Abstract. The object of the present paper is to study η-recurrent $*$-Ricci tensor, $*$-Ricci semisymmetric and globally φ-$*$-Ricci symmetric contact metric generalized (κ, μ)-space form. Besides these, $*$-Ricci soliton and gradient $*$-Ricci soliton in contact metric generalized (κ, μ)-space form have been studied.

1. Introduction

In 1995 Blair et al. [7] introduced the notion of a contact metric manifold with characteristic vector field ξ belonging to the (κ, μ)-nullity distribution, called a (κ, μ)-contact metric manifold. A contact metric manifold M is said to be (κ, μ)-space if its curvature tensor R satisfies the condition

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \tag{1}$$

for any $X, Y \in TM$, where κ and μ are constants on M and $2h = L_\xi \varphi$ (here L is the usual Lie derivative). In 2000, Koufogiorgos and Tsichlias [21] generalized the notion of a (κ, μ)-contact metric manifold by taking the constants κ and μ in (1) to be smooth functions on M, called a generalized (κ, μ)-contact metric manifold and further such type of manifolds have been studied by several authors [10, 12, 22, 25, 26].

A (κ, μ)-space M of dimension greater than 3 with constant φ-sectional curvature c is called (κ, μ)-space form [20] and the curvature tensor R of such a type of manifold is given by [20]

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}R_1(X,Y)Z + \frac{c-1}{4}R_2(X,Y)Z + \left(\frac{c+3}{4} - \kappa\right)R_3(X,Y)Z \\ & + R_4(X,Y)Z + \frac{1}{2}R_5(X,Y)Z + (1-\mu)R_6(X,Y)Z, \end{aligned} \tag{2}$$

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where R_1, R_2, R_3, R_4, R_5, R_6 are defined as

$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$

$R_2(X,Y)Z = g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z,$

$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$

$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$

$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX,$

$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,$

for all vector fields X, Y, Z on M. As a generalization of (κ,μ)-space form, in [9] Carriazo et al. introduced and studied the notion of generalized (κ,μ)-space form by providing several interesting examples. A generalized (κ,μ)-space form is an almost contact metric manifold (M,φ,ξ,η,g) whose curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1R_1(X,Y)Z + f_2R_2(X,Y)Z + f_3R_3(X,Y)Z \\ & + f_4R_4(X,Y)Z + f_5R_5(X,Y)Z + f_6R_6(X,Y)Z, \end{aligned} \tag{3}$$

where R_1, R_2, R_3, R_4, R_5, R_6 are defined as in (2) and f_1, f_2, f_3, f_4, f_5, f_6 are differentiable functions on M. This manifold is denoted by $M(f_1,\ldots,f_6)$.

If, in particular, $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$, $f_3 = \left(\frac{c+3}{4} - \kappa\right)$, $f_4 = 1$, $f_5 = \frac{1}{2}$ and $f_6 = (1-\mu)$, then the generalized (κ,μ)-space form turns out to the notion of (κ,μ)-space forms. In this connection it may be noted that the generalized (κ,μ)-space form is the generalization of the generalized Sasakian space form introduced by Alegre et al. [1] and further such manifolds have been studied by many authors [2–4, 18, 29, 31]. The generalized (κ,μ)-space forms have also been studied by Prakasha et al. [23] and Premalatha and Nagaraja [24]. Very recently, Shanmukha et al. [27] and Hui et al. [16] studied some interesting results on generalized (κ,μ)-space forms.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [8]. On the manifold M, a Ricci soliton is a triple (g,V,λ) with g is a Riemannian metric, V a vector field, called potential vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda = 0, \tag{4}$$

where L is the Lie derivative. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive. If the vector field V is the gradient of a smooth function $-f$, then g is called a gradient Ricci soliton and equation (4) takes the form

$$\nabla\nabla f = S + \lambda g, \tag{5}$$

where ∇ denotes the Riemannian connection. In 2014 Kaimakamis and Panagiotidou [17] introduced and studied the notion of $*$-Ricci solitons within the framework of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (4) with the $*$-Ricci tensor S^*. A Riemannian metric g on a manifold M is called a $*$-Ricci soliton if there exist a constant λ and a vector field V such that

$$L_V g + 2S^* + 2\lambda = 0 \tag{6}$$

for all vector fields X, Y on M. Moreover, if the vector field V is a gradient of a smooth function $-f$, then we say that it is gradient $*$-Ricci soliton and equation (6) takes the form

$$\nabla\nabla f = S^* + \lambda g. \tag{7}$$

Note that a $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case the manifold becomes $*$-Einstein. Here, it is suitable to mention that the notion of $*$-Ricci tensor was first introduced by Tachibana [30] on almost Hermitian manifolds and further studied by Hamada [14] on real hypersurfaces of non-flat complex space forms.

Motivated by the above mentioned works, the present paper is organized as follows: After preliminaries in Section 3, we consider η-recurrent $*$-Ricci tensor contact metric generalized (κ,μ)-space forms. Section 4 is devoted to study of $*$-Ricci semisymmetric contact metric generalized (κ,μ)-space forms. In this section we prove that if a contact metric generalized (κ,μ)-space form is $*$-Ricci semisymmetric, then either $f_1 = f_3$ or $f_1 + (2n+1)f_2 = 0$. Section 5 deals with globally φ-$*$-Ricci symmetric contact metric generalized (κ,μ)-space form. Finally, we study $*$-Ricci soliton and gradient $*$-Ricci soliton in contact metric generalized (κ,μ)-space form.

2. Preliminaries

In this section we present some general definitions and basic formulas which will be used later. For more background on almost contact metric manifolds, we recommend the reference [5, 6].

A $(2n+1)$-dimensional smooth connected manifold M is called almost contact manifold if it admits a triple (φ,ξ,η), where φ is tensor field of type $(1,1)$, ξ is a global vector field and η is a 1-form, such that

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta\circ\varphi = 0, \tag{8}$$

for all $X, Y \in TM$. An almost contact structure is said to be normal if the induced almost complex structure J on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where t is the coordinate of R and f is a smooth function on $M \times R$. Let g be a compatible Riemannian metric with (φ,ξ,η), that is,

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

or equivalently

$$g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g(X,\xi) = \eta(X),$$

for all vector fields $X, Y \in TM$. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y) \tag{10}$$

for all $X, Y \in TM$. It is well known that on a contact metric manifold $(M, \varphi, \xi, \eta, g)$, the tensor h, defined by $2h = L_\xi \varphi$, is symmetric and satisfies the following relations [6]

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \eta \circ h = 0, \quad tr(h) = tr(\varphi h) = 0, \tag{11}$$

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{12}$$

The vector field ξ is a Killing vector with respect to g if and only if $h = 0$. A contact metric manifold $(M, \varphi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a K-contact manifold. By a generalized (κ, μ)-manifold, we mean a three dimensional contact metric manifold such that

$$R(X, Y)\xi = \kappa I + \mu h[\eta(Y)X - \eta(X)Y] \tag{13}$$

for all $X, Y \in TM$, where κ, μ are smooth non-constant real functions on M. In the special case where κ, μ are constants, then $(M, \varphi, \xi, \eta, g)$ is called a (κ, μ)-manifold. We note that in any Sasakian manifold $h = 0$ and $\kappa = 1$.

Lemma 2.1 ([21, 22])**.** *In any generalized (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ the following formulas are valid*:

$$h^2 = (\kappa - 1)\varphi^2, \quad \kappa \leq 1, \tag{14}$$

$$\xi\kappa = 0, \quad \xi r = 0, \qquad hD\mu = D\kappa, \tag{15}$$

where D is the gradient operator of g.

In a $(2n+1)$-dimensional generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$, the following relations hold [9, 16]

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{16}$$

$$\begin{aligned} S(X, Y) = {} & [2nf_1 + 3f_2 - f_3]g(X, Y) + [(2n-1)f_4 - f_6]g(hX, Y) \\ & - [3f_2 + (2n-1)f_3]\eta(X)\eta(Y), \end{aligned} \tag{17}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \quad \text{and} \tag{18}$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3]. \tag{19}$$

We know that a contact metric manifold is K-contact if and only if $h = 0$. Therefore, a generalized (κ, μ)-space form with such a structure is actually a generalized Sasakian-space form.

In [9], the following theorems have been proved:

Theorem 2.2 ([9])**.** *Let $M(f_1, \ldots, f_6)$ be a generalized (κ, μ)-space form. If M is a K-contact manifold, then $f_1 - f_3 = 1$. Moreover, M is Sasakian.*

Theorem 2.3 ([9])**.** *If $M(f_1, \ldots, f_6)$ is a contact metric generalized (κ, μ)-space form, then it is a generalized (κ, μ)-space, with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$.*

On the other hand, let $M(\varphi,\xi,\eta,g)$ be an almost contact metric manifold with Ricci tensor S. The $*$-Ricci tensor and $*$-scalar curvature of M are defined respectively by

$$S^*(X,Y)=\sum_{i=1}^{2n+1}R(X,e_i,\varphi e_i,\varphi Y),\qquad r^*=\sum_{i=1}^{2n+1}S^*(e_i,e_i), \tag{20}$$

for all $X,Y\in TM$, where $e_1,\dots,e_{2n+1}$ is an orthonormal basis of the tangent space TM. By using the first Bianchi identity and (20) we get

$$S^*(X,Y)=\frac{1}{2}\sum_{i=1}^{2n+1}g(\varphi R(X,\varphi Y)e_i,e_i). \tag{21}$$

An almost contact metric manifold is said to be $*$-Einstein if S^* is a constant multiple of the metric g. One can see $S^*(X,\xi)=0$ for all $X\in TM$. It should be remarked that S^* is not symmetric, in general. Thus the condition $*$-Einstein automatically requires a symmetric property of the $*$-Ricci tensor [15].

3. η-recurrent contact metric generalized (κ,μ)-space forms

Before entering the main results, first we make an effort to find the expression of $*$-Ricci tensor on contact metric generalized (κ,μ)-space form.

Lemma 3.1. *In a contact metric generalized (κ,μ)-space form $M(f_1,\dots,f_6)$, the $*$-Ricci tensor is given by*

$$S^*(X,W)=[f_1+(2n+1)f_2]g(X,W)-[f_1+(2n+1)f_2]\eta(X)\eta(W). \tag{22}$$

Proof. Since M is a contact metric manifold, substituting $Z=\varphi Z$ in (3) and taking inner product with φW, and then using (8) and (11) the resultant equation becomes

$$\begin{aligned}R(X,Y,\varphi Z,\varphi W)=\;&f_1\{g(Y,\varphi Z)g(X,\varphi W)-g(X,\varphi Z)g(Y,\varphi W)\}\\&+f_2\{\eta(Z)\eta(X)g(\varphi Y,\varphi W)-g(X,Z)g(\varphi Y,\varphi W)\\&+g(Y,Z)g(\varphi X,\varphi W)-\eta(Z)\eta(Y)g(\varphi X,\varphi W)\\&-2g(X,\varphi Y)g(Z,\varphi W)\}+f_4\{g(Y,\varphi Z)g(hX,\varphi W)\\&-g(X,\varphi Z)g(hY,\varphi W)+g(hY,\varphi Z)g(X,\varphi W)\\&-g(hX,\varphi Z)g(Y,\varphi W)\}+f_5\{g(hY,\varphi Z)g(hX,\varphi W)\\&-g(hX,\varphi Z)g(hY,\varphi W)+g(hX,Z)g(hY,W)\\&-g(hY,Z)g(hX,W)\}.\end{aligned} \tag{23}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y=Z=e_i$ in (23) and taking summation over $1\le i\le 2n+1$, we obtain

$$R(X,e_i,\varphi e_i,\varphi W)=f_1g(\varphi X,\varphi W)+f_2\{g(X,\varphi^2W)+(2n+2)g(\varphi X,\varphi W)\}$$

$$+ f_4\{g(\varphi X, h\varphi W) + tr(\varphi h)g(X, \varphi W) + g(\varphi hX, \varphi W)\}$$
$$+ f_5\{tr(h\varphi)g(hX, \varphi W) + g(\varphi hX, h\varphi W) + g(hX, hW)$$
$$- tr(h)g(hX, W)\}. \tag{24}$$

Since in a contact metric manifold $tr(h) = tr(\varphi h) = 0$ and $h\varphi + \varphi h = 0$, we have from (8) that

$$R(X, e_i, \varphi e_i, \varphi W) = f_1 g(\varphi X, \varphi W) + f_2(2n+1)g(\varphi X, \varphi W).$$

From definition of $*$-Ricci tensor and (9), we have (22), completing the proof. □

From Theorem 3.1, we conclude that

Corollary 3.2. *A contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is an $*$-η-Einstein manifold.*

Definition. A $(2n+1)$-dimensional contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is said to have η-recurrent Ricci tensor if there exists a non-zero 1-form A such that

$$(\nabla_W S)(\varphi X, \varphi Y) = A(W)S(X, Y).$$

If the 1-form vanishes on $M(f_1, \ldots, f_6)$, then the space form is said to have η-parallel Ricci tensor. The notion of η-parallel Ricci tensor was introduced by Kon in the context of Sasakian geometry [19].

Analogous to the Definition above, we define the following:

Definition. A $(2n+1)$-dimensional contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is said to have η-recurrent $*$-Ricci tensor if there exists a non-zero 1-form A such that

$$(\nabla_W S^*)(\varphi X, \varphi Y) = A(W)S^*(X, Y). \tag{25}$$

If the 1-form vanishes on $M(f_1, \ldots, f_6)$, then the space form is said to have η-parallel $*$-Ricci tensor.

In view of (22), we have

$$(\nabla_W S^*)(\varphi X, \varphi Y) = d(f_1 + (2n+1)f_2)(W)\{g(X, Y) - \eta(X)\eta(Y)\}$$
$$- (f_1 + (2n+1)f_2)\{(\nabla_W \eta)(X)\eta(Y) - (\nabla_W \eta)(Y)\eta(X)\}. \tag{26}$$

Suppose contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ has an η-recurrent $*$-Ricci tensor. Then in view of (25) and (26) it follows that

$$d(f_1 + (2n+1)f_2)(W)\{g(X, Y) - \eta(X)\eta(Y)\} - (f_1 + (2n+1)f_2)$$
$$\{(\nabla_W \eta)(X)\eta(Y) - (\nabla_W \eta)(Y)\eta(X)\} = A(W)\{[f_1 + (2n+1)f_2]$$
$$g(X, W) - [f_1 + (2n+1)f_2]\eta(X)\eta(Y)\}. \tag{27}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $X = Y = e_i$ in (27) and taking summation over $1 \leq i \leq 2n+1$, we obtain

$$2n\; d(f_1 + (2n+1)f_2)(W) - 2(f_1 + (2n+1)f_2)(\nabla_W \eta)(\xi) = 2nA(W)(f_1 + (2n+1)f_2). \tag{28}$$

We know that $\eta(\xi) = 1$. Therefore

$$(\nabla_W \eta)\xi = 0. \tag{29}$$

So (28) and (29) gives us

$$d(f_1 + (2n+1)f_2)(W) = A(W)(f_1 + (2n+1)f_2). \tag{30}$$

Let $f_1 + (2n+1)f_2 = f$. Then (30) reduces to

$$df(W) = fA(W). \tag{31}$$

From (31) it follows that

$$df(Y)A(W) + (\nabla_Y A)(W)f = d^2 f(W, Y). \tag{32}$$

Interchanging Y and W in (32) we get

$$df(W)A(Y) + (\nabla_W A)(Y)f = d^2 f(Y, W). \tag{33}$$

Subtracting (33) from (32) we get

$$(\nabla_W A)(Y) - (\nabla_Y A)(W) = 0. \tag{34}$$

Hence 1-form A is closed. Thus we have the following:

Theorem 3.3. *If a contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ has an η-recurrent $*$-Ricci tensor, then the 1-form A is closed.*

Since A is non-zero, equation (30) leads us to state the following:

Corollary 3.4. *If a contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ has an η-recurrent $*$-Ricci tensor, then $f_1 + (2n+1)f_2$ can never be a non-zero constant.*

Corollary 3.5. *A contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ has an η-parallel $*$-Ricci tensor if and only if $f_1 + (2n+1)f_2$ is constant.*

4. $*$-Ricci semisymmetric contact metric generalized (κ, μ)-space forms

A contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is called Ricci semisymmetric if $R(X, Y) \cdot S = 0$ for all $X, Y \in TM$. Analogous to this definition, we define $*$-Ricci semisymmetric by $R(X, Y) \cdot S^* = 0$.

Let us consider a contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ satisfying the condition

$$R(X, Y) \cdot S^* = 0. \tag{35}$$

From (35) we have

$$S^*(R(X,Y)U,V)+S^*(U,R(X,Y)V)=0.$$

Substituting $X=U=\xi$ in the foregoing equation we get

$$S^*(R(\xi,Y)\xi,V)+S^*(\xi,R(\xi,Y)V)=0. \tag{36}$$

Using $S^*(X,\xi)=0$ and (16) in (36) we obtain

$$(f_1-f_3)S^*(Y,V)+(f_4-f_6)S^*(hY,V)=0. \tag{37}$$

In view of (22), equation (37) takes the form

$$\begin{aligned}&(f_1-f_3)(f_1+(2n+1)f_2)\{g(Y,V)-\eta(Y)\eta(V)\}\\&+(f_4-f_6)(f_1+(2n+1)f_2)g(hY,V)=0.\end{aligned} \tag{38}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y=V=e_i$ in (38) and taking summation over $1\leq i\leq 2n+1$, we get

$$2n(f_1-f_3)(f_1+(2n+1)f_2)+(f_4-f_6)(f_1+(2n+1)f_2)tr(h)=0. \tag{39}$$

Since M is a contact metric manifold, in view of (11), we have

$$2n(f_1-f_3)(f_1+(2n+1)f_2)=0, \tag{40}$$

which gives either $(f_1-f_3)=0$ or $f_1+(2n+1)f_2=0$. Thus we can state the following:

Theorem 4.1. *If a contact metric generalized (κ,μ)-space form $M(f_1,\ldots,f_6)$ is $*$-Ricci semisymmetric, then either $f_1=f_3$ or $f_1+(2n+1)f_2=0$.*

5. Globally φ-$*$-Ricci symmetric contact metric generalized (κ,μ)-space forms

Definition. A contact metric generalized (κ,μ)-space form $M(f_1,\ldots,f_6)$ is said to be globally φ-$*$-Ricci symmetric if the $*$-Ricci operator Q^* satisfies

$$\varphi^2((\nabla_X Q^*)(Y))=0,$$

for all vector fields $X,Y\in TM$ and $S^*(X,Y)=g(Q^*X,Y)$. In particular, if X,Y are orthogonal to ξ, then the space form is said to be locally φ-$*$-Ricci symmetric.

Let us suppose that a contact metric generalized (κ,μ)-space form $M(f_1,\ldots,f_6)$ is globally φ-$*$-Ricci symmetric. Then by definition

$$\varphi^2((\nabla_X Q^*)(Y))=0. \tag{41}$$

Making use of (8) in (41), we have

$$-(\nabla_X Q^*)(Y)+\eta((\nabla_X Q^*)(Y))\xi=0. \tag{42}$$

Equivalently

$$-g((\nabla_X Q^*)(Y),Z)+\eta((\nabla_X Q^*)(Y))\eta(Z)=0, \tag{43}$$

i.e.,

$$-g(\nabla_X Q^* Y, Z) + g(Q^* \nabla_X Y, Z) + \eta((\nabla_X Q^*)(Y))\eta(Z) = 0. \tag{44}$$

Putting $Y = \xi$ in (44) and using $Q^*\xi = 0$ (follows from (22)), we obtain

$$g(Q^* \nabla_X \xi, Z) + \eta((\nabla_X Q^*)\xi)\eta(Z) = 0. \tag{45}$$

Making use of (12) in (45), we obtain

$$-S^*(\varphi X, Z) - S^*(\varphi h X, Z) + \eta((\nabla_X Q^*)\xi)\eta(Z) = 0. \tag{46}$$

Replacing Z by φZ in (46) and applying (8), we obtain

$$S^*(\varphi X, \varphi Z) + S^*(\varphi h X, \varphi Z) = 0. \tag{47}$$

Again replacing X by hX in (47) and using (14) with $\kappa = f_1 - f_3$, we have

$$S^*(\varphi h X, \varphi Z) = (f_1 - f_3 - 1)S^*(\varphi X, \varphi Z). \tag{48}$$

Making use of (48) in (47), we have

$$(f_1 - f_3)S^*(\varphi X, \varphi Z) = 0. \tag{49}$$

Substituting φX for X and φZ for Z in (49) and applying (8), we obtain

$$(f_1 - f_3)S^*(X, Z) = 0, \tag{50}$$

which gives either $(f_1 - f_3) = 0$ or $S^*(X, Z) = 0$. Thus we can state the following:

Theorem 5.1. *If a contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is globally φ-$*$-Ricci symmetric, then either $f_1 = f_3$ or $M(f_1, \ldots, f_6)$ is $*$-Ricci flat.*

6. $*$-Ricci solitons and gradient $*$-Ricci solitons in contact metric generalized (κ, μ)-space forms

In [11], Ghosh studied gradient almost Ricci soliton and in [28] Sharma studied Ricci soliton in (κ, μ)-contact metric manifolds. Further, De and Mandal studied Ricci solitons on generalized (κ, μ)-contact metric manifolds [10]. Very recently, Ghosh and Patra [13] studied $*$-Ricci solitons with in the frame work of Sasakian geometry.

Now we investigate $*$-Ricci soliton when $V = \xi$. We have from (6) that

$$L_\xi g(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0. \tag{51}$$

This can be written as

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0. \tag{52}$$

Using (12) in (52) we get

$$\begin{aligned} -\{g(\varphi X, Y) + g(\varphi h X, Y) + g(X, \varphi Y) + g(X, \varphi h Y)\} \\ + 2S^*(X, Y) + 2\lambda g(X, Y) = 0. \end{aligned} \tag{53}$$

Since M is a contact metric manifold, making use of (9) and (11) in (53), it follows that

$$S^*(X,Y) - g(\varphi hX,Y) + \lambda g(X,Y) = 0. \tag{54}$$

Since trace of φh vanishes, from (54) we have the $*$-scalar curvature $r^* = -\lambda(2n+1)$, a constant. It is known that in a contact metric generalized (κ,μ)-space form $M(f_1,\dots,f_6)$, the $*$-Ricci tensor is given by

$$S^*(X,Y) = [f_1 + (2n+1)f_2]g(X,Y) - [f_1 + (2n+1)f_2]\eta(X)\eta(Y). \tag{55}$$

Substituting X by φX in (54) and (55) and equating right hand sides of the resulting equation, it gives

$$[f_1 + (2n+1)f_2 + \lambda]g(\varphi X,Y) = g(hX,Y). \tag{56}$$

Interchanging X and Y we get from the foregoing equation that

$$[f_1 + (2n+1)f_2 + \lambda]g(\varphi Y,X) = g(hY,X). \tag{57}$$

Adding (56) and (57) we get $h = 0$, which implies M is a K-contact manifold. From Theorem 2.2, we conclude that M is Sasakian. Now, subtracting (56) from (57) we get

$$\lambda = -[f_1 + (2n+1)f_2]. \tag{58}$$

Making use of (58) and $h = 0$ in (54), we obtain

$$S^*(X,Y) = [f_1 + (2n+1)f_2]g(X,Y). \tag{59}$$

From (59), it follows that M is $*$-Einstein. This leads to the following:

Theorem 6.1. *If a contact metric generalized (κ,μ)-space form $M(f_1,\dots,f_6)$ admits a $*$-Ricci soliton (g,ξ,λ), then M is Sasaki-$*$-Einstein.*

Let $M(f_1,\dots,f_6)$ be a contact metric generalized (κ,μ)-space form with g as a gradient $*$-Ricci soliton. Then the equation (7) can be written as

$$\nabla_Y Df = Q^*Y + \lambda Y, \tag{60}$$

for all vector fields Y, where D denotes the gradient operator of g. From (60), it follows that

$$R(X,Y)Df = (\nabla_X Q^*)Y - (\nabla_Y Q^*)X. \tag{61}$$

Making use of (16) in (60) we have

$$\begin{aligned} g(R(\xi,Y)Df,\xi) = {} & (f_1 - f_3)\{g(Y,Df) - \eta(Df)\eta(Y)\} \\ & + (f_4 - f_6)g(Df,hY). \end{aligned} \tag{62}$$

From (22) it follows that

$$\begin{aligned} g((\nabla_\xi Q^*)Y - (\nabla_Y Q^*)\xi,\xi) = & -[f_1 + (2n+1)f_2]\{(\nabla_\xi \eta)(Y) + \eta(Y)g(\nabla_\xi \xi,\xi)\} \\ & + [f_1 + (2n+1)f_2]\{(\nabla_Y \eta)(\xi) + g(\nabla_Y \xi,\xi)\}. \end{aligned} \tag{63}$$

We know that $g(\nabla_\xi\xi,\xi) = 0$, $(\nabla_X\eta)(\xi) = 0$, $\eta(\nabla_X\xi) = 0$ and $(\nabla_\xi\eta)(Y) = g(Y,\nabla_\xi\xi) = 0$ on contact metric manifold. Substituting these equation in (63), it follows that

$$g((\nabla_\xi Q^*)Y - (\nabla_Y Q^*)\xi,\xi) = 0. \tag{64}$$

In view of (61) and (62), (64) yields

$$(f_1 - f_3)\{g(Y,Df) - \eta(Df)\eta(Y)\} + (f_4 - f_6)g(Df,hY) = 0,$$

which gives

$$(f_1 - f_3)Df + (f_4 - f_6)hDf = (f_1 - f_3)\eta(Df)\xi. \tag{65}$$

Recall that for any contact metric manifold we have [6]:

$$\nabla_\xi h = \varphi - \varphi h^2 - \varphi l, \tag{66}$$

where $l = R(X,\xi)\xi$. On the other hand, from (16) it follows that

$$l = -(f_1 - f_3)\varphi^2 + (f_4 - f_6)h. \tag{67}$$

Making use of this and (14) with $\kappa = f_1 - f_3$ in (66) we obtain

$$\nabla_\xi h = (f_4 - f_6)h\varphi. \tag{68}$$

In view of (60) one obtains

$$\nabla_\xi Df = \lambda\xi. \tag{69}$$

Let us assume that $f_1 - f_3$ is constant. Since $f_1 - f_3$ is constant, we have from Theorem 4.7 in [9] that $f_4 - f_6$ is also constant. Now, differentiating (65) along ξ and taking into account of (12), (68) and (69) we get

$$(f_4 - f_6)^2 h\varphi Df = 0, \tag{70}$$

where $(\nabla_\xi\eta)(Df) = g(Df,\nabla_\xi\xi) = 0$ is used. Operating equation (70) by h, recalling (14) with $\kappa = f_1 - f_3$ it follows that

$$(f_4 - f_6)^2(f_1 - f_3 - 1)\varphi Df = 0. \tag{71}$$

Thus, we have either

(i) $f_4 - f_6 = 0$, or (ii) $f_1 - f_3 = 1$, or (iii) $\varphi Df = 0$.

Case (i). Since $f_4 - f_6 = 0$ it follows from Theorem 2.3 that M is $(\kappa,0)$-space, with $\kappa = f_1 - f_3$ constant.

Case (ii). In this case $f_1 - f_3 = 1$. In [9] Carriazo et al. proved that if $M(f_1,\ldots,f_6)$ is a contact metric generalized (κ,μ)-space form with $f_1 - f_3 = 1$, then M is a Sasakian manifold. Hence we conclude that M is a Sasakian manifold.

Case (iii). Here $\varphi Df = 0$. Operating this by φ yields $Df = (\xi f)\xi$. Using this in (60) we obtain

$$S^*(X,Y) + \lambda g(X,Y) = g(Y(\xi f)\xi + (\xi f)\nabla_Y\xi, X). \tag{72}$$

Taking $X = \xi$ in (72) and applying $S^*(Y, \xi) = 0$ we get

$$Y(\xi f) = \lambda\eta(Y). \tag{73}$$

In view of (72) and (73) we have

$$S^*(X, Y) + \lambda g(X, Y) = (\xi f)g(\nabla_Y \xi, X) + \lambda\eta(X)\eta(Y). \tag{74}$$

Making use of (74) in (60), we obtain

$$\nabla_Y Df = (\xi f)\nabla_Y \xi + \lambda\eta(Y)\xi. \tag{75}$$

In view of (75) we compute $R(X, Y)Df$ (keeping in mind that $Df = (\xi f)\xi$) and obtain

$$g(R(X, Y)(\xi f)\xi, \xi) = 2\lambda d\eta(X, Y). \tag{76}$$

Thus we get $\lambda = 0$. Therefore from equation (73) it follows that $Y(\xi f) = 0$, that is, $\xi f = c$, where c is a constant. Thus the relation $Df = (\xi f)\xi$ gives $df = c\eta$. Its exterior derivative implies that $cd\eta = 0$, which implies $c = 0$. Hence f is constant. Consequently (60) reduces to $S^*(X, Y) = 0$. Hence it is $*$-Ricci flat. So we have the following:

Theorem 6.2. *If the metric g of a contact metric generalized (κ, μ)-space form $M(f_1, \ldots, f_6)$ is a gradient $*$-Ricci soliton with $f_1 - f_3$ is a constant, then either M is a $(\kappa, 0)$-space with $\kappa = f_1 - f_3$, or M is Sasakian Manifold, or it is $*$-Ricci flat.*

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