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# APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS TO CERTAIN CLASSES OF STARLIKE FUNCTIONS

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Abstract. Let $p$ be an analytic function defined on the open unit disk $\mathbb{D}$. We obtain certain differential subordination implications such as $\psi(p) := p^{\lambda}(z)(\alpha+\beta p(z)+\gamma/p(z)+\delta zp'(z)/p^{j}(z)) \prec h(z)$ $(j=1,2)$ implies $p \prec q$, where $h$ is given by $\psi(q)$ and $q$ belongs to $\mathcal{P}$, by finding the conditions on $\alpha$, $\beta$, $\gamma$, $\delta$ and $\lambda$. Further as an application of our derived results, we obtain sufficient conditions for normalized analytic function $f$ to belong to various subclasses of starlike functions, or to satisfy $|\log(zf'(z)/f(z))| < 1$, $|(zf'(z)/f(z))^2 - 1| < 1$ and $zf'(z)/f(z)$ lying in the parabolic region $v^2 < 2u - 1$.

## 1. Introduction

The set of analytic functions $f$ defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is denoted by $\mathcal{A}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{A}_0$ be the class of all analytic functions $p(z)$, which does not vanish anywhere in $\mathbb{D}$, with the normalization $p(0) = 1$. Let $\mathcal{P}$ denote the subclass of $\mathcal{A}_0$ consisting of the functions $p$ with the positive real part. This class is known as Carathéodory class of functions. For two analytic functions $f$ and $g$, we have $f$ subordinate to $g$, written as $f \prec g$, if there is a Schwarz function $w$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Ma and Minda [8] introduced the classes:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

and

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\},$$

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where $\phi$ is an analytic univalent function with positive real part in $\mathbb{D}$ such that $\phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$. For different choices of $\phi$, $\mathcal{S}^*(\phi)$ reduces to well known classes. For example, when $\phi(z) := (1+Az)/(1+Bz)$, $\mathcal{S}^*((1+Az)/(1+Bz)) =: \mathcal{S}^*[A, B]$, is the class of Janowski starlike functions [5], where $-1 \leq B < A \leq 1$. For $A = 1$ and $B = -1$, this class reduces to the class of normalized starlike functions, $\mathcal{S}^*((1+z)/(1-z)) =: \mathcal{S}^*[-1, 1]$ and for $A = 1 - 2\nu$ $(0 \leq \nu < 1)$ and $B = -1$, Robertson [17] introduced the class of starlike functions of order $\nu$, $\mathcal{S}^*((1 + (1 - 2\nu)z)/(1 - z)) =: \mathcal{S}^*(\nu)$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$, is the class of starlike functions. Another interesting class is the class of starlike functions of reciprocal order $\nu$, in $\mathbb{D}$, which is given by

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) > \nu \quad (z \in \mathbb{D}).$$

For $\phi(z) := \sqrt{1+z}$, Sokół and Stankiewicz [20] introduced the class of analytic functions associated with lemniscate of Bernoulli, $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$. Functions satisfying $|\log(zf'(z)/f(z))| < 1$, belongs to the class $\mathcal{S}^*_e :=\mathcal{S}^*(e^z)$, introduced by Mendiratta *et al*. [10]. The class of strongly starlike functions of order $\eta$ is introduced in [1] and [22]. We denote it by

$$\mathcal{SS}^*(\eta) := \{f \in \mathcal{A} : |\arg(zf'(z)/f(z))| < \eta\pi/2\}, \quad (0 < \eta \leq 1).$$

It is obtained when $\phi(z) := ((1 + z)/(1 - z))^\eta$. The speciality of this class lies in the fact that it helps to study the function in terms of argument estimation. For $\eta = 1$, it reduces to the class of normalized starlike functions $\mathcal{S}^*$. For $\phi(z) := ((1 + cz)/(1 - z))^{(\eta_1+\eta_2)/2}$, Liu and Srivastava [7] introduced the class

$$\begin{aligned} \mathcal{SS}^*(\eta_1, \eta_2) = \{f \in \mathcal{A} : -\eta_2\pi/2 < |\arg(zf'(z)/f(z))| < \eta_1\pi/2\} \\ (0 < \eta_1, \eta_2 \leq 1), \end{aligned} \tag{1}$$

where

$$\eta = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}, \qquad c = e^{\eta\pi i}. \tag{2}$$

As per the definition of the class $\mathcal{SS}^*(\eta_1, \eta_2)$ in (1), we give the name *oblique sector* function to the above defined function $((1 + cz)/(1 - z))^{(\eta_1+\eta_2)/2}$. Note that $\mathcal{SS}^*(\eta, \eta)$ reduces to the class $\mathcal{SS}^*(\eta)$. The class $S_P :=\mathcal{S}^*(\phi_{PAR}(z))$, introduced by Rønning [18], is the class of parabolic starlike functions, where

$$\phi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \quad \mathrm{Im}\sqrt{z} \geq 0,$$

consists of functions $f \in \mathcal{A}$, such that $\mathrm{Re}(zf'(z)/f(z)) > |zf'(z)/f(z) - 1|$. Let $\mathcal{N}(\kappa)$ be the subclass of $\mathcal{A}$ consisting of the functions $f(z)$ which satisfy the following

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \kappa \quad (\kappa > 1; z \in \mathbb{D}).$$

Uralegaddi *et al*. [24] investigated the class $\mathcal{N}(\kappa)$ for $1 < \kappa < 4/3$.

Let $\psi$ be analytic in a domain $D \subset \mathbb{C}^2$ and $h$ be univalent in $D$. Then an analytic function $p$ is the solution of the first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \quad (p(z), zp'(z)) \in D \quad \text{for } z \in \mathbb{D}. \tag{3}$$

Then the dominant of the differential subordination (3) is the univalent function $q$, if $p \prec q$ for all the solutions $p$ of (3). The best dominant of all the dominants of differential subordination (3) is the function $\tilde{q}$ whenever $\tilde{q} \prec q$. Miller and Mocanu [11] discussed the general theory of the first-order differential subordinations. Motivated by this result, many authors [19], [21] established several generalizations of first order differential subordination. Recently, the authors [16] obtained some sufficient conditions for analytic functions in $\mathbb{D}$ to satisfy the subordination $p(z) \prec q(z)$ and in specific to have positive real part. Many authors [2], [3], [4], [7] and [13] have evolved this concept of finding conditions on the parameters involved in the first order differential subordination as given in (3) in order to prove $p(z) \prec ((1+z)/(1-z))^{\eta}$, $(0 < \eta \leq 1)$ by satisfying the condition $|\arg(\psi(p(z), zp'(z))| \leq \arg(h(z))$, after estimating the argument of $h(z)$.

In this paper, we obtain certain differential subordination implications by finding conditions on the parameters involved in it. Mainly, our results involve two admissible classes of analytic functions:

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta \frac{zp'(z)}{p(z)}\right) \prec h(z)$$

and

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta \frac{zp'(z)}{p^2(z)}\right) \prec h(z),$$

which implies $p(z) \prec q(z)$, where $h$ is univalent and $q(z) \in \mathcal{P}$. Also, we study these classes in terms of argument estimation of the class for the oblique sector function $q$.

In Section 3, we also obtain sufficient conditions for functions belonging in $\mathcal{S}^*(\nu)$ $(\nu = 0, 1/2)$, $\mathcal{S}^*_e$, $\mathcal{S}^*_L$, $\mathcal{S}^*[A, B]$, $(-1 < B < A \leq 1)$ and $S_P$ as an application of our derived results in Section 2. In some cases, our results become the general case of the results obtained by Ravichandran and Kumar [16], Nunokawa *et al*. [14], Obradowič and Tuneski [15] and Cho *et al*. [4]. We also extend the result obtained by Mocanu [12].

Miller and Mocanu [11] gave the following result which is required to prove our main results.

**Lemma 1.1.** *Let $q(z)$ be univalent in the unit disk $\mathbb{D}$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\mathbb{D})$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that* (i) *$Q(z)$ is starlike univalent in $\mathbb{D}$, and* (ii) Re $(zh'(z)/Q(z)) > 0$ *for $z \in \mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and satisfies*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{4}$$

*then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

## 2. Results involving differential subordination implications

We begin with the following main results:

**Theorem 2.1.** *Let $\lambda$ be a real number and $\alpha, \beta, \gamma$ and $\delta(\neq 0)$ be complex numbers. Suppose $q(z) \in \mathcal{A}_0$ be univalent in $\mathbb{D}$ and satisfy the following conditions for $z \in \mathbb{D}$:*

(1) $Q(z) := \delta z q'(z) q^{\lambda-1}(z)$ *be starlike (univalent).*

(2) $\operatorname{Re}\left(\dfrac{\alpha\lambda}{\delta} + \dfrac{\beta(\lambda+1)}{\delta} q(z) + \dfrac{\gamma(\lambda-1)}{\delta q(z)} + (\lambda-1)\dfrac{zq'(z)}{q(z)} + \left(1 + \dfrac{zq''(z)}{q'(z)}\right)\right)$
$> 0 =: \operatorname{Re}(H(z))$.

*If $p(z) \in \mathcal{A}_0$ satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q(z)}\right),$$

*then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

*Proof.* Let $\theta(\omega) = \omega^{\lambda}(\alpha + \beta\omega + \gamma/\omega)$, $\omega \neq 0$ and $\phi(\omega) = \delta\omega^{\lambda-1}$. Thus, $\phi(\omega) \neq 0$ and $\theta(\omega)$, $\phi(\omega)$ are analytic in $\mathbb{C} - \{0\}$. Let the function $Q(z)$ and $h(z)$ be given by

$$Q(z) := zq'(z)\phi(q(z)) = \delta z q'(z) q^{\lambda-1}(z)$$

and

$$h(z) := \theta(q(z)) + Q(z) = q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q(z)}\right).$$

Thus, we have $Q(z)$ is starlike and $\operatorname{Re}(zh'(z)/Q(z))$ reduces to

$$\operatorname{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right) > 0. \tag{5}$$

On substituting $q(z)$ as $p(z)$ in $\theta(q(z)) + Q(z)$, we get

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)).$$

The result follows by an application of Lemma 1.1. □

*Remark* 2.2. (i) Suppose that the function $f$ is analytic in a bounded domain $D$ and continuous on $\overline{D}$.

(ii) Suppose that $u(x, y)$ is a real part of non-constant analytic function $f$ on a bounded domain $D$ and $u(x, y)$ is lower/upper bounded in $D$. If either of the conditions (i), (ii) hold, then the real part of an analytic function $f$ attains its minimum/maximum value, respectively on the boundary of $D$, by Minimum/ Maximum Modulus Theorem.

As a consequence of the condition (ii) that has been used in this paper, we have to incorporate certain conditions on the parameters involved in the differential subordination implications in order to satisfy the two conditions of Lemma 1.1.

We give the following example to illustrate Remark 2.2.

**Example 2.3.** (i) Consider the function $f(z) = z/(1-z)$, $(z \in \mathbb{D})$. Now, to obtain the maximum/minimum value of the real part of $f(z)$ on the bounded domain $\mathbb{D}$, either of the above two conditions of Remark 2.2 should hold. Clearly, (i) fails as the function $f(z)$ is non continuous on $\overline{\mathbb{D}}$, so we apply (ii). For $z = x + iy$ $(x^2 + y^2 < 1)$, we have

$$\mathrm{Re}(f(z)) = \frac{x - x^2 - y^2}{(1-x)^2 + y^2} =: f(x, y).$$

The function $f(x, y)$ is unbounded when $x$ tends to 1 and $y$ tends or equal to 0. These are the only two possibilities for the function to be unbounded in $\mathbb{D}$. For this, we consider two cases:

Case 1: Let $y = 0$ in $f(x, y)$, then we get

$$f(x) = x/(1 - x).$$

Clearly, $f(x)$ is unbounded when $x$ tends to 1. As a result, we assume $x = 1-h$, $(h \to 0)$ in $f(x)$, we get

$$\lim_{h\to 0} f(1 - h) = \lim_{h\to 0} \frac{1 - h}{h} \to +\infty.$$

This shows that the real part of $f(z)$ is not upper bounded.

Case 2: Let $x = 1 - h$ in $f(x, y)$, $(h \to 0)$, we get

$$\lim_{h\to 0} f(1 - h, y) = \lim_{h\to 0} \frac{-h - h^2 + 2h - y^2}{h^2 + y^2} \to -1, \quad \forall y.$$

Thus, in particular for those $y$ such that $(1 - h)^2 + y^2 < 1$ holds, $f(x, y)$ is bounded.

We conclude that the function is lower bounded but not upper bounded. Thus, we can only obtain minimum of real part of $f(z)$ by evaluating on the boundary of $\mathbb{D}$. For $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$, we get

$$\mathrm{Re}(f(e^{i\theta})) = -1/2.$$

We infer that minimum value of real part of $f(z)$ is $-1/2$ and maximum value can not be obtained by evaluating on the boundary of $\mathbb{D}$ as it fails to be upper bounded.

(ii) Consider the Koebe function $K(z) := z/(1 - z)^2$. It maps the open unit disk $\mathbb{D}$ onto the entire complex plane minus the slit along the negative real axis from $w = -\infty$ to $w = -1/4$. Thus, real part of the Koebe function is unbounded, which can also be verified. For $z = x + iy$ $(x^2 + y^2 < 1)$, we have

$$\mathrm{Re}(K(z)) = \frac{x - 2x^2 + x^3 - 2y^2 + xy^2}{(1 - x)^2 + y^2} =: f(x, y).$$

It is trivial to say that the real part of $K(z)$ could only be unbounded when $z$ tends to 1. Equivalently, we can say $f(x, y)$ is unbounded when $x$ tends to 1 and y tends to or equal to 0. This is a complex plane so $x$ can tend to 1 from all the directions, unlike on the real line where it can only tend along the real axis. Thus, we consider two cases:

Case 1: Let $y = 0$ in $f(x, y)$ and proceeding same as in the case 1 in part (i), we assume $x = 1 - h$, $(h \to 0)$ in $f(x)$, we get

$$\lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{1-h}{h^2} \to +\infty.$$

Thus, $f(x, y)$ is not upper bounded in this case.

Case 2: Let $x = 1 - h$ in $f(x, y)$ $(h \to 0)$, we get

$$\lim_{h\to 0} f(1-h, y) = \frac{-1}{y^2} \to -\infty,\ y = 0.$$

Clearly, it is not lower bounded in this case.

Now, let $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$, we get

$$\operatorname{Re}(z/(1-z)^2) = -1/(4\sin^2\theta) := g(\theta).$$

We infer that the maximum value of $g(\theta)$ is $-1/4$, attained at $\theta = \pi/2$. Therefore, we conclude that $\max \operatorname{Re}(K(z)) = -1/4$, which is absurd as clearly from both the cases 1 and 2, we observe that real part of $K(z)$ is unbounded. Thus, we cannot evaluate the real part of $K(z)$ on the boundary of $\mathbb{D}$ to find the maximum value. This is due to the failure of both the conditions (i) and (ii) in Remark 2.2 for the Koebe function. Therefore, the maximum value of the real part of $K(z)$ is not attained on the boundary of $\mathbb{D}$.

Taking $\alpha, \beta, \gamma$ and $\delta$ to be real numbers in the above Theorem 2.1, we obtain the following results:

**Corollary 2.4.** *Let $0 \le \lambda \le 1$ and $\gamma\delta \le 0$. (i) For $0 \le \nu \le 1/2$, let $(\alpha + \nu\beta)/\delta \ge \nu/(2(1-\nu))$ and $1 + 2(1-\nu)\beta/\delta > 0$. (ii) For $1/2 \le \nu < 1$, let $\alpha + \nu\beta/\delta \ge (1-\nu)/(2\nu)$ and $2\beta/\delta > (\nu-1)/\nu^2$. If $p(z) \in \mathcal{A}_0$ satisfies*

$$\begin{aligned} &p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \\ &\prec \left(\frac{1+(1-2\nu)z}{1-z}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+(1-2\nu)z}{1-z}\right)\right. \\ &\quad \left. + \frac{\gamma(1-z)}{1+(1-2\nu)z} + \delta\left(\frac{(1-2\nu)z}{1+(1-2\nu)z} + \frac{z}{1-z}\right)\right) =: h(z), \end{aligned} \tag{6}$$

*then* $\operatorname{Re}\, p(z) > \nu$.

*Proof.* Let $q(z) = (1 + (1-2\nu)z)/(1-z)$ in Theorem 2.1. Then we have,

$$Q(z) = \frac{2(1-\nu)z}{(1-z)^{1+\lambda}(1+(1-2\nu)z)^{1-\lambda}},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + (1+\lambda)\frac{z}{1-z} - (1-\lambda)\frac{(1-2\nu)z}{1+(1-2\nu)z} =: K(z).$$

If $z = e^{i\theta}$, where $\theta \in [-\pi, \pi]$, we have

$$\mathrm{Re}(K(z)) = \left(\frac{1-\lambda}{2}\right)\left(\frac{\nu(1-\nu)}{\nu^2 + (1-2\nu)\cos^2(\theta/2)}\right). \tag{7}$$

Since $1-\lambda$, $\nu(1-\nu)$, $\nu^2 + (1-2\nu)\cos^2(\theta/2) \geq 0$, it follows that $Q$ is starlike (univalent) in $\mathbb{D}$. Also, from condition (2) of Theorem 2.1, we have

$$\begin{aligned} H(z) := \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\left(\frac{1+(1-2\nu)z}{1-z}\right) + \frac{\gamma(\lambda-1)}{\delta}\left(\frac{1-z}{1+(1-2\nu)z}\right) \\ + (\lambda-1)\left(\frac{(1-2\nu)z}{1+(1-2\nu)z} + \frac{z}{1-z}\right) + \left(1 + \frac{2z}{1-z}\right). \end{aligned}$$

For $z = e^{i\theta}$ ($\theta \in [-\pi, \pi]$), real part of $H(z)$ reduces to

$$1 + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\nu + \frac{\gamma(\lambda-1)}{\delta}A(\theta) - \frac{\lambda+1}{2} + (\lambda-1)(1-2\nu)B(\theta) =: L(\theta),$$

where $A(\theta) = (\nu\sin^2(\theta/2))/(\nu^2 + (1-2\nu)\cos^2(\theta/2))$ and $B(\theta) = (1-2\nu+\cos\theta)/(2(1-2\nu+\cos\theta+2\nu(\nu-\cos\theta)))$. Also, considering Remark 2.2, we need to have

$$\gamma/\delta < 1/2 \quad \text{and} \quad 1 + 2\beta(1-\nu)/\delta > 0, \tag{8}$$

which is trivial as the given conditions $\gamma\delta \leq 0$ and $2\beta/\delta > (\nu-1)/\nu^2$ imply the conditions in the equation (8), respectively. Now, to complete the proof it suffices to show that $L(\theta) \geq 0$ for the conditions given in the hypothesis. For this, we consider the following cases:

Case 1: Consider $0 \leq \nu \leq 1/2$, then $(1-2\nu) \geq 0$. As $0 \leq \lambda \leq 1$ and $\gamma\delta \leq 0$, we take into account the minimum value of $A(\theta) = 0$ and the maximum value of $B(\theta) = 1/(2(1-\nu))$, that are attained at $\theta = 0$ by the second derivative test. Thus, we get

$$L(\theta) \geq \lambda\left(\frac{\alpha}{\delta} + \frac{\beta\nu}{\delta} + \frac{1-2\nu}{2(1-\nu)} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\beta\nu}{\delta} - \frac{1-2\nu}{2(1-\nu)} \geq 0,$$

which is possible when $(\alpha + \beta\nu)/\delta \geq \nu/(2(1-\nu))$.

Case 2: Consider $1/2 \leq \nu < 1$ then $(1-2\nu) \leq 0$. Similarly, taking the range of $\lambda$ and $\gamma\delta$ into consideration we take into account the minimum value of $B(\theta) = -1/(2\nu)$ which is attained at $\theta = \pi$ by the second derivative test and let $A(\theta)$ to be same as in the case 1. Thus, we get

$$L(\theta) \geq \lambda\left(\frac{\alpha}{\delta} + \frac{\beta\nu}{\delta} - \frac{1-2\nu}{2\nu} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\beta\nu}{\delta} + \frac{1-2\nu}{2\nu} \geq 0,$$

which is possible when $(\alpha+\beta\nu)/\delta \geq (1-\nu)/(2\nu)$ and $\beta/\delta > (\nu-1)/2\nu^2$. With this, we complete the proof. □

We obtain the relation between the class of starlike of reciprocal order $\nu$ and $\mathcal{N}(\kappa)$ in the following corollary:

**Corollary 2.5.** *Let $f \in \mathcal{A}$. The function $f$ is starlike of reciprocal order $\nu$ if*

(a) $f \in \mathcal{N}((3-\nu)/2\nu)$, $0 < \nu \leq 3/4$.
(b) $f \in \mathcal{N}(\nu/2(1-\nu))$, $3/4 \leq \nu < 1$.

*Proof.* Let $p(z) = f(z)/(zf'(z))$, $\beta = \lambda = 0$, $\gamma = -\delta = 1$ and $\alpha = \min((\nu - 1)/(2\nu); \nu/(2(\nu-1)))$ in Corollary 2.4, we get, if $f$ satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+2(\nu-1)z}{1+(1-2\nu)z} - \frac{z}{1-z} =: S(z), \tag{9}$$

then $\operatorname{Re}(f(z)/(zf'(z))) > \nu$. To complete the proof it suffices to show that $\operatorname{Re}(S(z)) < (3-\nu)/(2\nu)$ for $0 < \nu \leq 3/4$ and $\operatorname{Re}(S(z)) < \nu/(2(1-\nu))$ for $3/4 \leq \nu < 1$. Since, $S(z)$ is upper bounded, evaluating it on the boundary of $\mathbb{D}$, we obtain

$$\operatorname{Re}(S(e^{i\theta})) = \frac{1-6\nu+4\nu^2+\cos(\theta)}{-2+4\nu-4\nu^2-2\cos(\theta)+4\nu\cos(\theta)} + \frac{1}{2} =: g(\theta).$$

A calculation shows that $g''(\theta)_{\theta=\pi} = (1-\nu)(4\nu-3)/(4\nu^3)$ and $g''(\theta)_{\theta=0} = \nu(4\nu-3)/(4(\nu-1)^3)$.

(a) For $0 < \nu \leq 3/4$, $\max g(\theta) = (3-\nu)/(2\nu)$, attained at $\theta = \pi$ and we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3-\nu}{2\nu},$$

equivalently, $f \in \mathcal{N}((3-\nu)/(2\nu))$. This completes the proof for part (a).

(b) For $3/4 \leq \nu < 1$, $\max g(\theta) = \nu/(2(1-\nu))$, attained at $\theta = 0$ and we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\nu}{(2(1-\nu))},$$

equivalently, $f \in \mathcal{N}(\nu/2(1-\nu))$. This completes the proof for part (b). □

Taking $\nu = 0$ in Corollary 2.4, we get the following result:

**Corollary 2.6.** *Let* $1 + 2\beta/\delta > 0$, $\gamma\delta \leq 0$, $\alpha\delta \geq 0$ *and* $0 \leq \lambda \leq 1$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right),$$

*then* $\operatorname{Re}\, p(z) > 0$.

Taking $\nu = 1/2$ in Corollary 2.4, we get the following result:

**Corollary 2.7.** *Let* $1 + \beta/\delta > 0$, $\gamma\delta \leq 0$, $-1 + (2\alpha+\beta)/\delta \geq 0$ *and* $0 \leq \lambda \leq 1$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right)$$

$$\prec \left(\frac{1}{1-z}\right)^{\lambda}\left(\alpha+\beta\frac{1}{1-z}+\gamma(1-z)+\frac{\delta z}{1-z}\right),$$

*then* $\operatorname{Re} p(z) > 1/2$.

Considering oblique sector function $q$ in Theorem 2.1, which is defined as

$$q(z)=\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}} \qquad (0<\eta_1,\eta_2\leq 1),$$

where $c$ and $\eta$ are as defined in the equation (2). We have $\operatorname{Re}(zQ'(z)/Q(z))>0$ for given $\eta$ and $\lambda$ from [7, Theorem 2, p. 6], where $Q$ is as defined in the condition (1) of Theorem 2.1. Therefore, $Q(z)$ is starlike (univalent) in $\mathbb{D}$. Also, from condition (2) of Theorem 2.1, we have

$$H(z)=\left(\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}}+\frac{\gamma(\lambda-1)}{\delta}\left(\frac{1-z}{1+cz}\right)^{\frac{\eta_1+\eta_2}{2}}+\frac{zQ'(z)}{Q(z)}\right).$$

In view of the fact that $q(z)\in\mathcal{P}$ implies $1/q(z)\in\mathcal{P}$, we get

$$\operatorname{Re}(H(z))>\alpha\lambda/\delta\geq 0,$$

provided $\beta(\lambda+1)/\delta$, $\gamma(\lambda-1)/\delta$, $\alpha\lambda/\delta\geq 0$. Therefore, both the conditions of Theorem 2.1 get satisfied and we get the result as follows:

**Corollary 2.8.** *Let* $\alpha\lambda/\delta\geq 0$, $\beta(\lambda+1)/\delta\geq 0$, $\gamma(\lambda-1)/\delta\geq 0$, $0<\eta_1,\eta_2\leq 1$ *and* $|\lambda|\leq 2/(\eta_1+\eta_2)$. *If* $p(z)\in\mathcal{A}_0$ *satisfies*

$$\begin{aligned} &p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\\ \prec\ &\left(\frac{1+cz}{1-z}\right)^{\frac{(\eta_1+\eta_2)\lambda}{2}}\left(\alpha+\beta\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}}+\gamma\left(\frac{1-z}{1+cz}\right)^{\frac{\eta_1+\eta_2}{2}}\right.\\ &\left.+\frac{\eta_1+\eta_2}{2}\left(\frac{(1+c)z}{(1+cz)(1-z)}\right)\right),\end{aligned}$$

*then* $p(z)\prec((1+cz)/(1-z))^{(\eta_1+\eta_2)/2}$.

*Remark* 2.9. Taking $\alpha=\gamma=0$ and $\delta=1$, Corollary 2.8 is the result obtained by [7].

Letting $\eta_1=\eta_2$ and $c=1$ in Corollary 2.8, we have the following result:

**Corollary 2.10.** *Let* $\alpha\lambda/\delta\geq 0$, $\beta(\lambda+1)/\delta\geq 0$, $\gamma(\lambda-1)/\delta\geq 0$, $0<\eta\leq 1$ *and* $|\lambda|\leq 1/\eta$. *If* $p(z)\in\mathcal{A}_0$ *satisfies*

$$\begin{aligned} p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec\ &\left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha+\beta\left(\frac{1+z}{1-z}\right)^{\eta}\right.\\ &\left.+\gamma\left(\frac{1-z}{1+z}\right)^{\eta}+\frac{2\delta\eta z}{1-z^2}\right)=:h(z), \end{aligned} \tag{10}$$

*then* $p(z) \prec ((1+z)/(1-z))^\eta$.

Taking $\lambda = 1$ and $\gamma = 0$ in Corollary 2.10, we have the following result:

**Corollary 2.11.** *Let* $\alpha\delta,\ \beta\delta \geq 0$. *If* $p(z) \in \mathcal{A}_0$ *and*

$$\alpha p(z) + \beta p(z)^2 + \delta z p'(z) \prec \alpha \left(\frac{1+z}{1-z}\right)^\eta + \beta \left(\frac{1+z}{1-z}\right)^{2\eta} + \frac{2\delta\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^\eta,$$

*then* $|\arg p(z)| < \eta\pi/2$.

*Remark* 2.12. Corollary 2.11 is the result obtained by Ravichandran and Kumar [16] for $\alpha\delta,\ \beta\delta > 0$.

Taking $\eta = 1$ in Corollary 2.10, we get the following result:

**Corollary 2.13.** *Let* $\alpha\lambda/\delta \geq 0$, $\beta(\lambda+1)/\delta \geq 0$, $\gamma(\lambda-1)/\delta \geq 0$ *and* $|\lambda| \leq 1$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^\lambda(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^\lambda \left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right),$$

*then* $\operatorname{Re} p(z) > 0$.

The argument estimation of the function $h$ as given by the right hand side of the subordination (10), gives the reformulation of Corollary 2.10 with $\gamma = 0$ and $0 \leq \lambda \leq 1/\eta$ as follows:

**Corollary 2.14.** *Let* $\alpha,\ \beta \geq 0$ *and* $\delta$ *be a positive real number. If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$\left|\arg\left(p^\lambda(z)\left(\alpha + \beta p(z) + \delta\frac{zp'(z)}{p(z)}\right)\right)\right| < \frac{\pi}{2}\zeta, \tag{11}$$

*where* (i) $\zeta = \eta\lambda$, *whenever* $0 \leq \eta \leq 1/2$, (ii) $\zeta = \eta\lambda + 1/2$, *whenever* $1/2 \leq \eta \leq 1$ *and* $\delta\eta \geq \alpha$, (iii) $\zeta = \eta\lambda + 2/\pi \tan^{-1}(\delta\eta/\alpha)$, *whenever* $1/2 \leq \eta \leq 1$ *and* $\alpha \geq \delta\eta$, *then*

$$|\arg p(z)| < \frac{\pi}{2}\eta.$$

*Proof.* Here, $h(z)$ is given by

$$h(z) := \left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right)^\eta + \frac{2\delta\eta z}{1-z^2}\right).$$

Consider

$$\begin{aligned} h(e^{i\theta}) &= \left(i\cot\frac{\theta}{2}\right)^{\eta\lambda}\left(\alpha + \beta\left(i\cot\frac{\theta}{2}\right)^\eta + i\frac{\delta\eta}{\sin\theta}\right) \\ &= \left|\cot\frac{\theta}{2}\right|^{\eta\lambda} e^{\pm i\eta\lambda\pi/2}\left(\alpha + \beta\left|\cot\frac{\theta}{2}\right|^\eta e^{\pm i\eta\pi/2} + i\frac{\delta\eta}{\sin\theta}\right). \end{aligned}$$

Note that the '+' sign comes for $0 < \theta < \pi$ and the '−' sign comes for $-\pi < \theta < 0$. Also, we observe that the real and the imaginary part of $h(e^{i\theta})$ is an even and odd function of $\theta$, respectively. Thus, we will consider $0 < \theta < \pi$. Then, we have the following

$$\begin{aligned}\arg h(e^{i\theta}) &= \frac{\pi}{2}\eta\lambda + \arg\left(\alpha + \beta\left|\cot\frac{\theta}{2}\right|^{n} e^{i\eta\pi/2} + i\frac{\delta\eta}{\sin\theta}\right)\\ &= \frac{\pi}{2}\eta\lambda + \tan^{-1}\left(\frac{\beta|\cot(\theta/2)|^{\eta}\sin(\eta\pi/2) + \delta\eta/\sin(\theta)}{\alpha + \beta|\cot(\theta/2)|^{\eta}\cos(\eta\pi/2)}\right)\\ &\geq \frac{\pi}{2}\eta\lambda + \tan^{-1}\left(\frac{\beta s^{\eta}\sin(\eta\pi/2) + \delta\eta}{\alpha + \beta s^{\eta}\cos(\eta\pi/2)}\right) =: \frac{\pi}{2}\eta\lambda + \tan^{-1} g(s) \qquad (12)\end{aligned}$$

for $\alpha$, $\beta \geq 0$, $\delta > 0$ and where $s = |\cot\theta/2|$, $(s_1 \leq s \leq s_2)$. Note that $s_1 \to 0$ and $s_2 \to \infty$. To complete the proof, it suffices to show that $\arg(he^{i\theta}) \geq \zeta\pi/2$, which equivalently implies $|\arg(p^{\lambda}(z)(\alpha + \beta p(z) + \delta zp'(z)/p(z)))| < \arg(h(z))$ which further yields $p(z) \prec ((1+z)/(1-z))^{\eta}$ as $p^{\lambda}(z)(\alpha + \beta p(z) + \delta zp'(z)/p(z)) \prec h(z)$ implies $p(z) \prec ((1+z)/(1-z))^{\eta}$ from Corollary 2.10. For this we consider two cases:

Case 1: Let $0 < \eta \leq 1/2$, then

$$g(s) \geq \frac{\beta s^{\eta}\sin(\eta\pi/2)}{\alpha + \beta s^{\eta}\cos(\eta\pi/2)} \geq \frac{\beta s_1^{\eta}\sin(\eta\pi/2)}{\alpha + \beta s_1^{\eta}\cos(\eta\pi/2)} \approx 0.$$

This completes the proof for the mentioned range of $\eta$ in this case.

Case 2: Let $1/2 \leq \eta \leq 1$, then

$$g(s) \geq \frac{\beta s^{\eta}\cos(\eta\pi/2) + \delta\eta}{\alpha + \beta s^{\eta}\cos(\eta\pi/2)} =: l(s).$$

When $\delta\eta \geq \alpha$, $l(s)$ attains its minimum value at $s = s_2$. Thus, we obtain

$$l(s) \geq \frac{\beta s_2^{\eta}\sin(\eta\pi/2) + \delta\eta}{\alpha + \beta s_2^{\eta}\cos(\eta\pi/2)} \approx 1.$$

When $\alpha \geq \delta\eta$, $l(s)$ attains its minimum value at $s = s_1$. Thus, we obtain

$$l(s) \geq \frac{\beta s_1^{\eta}\sin(\eta\pi/2) + \delta\eta}{\alpha + \beta s_1^{\eta}\cos(\eta\pi/2)} \approx \frac{\delta\eta}{\alpha}.$$

This completes the proof for the mentioned range of $\eta$ in this case. Thus, both the cases yield the desired condition $\arg h(e^{i\theta}) \geq \zeta\pi/2$ and hence the result. ☐

*Remark* 2.15. For $1/2 \leq \eta \leq 1$ and taking $\beta = 0$ in Corollary 2.14, we get the result obtained by Cho *et al.* [4], when restricting its range of $\eta$.

**Corollary 2.16.** *Let* $\delta > 0$, $|\lambda\mu| \leq 1$ *and* $\alpha\lambda \geq -(A + B)$, *where* $A = \beta(\lambda+1)/e^{|\mu|} \geq 0$ *and* $B = \gamma(\lambda-1)/e^{|\mu|} \geq 0$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec e^{\lambda\mu z}\left(\alpha + \beta e^{\mu z} + \gamma e^{-\mu z} + \delta\mu z\right),$$

*then* $p(z) \prec e^{\mu z}$, *where* $\mu$ *is a non-zero real number such that* $|\mu| \leq 1$.

*Proof.* This result follows from Theorem 2.1 by taking $q(z) = e^{\mu z}$. Then $Q(z) = \delta\mu z e^{\lambda\mu z}$ and by taking $z = x + iy$ $(x^2 + y^2 < 1)$, we get

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = 1 + \lambda\mu x.$$

A simple computation shows that $Q(z)$ is starlike (univalent) in $\mathbb{D}$ whenever $|\lambda\mu| \leq 1$. Also, in view of the fact that $\operatorname{Re}(e^{\mu z})$, $\operatorname{Re}(e^{-\mu z}) > 1/e^{|\mu|}$, it follows that

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right)\\
&= \operatorname{Re}\left(1 + \lambda\mu z + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}e^{\mu z} + \frac{\gamma(\lambda-1)}{\delta}e^{-\mu z}\right)\\
&> 1 + \lambda\mu x + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\frac{1}{e^{|\mu|}} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{e^{|\mu|}} =: S(x)
\end{aligned}$$

for $A$, $B > 0$ and $\delta \geq 0$. For the given range of $|\lambda\mu|$, we consider two cases:

Case 1: Let $-1 \leq \lambda\mu \leq 0$, then $1 + \lambda\mu x > 1 + \lambda\mu \geq 0$ and we obtain

$$S(x) > \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\frac{1}{e^{|\mu|}} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{e^{|\mu|}}. \tag{13}$$

Case 2: Let $0 \leq \lambda\mu \leq 1$, then $1 + \lambda\mu x > 1 - \lambda\mu \geq 0$ and we obtain the same equation (13) as in the case 1.

Further $S(x) \geq 0$ if $\alpha\lambda \geq (\gamma - \beta - \lambda(\beta + \gamma))/e^{|\mu|}$, which is trivial from the conditions in the hypothesis. Thus, both the conditions of Theorem 2.1 get satisfied and hence the result. □

Taking $\mu = 1$ in Corollary 2.16, we obtain:

**Corollary 2.17.** *Let $\delta > 0$, $|\lambda| < 1$ and $\alpha\lambda \geq -(A + B)$, where $A = \beta(\lambda + 1)/e \geq 0$ and $B = \gamma(\lambda - 1)/e \geq 0$. If $p(z) \in \mathcal{A}_0$ satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec e^{\lambda z}\left(\alpha + \beta e^{z} + \gamma e^{-z} + \delta z\right),$$

*then $p(z) \prec e^{z}$.*

**Corollary 2.18.** *Let $-2 \leq \lambda \leq 2$, $\gamma(\lambda-1) \geq 0$, $\beta(\lambda+1) \geq 0$, $\delta > \max(0; \sqrt{2}\gamma)$, $-(2\sqrt{2}\gamma + \delta)/4 < \alpha \leq -3\gamma/(2\sqrt{2})$. If $p(z) \in \mathcal{A}_0$ satisfies*

$$\begin{aligned}
p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec\ &(1+z)^{\lambda/2}\left(\alpha + \beta\sqrt{1+z}\right.\\
&\left. + \frac{\gamma}{\sqrt{1+z}} + \frac{\delta z}{2(1+z)}\right),
\end{aligned}$$

*then $p(z) \prec \sqrt{1+z}$.*

*Proof.* Now, to achieve the desired result we apply Theorem 2.1 for $q(z) = \sqrt{1+z}$ and we get $Q(z) = \delta z(1+z)^{\lambda/2-1}/2$. Then

$$\frac{zQ'(z)}{Q(z)} = 1 + \left(\frac{\lambda}{2} - 1\right)\frac{z}{1+z} =: K(z).$$

In view of the fact that $K(z)$ is bounded only if $\lambda \leq 2$, we evaluate it on the boundary. Therefore, $z = e^{i\theta}$ ($\theta \in [-\pi, \pi]$) yields

$$\mathrm{Re}(K(e^{i\theta})) = \frac{\lambda+2}{4} \geq 0$$

for $\lambda \geq -2$. Clearly $Q$ is starlike (univalent) in $\mathbb{D}$. Also, we have

(14)

$$\begin{aligned}
&\mathrm{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right)\\
&= \mathrm{Re}\left(1 + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\sqrt{1+z} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{\sqrt{1+z}} + \left(\frac{\lambda}{2}-1\right)\frac{z}{1+z}\right)\\
&\geq 1 + \frac{\alpha\lambda}{\delta} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{\sqrt{2}} + \frac{\lambda}{4} - \frac{1}{2}\\
&= \lambda\left(\frac{1}{4} + \frac{\alpha}{\delta} + \frac{\gamma}{\sqrt{2}\delta}\right) + \frac{1}{2} - \frac{\gamma}{\sqrt{2}\delta} =: \lambda(r) + s,
\end{aligned}$$

since $\mathrm{Re}(\sqrt{1+z}) > 0$, $\mathrm{Re}(1/\sqrt{1+z}) > 1/\sqrt{2}$, $\beta(\lambda+1)/\delta$ and $\gamma(\lambda-1)/\delta \geq 0$. Now, to complete the proof we need to find the conditions on the parameters such that $\lambda(r) + s \geq 0$. For this, it suffices to show that

$$\lambda \geq \frac{\frac{\gamma}{\sqrt{2}\delta} - \frac{1}{2}}{\frac{\alpha}{\delta} + \frac{\gamma}{\sqrt{2}\delta} + \frac{1}{4}}, \tag{15}$$

which is a valid expression when $\alpha > -(2\sqrt{2}\gamma + \delta)/4$ and $\delta > 0$. Also, from the inequality $-2 \leq \lambda \leq 2$ we have $\alpha \leq -3\gamma/(2\sqrt{2})$ and thus, equation (15) holds true. And the derived range of $\alpha$ is meaningful only if $\delta \geq \sqrt{2}\gamma$. This completes the proof. □

The following lemmas will be needed to prove some further results.

**Lemma 2.19.** *For* $-1 \leq B < A \leq 1$, *the function* $f(z) = (1+Az)/(1+Bz)$ *satisfies* $\min \mathrm{Re} f(z) = (1-A)/(1-B)$ *and* $\min \mathrm{Re}(1/f(z)) = (1+B)/(1+A)$.

*Proof.* Consider

$$\mathrm{Re}(f(e^{i\theta})) = \frac{1 + (A+B)\cos\theta + AB}{1 + A^2 + 2A\cos\theta} =: K(\theta).$$

$K(\theta)$ attains its minimum value at $\theta = \pi$ by the second derivative test and the minimum value is given by

$$K(\pi) = \frac{1-A}{1-B}. \tag{16}$$

Now, consider $f(-z) = (1 - Az)/(1 - Bz)$. In view of the fact that the image of unit disk under $f(z)$ is same as $f(-z)$, from equation (16), we get

$$\min \operatorname{Re}\left(\frac{1+Bz}{1+Az}\right) = \frac{1+B}{1+A}$$

as $-A$ playing the role as $B$ and $-B$ as $A$. This completes the proof. □

With the same technique we can also find the minimum and maximum value of the real part of the functions $(1-Bz)/(1+Bz)$ and $z(A-B)/((1+Az)(1+Bz))$, respectively as given in the following lemmas:

**Lemma 2.20.** *For $-1 \leq B < 1$, the minimum of real part of the function $f(z) = (1 - Bz)/(1 + Bz)$ is $(1 - |B|)/(1 + |B|)$.*

**Lemma 2.21.** *For $-1 < B < A \leq 1$. Consider the function $f(z) = (A - B)z/((1+Az)(1+Bz))$, then $\max \operatorname{Re} f(z) = (A-B)/(1+A)(1+B)$, whenever $(1 + AB)(1 - A)(1 - B) > 8AB$ and $\min \operatorname{Re} f(z) = (B - A)/((1 - A)(1 - B))$, whenever $(1 + AB)(1 + A)(1 + B) > 8AB$.*

**Corollary 2.22.** *Let $0 \leq \lambda \leq 1$ and $\gamma\delta \leq 0$. For $-1 < B < A \leq 1$, if $(1 + AB)(1 - A)(1 - B) > 8AB$, we assume $\alpha/\delta \geq (B - A)/((1 + A)(1 + B))$ and $\beta/\delta \geq (A - B)(1 - B)/((1 - A^2)(1 + B))$. Suppose $p(z) \in \mathcal{A}_0$ satisfies*

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\left(\frac{1+Bz}{1+Az}\right) + \delta\frac{(A-B)z}{(1+Az)(1+Bz)}\right),$$

*then $p(z) \prec (1 + Az)/(1 + Bz)$.*

*Proof.* The result is followed by taking $q(z) = (1+Az)/(1+Bz)$ in Theorem 2.1. Then, we have

$$Q(z) = \frac{\delta z(A-B)}{(1+Az)^{1-\lambda}(1+Bz)^{1+\lambda}},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + (\lambda - 1)\frac{Az}{1+Az} - (1+\lambda)\frac{Bz}{1+Bz} =: K(z).$$

As per Remark 2.2 for $-1 \leq \lambda \leq 1$, $K(e^{i\theta}) \geq 0$. Thus, clearly $Q(z)$ is starlike (univalent) in $\mathbb{D}$. Also, we have

(17)

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right) \\
&= \operatorname{Re}\Bigg(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\left(\frac{1+Az}{1+Bz}\right) + \frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+Bz}{1+Az}\right) \\
&\qquad + (\lambda-1)\frac{(A-B)z}{(1+Az)(1+Bz)} + \frac{1-Bz}{1+Bz}\Bigg) =: S(z).
\end{aligned}$$

Now, Lemmas 2.19, 2.20 and 2.21 yield

$$\begin{aligned} S(z) &\geq \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\left(\frac{1-A}{1-B}\right) + \frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+B}{1+A}\right) \\ &\quad + (\lambda-1)\left(\frac{A-B}{(1+A)(1+B)}\right) + \left(\frac{1-|B|}{1+|B|}\right) \\ &= \lambda\left(\frac{\alpha}{\delta} + \frac{A-B}{(1+A)(1+B)}\right) + \frac{\beta(\lambda+1)}{\delta}\left(\frac{1-A}{1-B}\right) \\ &\quad - \left(\frac{A-B}{(1+A)(1+B)}\right) + \frac{1-|B|}{1+|B|} + \frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+B}{1+A}\right) \end{aligned} \tag{18}$$

for $\beta\delta \geq 0$ and $\gamma\delta \leq 0$. To complete the proof it suffices to prove the second condition of Theorem 2.1. For this, we need to have equation (18) greater than or equal to 0, which is possible when $0 \leq \lambda \leq 1$, $\alpha/\delta \geq (B-A)/(1+A)(1+B)$ and $\beta/\delta \geq (A-B)(1-B)/((1-A^2)(1+B))$. Hence the result follows immediately. □

With $\lambda = 1$ and $q(z) = \phi_{PAR}(z) := 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z}))^2$ in Theorem 2.1, we have the following result:

**Corollary 2.23.** *Let* $\beta/\delta \geq \max(0; -\alpha/\delta)$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$\begin{aligned} &\alpha p(z) + \beta p(z)^2 + \delta z p'(z) \\ &\prec \left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\left(\alpha + \beta\left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right. \\ &\quad \left. + \frac{4\delta}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right), \end{aligned}$$

*then* $p(z) \prec 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$.

*Proof.* As we know that $\phi_{PAR}(z)$ is a convex function, thus by Alexander's theorem we have $Q(z) = \delta z q'(z)$ is starlike (univalent) in $\mathbb{D}$. Also, from condition (2) of Theorem 2.1, we have

$$\operatorname{Re}(H(z)) = \operatorname{Re}\left(\frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right) + 1 + \frac{zq''(z)}{q'(z)}\right). \tag{19}$$

Evaluating the equation (19) on $z = e^{i\theta}$ ($\theta \in [-\pi, \pi]$), we obtain

$$\begin{aligned} \operatorname{Re}(H(e^{i\theta})) &\geq \operatorname{Re}\left(\frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(1 + \frac{2}{\pi^2}(\log\left(i\cot(\theta/4)\right)^2)\right)\right) \\ &= \frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(\frac{1}{2} + \frac{2}{\pi^2}\log(\cot(\theta/4))^2\right) =: \frac{\alpha}{\delta} + \frac{2\beta}{\delta}g(\theta). \end{aligned} \tag{20}$$

For $\theta = \pi$, simple calculation shows that

$$\begin{aligned} g''(\theta) = {} & \csc^2(\theta/4)\log(\cot(\theta/4))/(4\pi^2) + (\csc^2(\theta/4)\sec(\theta/4)^2)/(4\pi^2) \\ & - (\log(\cot(\theta/4)))\sec^2(\theta/4)/(4\pi^2) > 0. \end{aligned}$$

Thus, $\mathrm{Re}(H(e^{i\theta})) \geq 0$ from equation (20), on substituting $g(\pi) = 1/2$, the minimum value of $g(\theta)$. Hence, both the conditions of Theorem 2.1 get satisfied and therefore the result follows. □

**Theorem 2.24.** *Let $\alpha, \beta, \gamma$ and $(0 \neq)\delta$ be complex numbers and $\lambda$ be a real number. Let $q(z) \in \mathcal{A}_0$ be univalent in $\mathbb{D}$ and satisfy the following conditions for $z \in \mathbb{D}$:*

(1) $Q(z) := \delta z q'(z) q^{\lambda-2}(z)$ *be starlike (univalent),*
(2) $\mathrm{Re}(H(z)) > 0$, *where*

$$H(z) = \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta} q(z) + \frac{\beta(\lambda+1)}{\delta} q^2(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right).$$

*If $p(z) \in \mathcal{A}_0$ satisfies*

$$p^\lambda(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec q^\lambda(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q^2(z)}\right),$$

*then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

The proof of this theorem is similar to that of Theorem 2.1 and therefore omitted.

*Remark* 2.25. Let $\lambda = \alpha = \gamma = \beta = 0$ and $\delta = 1$ in Theorem 2.24, we have the following result of Ravichandran and Kumar [16].

**Corollary 2.26.** *Let $q(z) \in \mathcal{A}_0$ be univalent in $\mathbb{D}$. Let $zq'(z)/q(z)^2$ be starlike. If $p(z) \in \mathcal{A}_0$ satisfies*

$$\frac{zp'(z)}{p(z)^2} \prec \frac{zq'(z)}{q(z)^2},$$

*then $p(z) \prec q(z)$. The dominant $q$ is the best dominant.*

We take $\alpha$, $\beta$, $\gamma$ and $\delta$ to be real numbers, for proving all the following results. For the next result, we further assume $\beta = 0$ and let $q(z)$ be the function defined by $q(z) = (1+(1-2\nu)z)/(1-z)$ for $0 \leq \nu < 1$ in Theorem 2.24, then we obtain function $Q$ as follows:

$$Q(z) = \frac{2(1-\alpha)z}{(1-z)^\lambda(1+(1-2\nu)z)^{2-\lambda}}.$$

Proceeding as in Corollary 2.4, we have $Q(z)$ starlike (univalent) in $\mathbb{D}$ for $1 \leq \lambda \leq 2$. If we take $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$, we have

$$\mathrm{Re}\left(\frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}q(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right)$$

$$= 1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\nu + (\lambda-2)(1-2\nu)B(\theta) - \frac{\lambda}{2} =: S(\theta), \tag{21}$$

where $B(\theta)$ is as defined in the proof of Corollary 2.4 and let $1+2\alpha(1-\nu)/\delta > 0$ as per Remark 2.2. In order to apply Theorem 2.24, we need to find the range of the parameters such that $S(\theta) \geq 0$. For this we consider two cases:

Case 1: Consider $0 \leq \nu \leq 1/2$, then $(1-2\nu) \geq 0$. Since $1 \leq \lambda \leq 2$, we take into account the maximum value of $B(\theta)$. Thus, we get

$$S(\theta) \geq \lambda\left(\frac{\gamma}{\delta}+\frac{\alpha\nu}{\delta}+\frac{1-2\nu}{2(1-\nu)}-\frac{1}{2}\right)+1-\frac{\gamma}{\delta}-\frac{1-2\nu}{1-\nu} =: \lambda(r)+s.$$

It suffices to find the conditions on the parameters for which $\lambda(r)+s \geq 0$. For this, either (i) we show

$$\lambda \geq \frac{\frac{\gamma}{\delta}+\frac{1-2\nu}{1-\nu}-1}{\frac{\gamma}{\delta}+\frac{\alpha\nu}{\delta}+\frac{1-2\nu}{2(1-\nu)}-\frac{1}{2}}, \tag{22}$$

which is a valid expression only if $(\gamma+\alpha\nu)/\delta > \nu/(2(1-\nu))$. Since $1 \leq \lambda$, the inequality (22) holds if $1+2\alpha(1-\nu)/\delta > 0$, or (ii) let $r = 0$, simple calculation yields $S(\theta) \geq 0$.

Case 2: Consider $1/2 \leq \nu < 1$, then $(1-2\nu) \leq 0$. Therefore, we take into account the minimum value of $B(\theta)$. Thus, we get

$$S(\theta) \geq \lambda\left(\frac{\gamma}{\delta}+\frac{\alpha\nu}{\delta}-\frac{1-2\nu}{2\nu}-\frac{1}{2}\right)+1-\frac{\gamma}{\delta}+\frac{1-2\nu}{\nu}.$$

Again, proceeding as in the case 1, we assume $(\gamma+\alpha\nu)/\delta \geq (1-\nu)/(2\nu)$ and $\alpha/\delta > -1/(2(1-\nu))$. Thus, $\alpha/\delta \geq \max(-1/(2(1-\nu)); (\nu-1)/2\nu^2) = (\nu-1)/(2\nu^2)$. Hence the result follows as:

**Corollary 2.27.** *Let* $0 \leq \nu < 1$ *and* $1 \leq \lambda \leq 2$. (i) *For* $0 \leq \nu \leq 1/2$, *let* $1+2(1-\nu)\alpha/\delta > 0$ *and* $(\gamma+\nu\alpha)/\delta \geq \nu/(2(1-\nu))$. (ii) *For* $1/2 \leq \nu < 1$, *let* $2\alpha/\delta > (\nu-1)/\nu^2$ *and* $(\gamma+\nu\alpha)/\delta \geq (1-\nu)/(2\nu)$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+(1-2\nu)z}{1-z}\right)^{\lambda}\left(\alpha+\frac{\gamma(1-z)}{1+(1-2\nu)z}\right.$$
$$\left.+\frac{2\delta(1-\nu)z}{(1+(1-2\nu z))^2}\right),$$

*then* $\operatorname{Re} p(z) > \nu$.

Taking $\nu = 0$ in the above Corollary 2.27, we have the following result:

**Corollary 2.28.** *Let* $1+2\alpha/\delta > 0$, $\gamma\delta \geq 0$ *and* $1 \leq \lambda \leq 2$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha+\gamma\frac{1-z}{1+z}+\frac{2\delta z}{(1+z)^2}\right),$$

*then* $\operatorname{Re} p(z) > 0$.

Taking $\nu = 1/2$ in Corollary 2.27, we obtain the following result:

**Corollary 2.29.** *Let* $1+\alpha/\delta > 0$, $(2\gamma+\alpha)/\delta \geq 1$ *and* $1 \leq \lambda \leq 2$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1}{1-z}\right)^{\lambda}(\alpha+\gamma(1-z)+\delta z),$$

*then* Re $p(z) > 1/2$.

*Remark* 2.30. By taking $\alpha = \gamma = 0$ and $\lambda = 1$ in Corollary 2.6, or by taking $\lambda = 2$, $\gamma = 0$ and $\alpha = \beta$ in Corollary 2.28, we have the following result of Nunokawa *et al*. [14]:

**Corollary 2.31.** *Let* $1 + 2\beta/\delta > 0$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$\beta p(z)^2 + \delta z p'(z) \prec \beta \left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

*then* Re $p(z) > 0$.

**Corollary 2.32.** *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \mathcal{R}(z),$$

*where* $\mathcal{R}$ *is the open door mapping, then* $p(z) \prec (1+z)/(1-z)$.

*Proof.* Let $\lambda = \gamma = \alpha = 0$ and $\beta = \delta = 1$ in Corollary 2.6 or by assuming $\lambda = \alpha = \delta = 1$ and $\gamma = 0$ in Corollary 2.28, we get

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2} =: \mathcal{R}(z).$$

This completes the proof. □

*Remark* 2.33. Corollary 2.32 is the result obtained by Nunokawa *et al*. [14].

For the following result, we one again assume $\beta = 0$ in Theorem 2.24.

**Corollary 2.34.** *Let* $\alpha\lambda/\delta \geq 0$, $\gamma(\lambda-1)/\delta \geq 0$ , $0 < \eta \leq 1$ *and* $-1 < \eta\lambda \leq 2$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$\begin{aligned} & p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \\ & \prec \left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha + \gamma\left(\frac{1-z}{1+z}\right)^{\eta} + \frac{2\delta\eta z}{(1-z)^{1-\eta}(1+z)^{1+\eta}}\right) \qquad (23) \\ & =: h(z), \end{aligned}$$

*then* $p(z) \prec ((1+z)/(1-z))^{\eta}$.

*Proof.* Let $q(z) = ((1+z)/(1-z))^{\eta}$ in Theorem 2.24, then $Q(z)$ is given by

$$Q(z) = \frac{2\delta\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\eta(\lambda-1)}$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{2z^2}{1-z^2} + \frac{2\eta(\lambda-1)z}{1-z^2} =: K(z).$$

Since $K(z)$ is lower bounded for $-1 < \eta\lambda \leq 2$, we evaluate $K(z)$ on the boundary of $\mathbb{D}$ which implies Re $K(e^{i\theta}) \geq 0$. Thus, $Q(z)$ is starlike (univalent) in $\mathbb{D}$. Also, from condition (2) of Theorem 2.24, we have

$$\operatorname{Re}(H(z)) = \operatorname{Re}\left(1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2z(z+(\lambda-1)\eta)}{1-z^2}\right).$$

Some computation shows that for $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$, we obtain

$$\operatorname{Re}(H(e^{i\theta})) = \frac{\gamma(\lambda-1)}{\delta} \geq 0,$$

when $\alpha\lambda/\delta$, $\gamma(\lambda-1)/\delta \geq 0$. □

Taking $\eta = 1$ in Corollary 2.34, we obtain the following result:

**Corollary 2.35.** *Let $\alpha\lambda/\delta$, $\gamma(\lambda-1)/\delta \geq 0$ and $-1 < \lambda \leq 2$. If $p(z) \in \mathcal{A}_0$ satisfies*

$$p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{(1+z)^2}\right),$$

*then* Re $p(z) > 0$.

Here, we derive the argument relation between

$$p^{\lambda}(z)\left(\alpha + \gamma/p(z) + \delta zp'(z)/p^2(z)\right)$$

and $h(z)$, as defined by the equation (23), such that the subordination (23) holds for $0 < \eta \leq 1$. We assume $-1 < \lambda \leq 0$.

**Corollary 2.36.** *Let $\alpha \geq -\delta > 0$ and $\gamma \geq 0$. If $p(z) \in \mathcal{A}_0$ satisfies*

$$\left|\arg\left(p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right)\right)\right| < \frac{\pi}{2}\zeta, \tag{24}$$

*where*

(i) $\zeta = -\eta\lambda$ *whenever* $0 < \eta \leq 1/2$,

(ii) $\zeta = -\eta\lambda + (2/\pi)\tan^{-1}(-\delta\eta\cos(\eta\pi/2)/\alpha)$ *whenever* $1/2 \leq \eta \leq 1$, *then*

$$|\arg p(z)| < \frac{\pi}{2}\eta.$$

*Proof.* A calculation shows that

$$\begin{aligned} h(e^{i\theta}) &= (i\cot\theta/2)^{\eta\lambda}\left(\alpha + \gamma\,(i\cot\theta/2)^{-\eta} + i\frac{\delta\eta}{\sin\theta}(i\cot\theta/2)^{-\eta}\right) \\ &= |\cot\theta/2|^{\eta\lambda}e^{\pm i\eta\lambda\pi/2}\biggl(\alpha + \gamma|\cot\theta/2|^{-\eta}e^{\mp i\eta\pi/2} \\ &\quad + \frac{\delta\eta}{\sin\theta}|\cot\theta/2|^{-\eta}e^{i\pi/2(1\mp\eta)}\biggr). \end{aligned}$$

As the '+' sign comes for $0 < \theta < \pi$ and the '−' sign comes for $-\pi < \theta < 0$. Also, we observe that the real and the imaginary part of $h(e^{i\theta})$ is an even and odd function of $\theta$, respectively. Thus, we will consider here $-\pi < \theta < 0$. Then, we have the following

$$\begin{aligned}\arg h(e^{i\theta}) &= -\frac{\pi}{2}\eta\lambda + \tan^{-1}\left(\frac{\gamma|\cot(\theta/2)|^{-\eta}\sin(\eta\pi/2) + \frac{\delta\eta}{\sin\theta}\cos(\eta\pi/2)}{\alpha + \gamma|\cot(\theta/2)|^{-\eta}\cos(\eta\pi/2) - \frac{\delta\eta}{\sin\theta}\sin(\eta\pi/2)}\right)\\ &\geq -\frac{\pi}{2}\eta\lambda + \tan^{-1}\left(\frac{\gamma s^{-\eta}\sin(\eta\pi/2) - \delta\eta\cos(\eta\pi/2)}{\alpha + \gamma s^{-\eta}\cos(\eta\pi/2)}\right)\\ &=: -\frac{\pi}{2}\eta\lambda + \tan^{-1}(g(s))\end{aligned}$$

for $\alpha \geq -\delta >$, $\gamma \geq 0$ and where $s = |\cot(\theta/2)|$, $(s_1 \leq s \leq s_2)$, $s_1 \to 0$ and $s_2 \to \infty$. Now, we consider two cases:

Case 1: Let $0 < \eta \leq 1/2$, then

$$g(s) \geq \left(\frac{\gamma s^{-\eta}\sin(\eta\pi/2)}{\alpha + \gamma s^{-\eta}\cos(\eta\pi/2)}\right) \geq \left(\frac{\gamma s_2^{-\eta}\sin(\eta\pi/2)}{\alpha + \gamma s_2^{-\eta}\cos(\eta\pi/2)}\right) \approx 0.$$

Case 2: Let $1/2 \leq \eta \leq 1$, then

$$g(s) \geq \left(\frac{\gamma s^{-\eta}\cos(\eta\pi/2) + \delta\eta\cos(\eta\pi/2)}{\alpha + \gamma s^{-\eta}\cos(\eta\pi/2)}\right) =: l(s)$$

as $\alpha \geq -\delta$, then

$$l(s) \geq \left(\frac{\gamma s_2^{-\eta}\cos(\eta\pi/2) + \delta\eta\cos(\eta\pi/2)}{\alpha + \gamma s_2^{-\eta}\cos(\eta\pi/2)}\right) \approx \frac{-\delta\eta\cos(\eta\pi/2)}{\alpha}.$$

Thus from both the cases, we get $\arg(h(e^{i\theta})) \geq \zeta\pi/2$, where $\zeta$ is as given in the hypothesis. We observe that condition (24) concludes that the subordination (23) holds. Also, the hypothesis of Corollary 2.34 gets satisfied, as a result we get

$$p(z) \prec ((1+z)/(1-z))^{\eta},$$

equivalently $|\arg(p(z))| < \eta\pi/2$. This completes the proof. □

The next result follows from Theorem 2.24 by taking $q(z) = e^{\mu z}$. Again assume $\beta = 0$.

**Corollary 2.37.** *Let* $|(\lambda-1)\mu| \leq 1$, $\alpha\lambda/\delta \geq 0$ *and* $\gamma(\lambda-1)/\delta \geq 0$. *If* $p(z) \in \mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec e^{\mu\lambda z}\left(\alpha + \frac{\gamma}{e^{\mu z}} + \delta\frac{\mu z}{e^{\mu z}}\right),$$

*then* $p(z) \prec e^{\mu z}$, *where* $\mu$ *is a non-zero real number such that* $|\mu| \leq 1$.

The proof of this corollary is on the similar lines of the proof of Corollary 2.16 and therefore omitted.

Taking $\mu = 1$ in the above Corollary 2.37, we obtain the following result:

**Corollary 2.38.** *Let* $|\lambda-1|\leq 1$, $\alpha\lambda/\delta\geq 0$ *and* $\gamma(\lambda-1)\geq 0$. *If* $p(z)\in\mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec e^{\lambda z}\left(\alpha+\gamma e^{-z}+\delta z e^{-z}\right),$$

*then* $p(z)\prec e^{z}$.

**Corollary 2.39.** *Let* $-1\leq\lambda\leq 3$, $\alpha\lambda/\delta\geq 0$, $\beta(\lambda+1)/\delta\geq 0$ *and* $-1/4<\gamma/\delta\leq 0$. *If* $p(z)\in\mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec (1+z)^{\lambda/2}\left(\alpha+\beta\sqrt{1+z}+\frac{\gamma}{\sqrt{1+z}}+\frac{\delta z}{2(1+z)^{3/2}}\right),$$

*then* $p(z)\prec\sqrt{1+z}$.

*Proof.* The result follows from Theorem 2.24 by taking $q(z)=\sqrt{1+z}$.

Clearly, $Q(z)=\delta(1+z)^{(\lambda-3)/2}z/2$ is starlike (univalent) in $z\in\mathbb{D}$ on the similar lines of the proof of Corollary 2.18. Also, we have

(25)

$$\begin{aligned}&\operatorname{Re}\left(1+\frac{\gamma(\lambda-1)}{\delta}+\frac{\alpha\lambda}{\delta}q(z)+\frac{\beta(\lambda+1)}{\delta}q^2(z)+(\lambda-2)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)\\ &=\operatorname{Re}\left(1+\frac{\gamma(\lambda-1)}{\delta}+\frac{\alpha\lambda}{\delta}\sqrt{1+z}+\frac{\beta(\lambda+1)}{\delta}(1+z)+\frac{\lambda}{4}+(\lambda-3)\frac{z}{2(1+z)}\right)\\ &\geq 1+\frac{\gamma(\lambda-1)}{\delta}+\frac{\lambda-3}{4}=\lambda\left(\frac{\gamma}{\delta}+\frac{1}{4}\right)-\frac{\gamma}{\delta}+\frac{1}{4}\geq 0,\end{aligned}$$

by proceeding as in the proof of Corollary 2.18 for the given range of constants in the hypothesis. And hence the result. ☐

Taking $\beta=0$ in Theorem 2.24, we have the following result:

**Corollary 2.40.** *Let* $\alpha\delta\geq 0$ *and* $0\leq\lambda\leq 2$. *For* $-1<B<A\leq 1$, *if* $(1+AB)(1-A)(1-B)>8AB$, *let* $(B-A)/((1+A)(1+B))\leq\gamma/\delta\leq 2(B-A)/((1+A)(1+B))$. *Suppose* $p(z)\in\mathcal{A}_0$ *satisfies*

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec\left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha+\gamma\left(\frac{1+Bz}{1+Az}\right)+\delta\frac{(A-B)z}{(1+Az)^2}\right),$$

*then* $p(z)\prec(1+Az)/(1+Bz)$.

*Proof.* The result is followed by taking $q(z)=(1+Az)/(1+Bz)$ in Theorem 2.24. Then, we have

$$Q(z)=\frac{\delta z(A-B)}{(1+Az)^{2-\lambda}(1-Bz)^{\lambda}},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + (\lambda - 2)\frac{Az}{1+Az} - \frac{\lambda Bz}{1+Bz}, \quad 0 \le \lambda \le 2, \tag{26}$$

which is clearly bounded below for the mentioned range of $\lambda$. Thus, computing equation (26) on the boundary of $\mathbb{D}$, we get $Q(z)$ is starlike (univalent) for the given range of $\lambda$. Also, from condition (2) of Theorem 2.24, we have $\operatorname{Re}(H(z))$ as follows:

$$\operatorname{Re}\left(\frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1+Az}{1+Bz}\right) + (\lambda-2)\left(\frac{A-Bz}{(1+Az)(1+Bz)}\right) + \left(\frac{1-Bz}{1+Bz}\right)\right). \tag{27}$$

Now, by using Lemmas 2.19, 2.20 and 2.21, we obtain

$$\begin{aligned}\operatorname{Re}(H(z)) \ge{}& \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1-A}{1-B}\right) + \left(\frac{1-|B|}{1+|B|}\right)\\ &+ (\lambda-2)\left(\frac{A-B}{(1+A)(1+B)}\right) \quad (\alpha\delta \ge 0)\\ ={}& \lambda\left(\frac{\gamma}{\delta} + \frac{A-B}{(1+A)(1+B)}\right) + \frac{\alpha\lambda}{\delta}\left(\frac{1-A}{1-B}\right) + \frac{1-|B|}{1+|B|}\\ &- \left(\frac{\gamma}{\delta} + \frac{2(A-B)}{(1+A)(1+B)}\right). \end{aligned} \tag{28}$$

To achieve the desired result it is required to show that equation (28) becomes greater than or equal to 0, which is possible when $(B-A)/((1+A)(1+B)) \le \gamma/\delta \le (2(B-A))/((1+A)(1+B))$. This completes the proof. ☐

## 3. Sufficient conditions

On substituting $p(z) = zf'(z)/f(z)$ in Corollaries 2.6, 2.13, 2.28 and 2.35, respectively, we have the following example.

**Example 3.1.** The following are sufficient conditions for $f \in \mathcal{S}^*$.

(i) Let (a) $1 + 2\beta/\delta > 0$, $\gamma\delta \le 0$, $\alpha\delta \ge 0$ and $0 \le \lambda \le 1$, or (b) $\alpha\lambda/\delta \ge 0$, $\beta(\lambda+1)/\delta \ge 0$, $\gamma(\lambda-1)/\delta \ge 0$ and $|\lambda| \le 1$. If $f \in \mathcal{A}$ satisfies

$$\begin{aligned}&\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\\ &\prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right).\end{aligned}$$

(ii) Let (a) $1+2\alpha/\delta > 0$, $\gamma\delta \ge 0$ and $1 \le \lambda \le 2$, or (b) $\alpha\lambda/\delta$, $\gamma(\lambda-1)/\delta \ge 0$ and $-1 \le \lambda \le 2$. If $f(z) \in \mathcal{A}$ satisfies

$$\begin{aligned}&\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)\\ &\prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \gamma\frac{1-z}{1+z} + \frac{2\delta z}{(1+z)^2}\right).\end{aligned}$$

*Remark* 3.2. Let $\alpha = \gamma = 0$ and $\lambda = 1$ in Example 3.1(i)(a). Then we have the following result of Ravichandran and Kumar [16]:

**Corollary 3.3.** *Let* $1 + 2\beta/\delta > 0$. *If* $f(z) \in \mathcal{A}$ *and*

$$\frac{zf'(z)}{f(z)}\left((\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

*then* $f(z) \in \mathcal{S}^*$.

*Remark* 3.4. Let $\lambda = \alpha = \gamma = 0$ and $\delta = 1$ in Example 3.1(ii)(b). Then we have the following result of Obradowič and Tuneski [15]:

**Corollary 3.5.** *If* $f(z) \in \mathcal{S}$ *and*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{2z}{(1+z)^2},$$

*then* $f(z) \in \mathcal{S}^*$.

Kaplan [6] introduced the close-to-convex class $\mathcal{CC}$ as follows:

$$\mathcal{CC} = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \right\},$$

where $g(z)$ is convex (univalent) function in $\mathbb{D}$.

*Remark* 3.6. Let $p(z) = 2f'(z)/(2+z)$, $\alpha = \gamma = 0$ and $\lambda = 1$ in Corollary 2.6, we get the sufficient condition for $f$ to belong to the class $\mathcal{CC}$, as given below:

**Example 3.7.** Let $1 + 2\beta/\delta > 0$. If $f \in \mathcal{A}$ satisfies

$$\frac{2f'(z)}{(2+z)^2}\left(2\beta f'(z) - \delta z\right) + \frac{2\delta z f''(z)}{2+z} \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

then $f \in \mathcal{CC}$.

As an application of Corollaries 2.17 and 2.38, we have the following example:

**Example 3.8.** Let $f \in \mathcal{A}$. Then following are sufficient conditions for $f \in \mathcal{S}^*_e$.

(i) Let $\delta > 0$, $\alpha\lambda \geq -(A+B)$, where $A = \beta(\lambda+1)/e \geq 0$, $B = \gamma(\lambda-1)/e \geq 0$ and $|\lambda| \leq 1$. If $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec e^{\lambda z}\left(\gamma e^{-z} + \alpha + \beta e^{z} + \delta z\right).$$

(ii) Let $|\lambda - 1| \leq 1$, $\alpha\lambda/\delta \geq 0$ and $\gamma(\lambda-1) \geq 0$. If $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \beta\frac{zf'(z)}{f(z)} + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)$$
$$\prec e^{\lambda z}\left(\alpha + \beta e^{z} + \gamma e^{-z} + \delta z e^{-z}\right).$$

Taking $p(z) = zf'(z)/f(z)$ in Corollaries 2.7 and 2.29, respectively, we have the following example:

**Example 3.9.** Let $f \in \mathcal{A}$. Then we have the following sufficient conditions for $f \in \mathcal{S}^*(1/2)$.

(i) Let $1 + \beta/\delta > 0$, $\gamma\delta \leq 0$, $-1 + (2\alpha + \beta)/\delta \geq 0$ and $0 \leq \lambda \leq 1$ and $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \left(\alpha + \gamma \frac{f(z)}{zf'(z)} + (\beta - \delta)\frac{zf'(z)}{f(z)} + \delta \left(1 + \frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec \left(\frac{1}{1-z}\right)^{\lambda} \left(\alpha + \beta \frac{1}{1-z} + \gamma(1-z) + \frac{\delta z}{1-z}\right).$$

(ii) Let $1 + \alpha/\delta > 0$, $(2\gamma + \alpha)/\delta \geq 1$ and $1 \leq \lambda \leq 2$ and $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \left(\alpha + \gamma \frac{f(z)}{zf'(z)} + \delta \left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)$$
$$\prec \left(\frac{1}{1-z}\right)^{\lambda} (\alpha + (1-z)\gamma + \delta z).$$

*Remark* 3.10. Let $\lambda = \alpha = \gamma = 0$, $\delta = \beta = 1$ and $\lambda = \alpha = \delta = 1$, $\gamma = 0$, respectively in Example 3.9(i) and (ii), we obtain the following known result of Marx [9] and Strohhäcker [23].

**Corollary 3.11.** *A convex function is starlike of order* 1/2*.*

**Definition.** A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be $\rho$-convex in $\mathbb{D}$ if it is analytic and $f(z)f'(z)/z \neq 0$ and also it satisfies

$$\operatorname{Re}\left(\rho\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\rho)\frac{zf'(z)}{f(z)}\right) > 0. \tag{29}$$

The set of all such functions is denoted by $\rho$-CV.

Mocanu [12] gave the following result for $\rho$-convex functions.

**Corollary 3.12.** *Let $\rho$ be an arbitrary real number, and suppose that $f(z)$ is $\rho$-convex. If $\rho \geq 1$, then $f(z)$ is convex.*

Taking $\lambda = \alpha = \gamma = 0$, $\beta = 1$ and $\delta = \rho$ in Example 3.9(i), we get the following result:

**Corollary 3.13.** *Let $1/\rho \geq 1$ and if $f$ satisfies*

$$(1-\rho)\frac{zf'(z)}{f(z)} + \rho\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+\rho z}{1-z}, \tag{30}$$

*then* $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1}{1-z}$.

The above Definition, Corollaries 3.11, 3.12 and 3.13 yield the following result:

**Corollary 3.14.** *A $\rho$-convex function is starlike of order* $1/2$ *for* $\rho > 0$.

As an application of Corollaries 2.18 and 2.39, respectively, we have the following example.

**Example 3.15.** Let $f \in \mathcal{A}$. Then following are sufficient conditions for $f \in \mathcal{S}^*_L$.

(i) Let $-2 \leq \lambda \leq 2$, $\gamma(\lambda - 1) \geq 0$, $\beta(\lambda + 1) \geq 0$, $\delta > \max(0, \sqrt{2}\gamma)$ and $-(2\sqrt{2}\gamma + \delta)/4 < \alpha \leq -3\gamma/(2\sqrt{2})$ and $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta - \delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec (1+z)^{\lambda/2}\left(\frac{\gamma}{\sqrt{1+z}} + \alpha + \beta\sqrt{1+z} + \delta\left(\frac{z}{2(1+z)}\right)\right).$$

(ii) Let $-1 \leq \lambda \leq 3$, $\alpha\lambda/\delta \geq 0$, $\beta(\lambda+1)/\delta \geq 0$ and $-1/4 < \gamma/\delta \leq 0$ and $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \beta\frac{zf'(z)}{f(z)} + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)$$
$$\prec (1+z)^{\lambda/2}\left(\alpha + \beta\sqrt{1+z} + \frac{\gamma}{\sqrt{1+z}} + \frac{\delta z}{2(1+z)^{3/2}}\right).$$

Let $p(z) = zf'(z)/f(z)$ in Corollaries 2.22 and 2.40. Then we have the following example.

**Example 3.16.** Let $f \in \mathcal{A}$. Then following are sufficient conditions for $f \in \mathcal{S}^*[A, B]$, $-1 < B < A \leq 1$.

(i) Let $0 \neq f \in \mathcal{S}$. Let $0 \leq \lambda \leq 1$. If $(1 + AB)(1 - A)(1 - B) > 8AB$, let $\alpha/\delta \geq (B-A)/((1+A)(1+B))$, $\beta/\delta \geq (A-B)(1-B)/((1-A^2)(1+B))$. If $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta - \delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha + (\beta - \delta)\left(\frac{1+Az}{1+Bz}\right) + \delta\frac{(A-B)z}{(1+Az)(1+Bz)}\right).$$

(ii) Let $\alpha\delta \geq 0$, and $0 \leq \lambda \leq 2$. If $(1 + AB)(1 - A)(1 - B) > 8AB$, let $(B - A)/((1 + A)(1 + B)) \leq \gamma/\delta \leq 2(B - A)/((1 + A)(1 + B))$. If $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)$$
$$\prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha + \gamma\left(\frac{1+Bz}{1+Az}\right) + \delta\left(\frac{(A-B)z}{(1+Az)^2}\right)\right).$$

From Corollary 2.23, we get the following example by taking

$$p(z) = zf'(z)/f(z).$$

**Example 3.17.** Let $f \in \mathcal{A}$. Then following are sufficient conditions for $f \in \mathcal{S}_P$.

(i) Let $\beta/\delta \geq \max(0; -\alpha/\delta)$. If $f$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\alpha + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec \left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\left(\alpha + \beta\left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right.$$
$$\left. + \frac{4\delta}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right).$$

*Remark* 3.18. As an application of the above results and by Noshiro-Warschawski, an analytic function $f$ is univalent if we substitute $p(z) = f'(z)$ in Corollaries 2.6, 2.28, 2.7, 2.29, 2.18, 2.39, 2.23, 2.17, 2.38, 2.13 and 2.35, respectively.

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