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A PARALLEL ITERATIVE METHOD FOR A FINITE FAMILY OF BREGMAN STRONGLY NONEXPANSIVE MAPPINGS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we introduce a parallel iterative method for finding a common fixed point of a finite family of Bregman strongly nonexpansive mappings in a real reflexive Banach space. Moreover, we give some applications of the main theorem for solving some related problems. Finally, some numerical examples are developed to illustrate the behavior of the new algorithms with respect to existing algorithms.

1. Introduction

Many kinds of problems in mathematics and physical sciences can be recast in terms of the problem for finding a fixed point for Bregman strongly nonexpansive mappings in Banach spaces, for instance, convex feasibility problems, null point problems for maximal monotone mappings, generalized mixed equilibrium problems, variational inequality problems (see Section 4). Due to the practical importance of these problems, iterative algorithms for finding fixed points of Bregman strongly nonexpansive mapping continue to be flourishing topic of interest in nonlinear analysis. Many authors have studied iterative methods to find the fixed point of Bregman strongly nonexpansive mappings and some related problems, see for instance, Chidume et.al. [14], Duan et.al. [17], Eskandani et al. [18], Kassay [20], Reich et al. [35–38], Suantai et al. [41], Tuyen [43, 44], Wang et al. [45], Zegye [48].

Let X be a real reflexive Banach space, X^* be the dual space of X, and $T_i : X \to X$, $i = 1, 2, \ldots, N$, be Bregman strongly nonexpansive mappings

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which satisfy $F(T_i) = \hat{F}(T_i)$ for all $i \in \{1, 2, \ldots, N\}$ and

$$F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset,$$

where $F(T_i)$ (resp. $\hat{F}(T_i)$) is the set of fixed points (resp. asymptotic fixed points) of T_i

Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X.

In 2010, Reich et al. introduced two algorithms for finding an element in F by using Bregman distance. They proved the strong convergence of the following two algorithms in a real reflexive Banach space.

$$(1) \qquad \begin{cases} x_0 \in X, \\ y_n^i = T_i(x_n + e_n^i),\ i = 1, 2, \ldots, N, \\ C_n^i = \{z \in X : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n := \bigcap_{i=1}^{N} C_n^i, \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0),\ n \geq 0, \end{cases}$$

and

$$(2) \qquad \begin{cases} x_0 \in X, \\ C_0^i = X,\ i = 1, 2, \ldots, N, \\ y_n^i = T_i(x_n + e_n^i),\ i = 1, 2, \ldots, N, \\ C_{n+1}^i = \{z \in C_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_{n+1} := \bigcap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = \mathrm{proj}^f_{C_{n+1}}(x_0),\ n \geq 0, \end{cases}$$

where the sequences of errors $\{e_n^i\} \subset X$ satisfy $\|e_n^i\| \to 0$ for all $i = 1, 2, \ldots, N$.

We can see that in the iterative methods (1) and (2), it is difficult to find the element x_{n+1}. Indeed, for each iteration step, in (1), we have to find the Bregman projection of x_0 onto the set of the intersection of N closed and convex subsets of X. Particularly, in (4), we have to find the Bregman projection of x_0 on the set of the intersection of $N(n+1)$ closed and convex subsets of X.

For $N = 1$, in 2012, by using Halpern's iterative method, Suantai et al. [41] gave the following iterative method: for fixed $u \in X$ and starting point $x_1 \in X$,

$$(3) \qquad x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)), \forall n \geq 1,$$

to find a fixed point of a Bregman strongly nonexpansive mapping T on X, where f^* is a Fenchel conjugate function of f. They proved that if the sequence $\{\alpha_n\} \subset (0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ defined by (3) converges strongly to $\mathrm{proj}_F^f(u)$, where $F = F(T)$ and proj_F^f is the Bregman projection of $u \in \mathrm{int\,dom}\, f$ onto F.

In 2014, to find a common fixed point of a finite family of Bregman strongly nonexpansive mappings $T_1, \ldots, T_N$, Zegeye [48] introduced the following iterative method

$$x_{n+1} = \mathrm{proj}_C^f(\triangledown f^*(\alpha_n \triangledown f(u) + (1 - \alpha_n) \triangledown f(Tx_n))), \tag{4}$$

where $T = T_N T_{N-1} \cdots T_1$. He proved that if the sequence $\{\alpha_n\} \subset (0, 1)$ satisfying the conditions (C1) and (C2) and

$$F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset,$$

then the sequence $\{x_n\}$ generated by (4) converges strongly to $\mathrm{proj}_F^f(u)$.

We can see that in the iterative method (4), to compute the element x_{n+1}, we have to compute $T(x_n)$ by $T(x_n) = T_N(T_{N-1}(\cdots(T_1(x_n))))$. For instance if T_i, $i = 1, 2, \ldots, N$ is huge size matrix, then the computation process the element $T(x_n)$ is not easy. So, there exists an open question is posed as follows:

Can we introduce a new parallel algorithm for finding a common fixed point of a finite family of Bregman nonexpansive mappings without using the product mapping, that is, we only have to compute $T_i(x_n)$ at each iteration step?

The purpose of this work is to give a new parallel iterative method for finding a common fixed point of a finite family of Bregman strongly nonexpansive mappings by using Bregman distance tool and the Halpern's iteration [19] to answer the above open question.

Furthermore, in Section 4, we give some applications of our main result to solving convex feasibility problems, problem of finding a common zero of maximal monotone mappings, generalized mixed equilibrium problems, problem of finding a common zero of Bregman inverse strongly monotone mappings and variational inequality problems. Finally, in Section 5, we give two numerical examples to illustrate the obtained results and show its performance.

2. Preliminaries

Let X be a real Banach space, X^* be the dual space of X and C be a nonempty, closed and convex subset of X. We denote the norm in X and X^* by $\|\cdot\|$ and $\|\cdot\|_*$, respectively, and we denote the value of the functional $x^* \in X^*$ at $x \in X$ by $\langle x^*, x \rangle$.

Let $f : X \to (-\infty, +\infty]$ be a function. We denote the domain of f by $\mathrm{dom}\, f$, that is, $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$, and the interior of a set K by $\mathrm{int}\, K$. The function f is called lower semi-continuous at $x_0 \in \mathrm{dom}\, f$ if $\liminf_{x \to x_0} f(x) \geq f(x_0)$ and f is called lower semi-continuous if it is lower semi-continuous at every point of its domain. For any $x \in \mathrm{int\,dom}\, f$ and $y \in X$, we defined by $f'(x,y)$ the right-hand derivative of f at x in the direction y, that is,

$$f'(x,y) = \lim_{t \downarrow 0} \frac{f(x+ty) - f(x)}{t},$$

The function f is called Gâteaux differentiable at x if

$$\lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

exists for any y. In this case $f'(x,y)$ coincides with $(\bigtriangledown f)(x)$, the value of the gradient $\bigtriangledown f$ of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for every $x \in$ int dom$f$.

The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$ and f is said to be uniformly Fréchet differentiable on a subset E of X if this limit is attained uniformly for $x \in E$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on int dom$f$, then f is continuous and its Gâteaux derivative $\bigtriangledown f$ is norm-to-weak* continuous (resp. continuous) on int dom$f$ (see [6]).

Let $f : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. For $x \in$ int dom$f$, the subdifferential of f at x is defined by

$$\partial f(x) = \{x^* \in X^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \ \forall y \in X\},$$

and the Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

Let X be a reflexive Banach space. Then a function $f : X \to (-\infty, +\infty]$ is called Legendre if it satisfies the following two conditions:

(L1) The interior of the domain of f is nonempty, f is Gâteaux differentiable on int dom$f$, and dom$\bigtriangledown f =$ int dom$f$;

(L2) The interior of the domain of f^* is nonempty, f^* is Gâteaux differentiable on int domf^*, and dom$\bigtriangledown f^* =$ int domf^*.

Since X is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [6]). This, with (L1) and (L2), imply the following equalities:

$$\bigtriangledown f = (\bigtriangledown f^*)^{-1}, \ \text{ran} \bigtriangledown f = \text{dom} \bigtriangledown f^* = \text{int}\,\text{dom}\, f^*$$

and

$$\text{ran} \bigtriangledown f^* = \text{dom} \bigtriangledown f = \text{int}\,\text{dom}\, f,$$

where ran $\bigtriangledown f$ denotes the range of $\bigtriangledown f$.

When the subdifferential of f is single-valued, it coincides with the gradient, that is, $\partial f = \bigtriangledown f$ (see [10]). By Bauschke et al. (see [4]) the conditions (L1) and (L2) also imply that the function f and f^* are strictly convex on the interior of their respective domains. If X is a smooth and strictly convex Banach space (see [15, 31]), then the Legendre function is $f(x) = \frac{1}{p}\|x\|^p$, $1 < p < +\infty$.

From now on we assume that X is a reflexive Banach space.

Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int dom} f \to [0, +\infty)$, defined by

$$D_f(y, x) = f(y) - f(x) - \langle \bigtriangledown f(x), y - x \rangle, \tag{5}$$

is called the Bregman distance with respect to f (see [12]). The Bregman distance has the following two important properties, called the three point identity: for any $x \in \text{dom}\, f$ and any $y, z \in \text{int}\,\text{dom}\, f$,

$$D_f(x, y) + D_f(y, z) - Df(x, z) = \langle \bigtriangledown f(z) - \bigtriangledown f(y), x - y \rangle, \tag{6}$$

and the four point identity: for any $y, \omega \in \text{dom}\, f$ and any $x, z \in \text{int}\,\text{dom}\, f$,

$$D_f(y, x) - D_f(y, z) - Df(\omega, x) + D_f(\omega, z) = \langle \bigtriangledown f(z) - \bigtriangledown f(x), y - \omega \rangle. \tag{7}$$

Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int dom} f$ onto the nonempty, closed and convex subset $C \subset \text{dom} f$ is the necessary unique vector $\text{proj}_C^f(x) \in C$ satisfying

$$D_f(\text{proj}_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

The normalized duality mapping $J : X \to 2^{X^*}$ of X is defined by

$$Jx := \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in X$.

It is known that X is strictly convex and reflective, then the duality mapping J of X is one-to-one and onto, and $J^{-1} : X^* \to 2^X$ is the duality mapping of X^*.

If X is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in X$, then $\bigtriangledown f(x) = 2Jx$ for all $x \in X$, where J is the normalized duality mapping from X into 2^{X^*}, and hence $D_f(x, y)$ is reduced to

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in X$, which is a Lyapunov function introduced by Alber in [1] and the Bregman projection $\text{proj}_C^f(x)$ is reduced to the generalized projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).$$

If $X = H$ is a Hilbert space, then J is the identity mapping and hence Bregman projection $\text{proj}_C^f(x)$ is reduced to the metric projection of H onto C.

Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then f is called:

(a) totally convex at $x \in \text{int dom} f$ if its modulus of total convexity at x, that is, the function $v_f : \text{int}\,\text{dom}\, f \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom}\, f, \|y - x\| = t\},$$

is positive, whenever $t > 0$;

(b) totally convex if it is totally convex at every point $x \in \text{int dom} f$;

(c) totally convex on bounded sets if $v_f(B,t)$ is positive for any nonempty bounded subset B of X and $t>0$, where the modulus of total convexity of the function f on the set B is the bi-function v_f : $\operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty)$, defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap \operatorname{int}\operatorname{dom} f\}.$$

A function $f : X \to (-\infty,+\infty]$ is called:

(a) cofinite if $\operatorname{dom} f^* = X^*$;

(b) coercive (see [48]) if the sublevel set of f is bounded which is equivalent to $\lim_{\|x\|\to+\infty} f(x) = \infty$, where the sublevel of f is defined by

$$lev_{\leq}^{f}(r) := \{x \in X : f(x) \leq r\}$$

for some $r \in R$;

(c) strongly coercive if $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$.

Let $f : X \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [1] and [12], we make use of the function $V_f : X \times X^* \to [0,\infty)$ associated with f, which is defined by

$$V_f(x,x^*) := f(x) - \langle x^*, x\rangle + f^*(x^*), \qquad \forall x \in X, x^* \in X^*.$$

Then V_f is nonexpansive and $V_f(x,x^*) = D_f(x, \bigtriangledown f^*(x^*))$ for all $x \in X$ and $x^* \in X^*$. Moreover, by the subdifferential inequality,

$$V_f(x,x^*) + \langle y^*, \bigtriangledown f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*) \tag{8}$$

for all $x \in X$ and $x^*, y^* \in X^*$ [26]. In addition, if $f : X \to (-\infty,+\infty]$ is a proper lower semicontinuous function, then $f^* : X^* \to (-\infty,+\infty]$ is a proper weak* lower semicontinuous and convex function (see [28]). Hence, V_f is convex in the second variable. Thus, for all $z \in X$, from $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, we have

$$\begin{aligned} D_f\left(z, \bigtriangledown f^*\left(\sum_{i=1}^N t_i \bigtriangledown f(x_i)\right)\right) &= V_f(z, \sum_{i=1}^N t_i \bigtriangledown f(x_i)) \\ &\leq \sum_{i=1}^N t_i V_f(z, \bigtriangledown f(x_i)) \\ &= \sum_{i=1}^N t_i D_f(z, \bigtriangledown f^* \bigtriangledown f(x_i)) \\ &= \sum_{i=1}^N t_i D_f(z, x_i), \end{aligned} \tag{9}$$

where $\{x_i\}_{i=1}^N \subset X$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a convex subset of int dom$f$ and let T be a self-mapping of C. A point p in the closure of C is said to be an asymptotic fixed point of T (see [13], [32]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T

will be denoted by $\hat{F}(T)$. A point $p \in C$ is called a strong asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C which converges strongly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\tilde{F}(T)$.

Recall that the mapping T is said to be Bregman quasi-nonexpansive [35] if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \le D_f(p, x),\ \forall x \in C,\ p \in F(T).$$

A mapping $T : C \to C$ is said to be Bregman relatively nonexpansive [35] if the following conditions are satisfied:

i) $F(T)$ is nonempty;
ii) $D_f(p, Tx) \le D_f(p, x),\ \forall x \in C,\ p \in F(T)$;
iii) $\hat{F}(T) = F(T)$.

In 2013, E. Naraghirad and J. C. Yao [30] introduced a new class of Bregman quasi-nonexpansive type mappings. A mapping $T : C \to C$ is said to be Bregman weak relatively nonexpansive if the following conditions are satisfied:

i) $F(T)$ is nonempty;
ii) $D_f(p, Tx) \le D_f(p, x),\ \forall x \in C,\ p \in F(T)$;
iii) $\tilde{F}(T) = F(T)$.

Remark 2.1.

a) It is easy to see that any Bregman relatively nonexpansive mapping is a Bregman-quasi nonexpansive mapping. It is also obvious that every Bregman relatively nonexpansive mapping is a Bregman weak relatively nonexpansive mapping, but the converse is not true in general [30, Example 1.1].
b) If X is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in X$, then the notations Bregman quasi-nonexpansive mapping, Bregman relatively nonexpansive mapping, Bregman relatively nonexpansive mapping and Bregman weak relatively nonexpansive mapping are called quasi nonexpansive mapping, relatively nonexpansive mapping, relatively nonexpansive mapping and weak relatively nonexpansive mapping, respectively.

A mapping $T : C \to C$ is called (quasi-)Bregman strongly nonexpansive (BSNE for short, see [28]) with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, Tx) \le D_f(p, x) \tag{10}$$

for all $p \in \hat{F}(T)$ and $x \in C$. If $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$, and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0, \tag{11}$$

then we have

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0. \tag{12}$$

A mapping $T : C \to C$ is called Bregman firmly nonexpansive (BFNE for short) if

$$\langle \triangledown f(Tx) - \triangledown f(Ty), Tx - Ty\rangle \leq \langle \triangledown f(x) - \triangledown f(y), Tx - Ty\rangle \tag{13}$$

for all $x, y \in C$. It is clear that from the definition of Bregman distance (5) that inequality (13) is equivalent to

$$\begin{aligned} &D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ &\leq D_f(Tx, y) + D_f(Ty, x). \end{aligned} \tag{14}$$

In [36] (see Lemma 1.3.2), Reich et al. proved that for any BFNE mapping T, $F(T) = \hat{F}(T)$ when the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X. In the case that, if T is a BFNE mapping, then T is a BSNE mapping with respect to a nonempty $F(T) = \hat{F}(T)$.

The following lemmas will be needed in the sequel for the proof of main results in this paper.

Lemma 2.2 ([2, Theorem 1.8]). *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable, then f is uniformly continuous on X.*

Lemma 2.3 ([33, Proposition 2.1]). *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on a bounded subset of X, then $\triangledown f$ is uniformly continuous on a bounded subset of X from the strong topology of X to the strong topology of X^*.*

Lemma 2.4 ([39, Proposition 2.2]). *If $x \in \operatorname{int}\operatorname{dom} f$, then the following statements are equivalent*:

(i) *The function f is totally convex at x*;

(ii) *For any sequence $\{y_n\} \subset domf$,*

$$\lim_{n\to\infty} D_f(y_n, x) = 0,$$

implies that $\lim_{n\to\infty} \|y_n - x\| = 0$.

Recall that the function f is called sequentially consistent (see [10]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded and $\lim_{n\to\infty} D_f(y_n, x_n) = 0$ implies that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 ([8, Lemma 2.1.2]). *The function f is totally convex on a bounded set if and only if it is sequentially consistent.*

Lemma 2.6 ([10, Corollary 4.4]). *Suppose that f is Gâteaux differentiable and totally convex on the* $\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $C \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex subset. If $\overline{x} \in C$, then the following conditions are equivalent*:

(i) $\overline{x} = \operatorname{proj}_C^f(x)$;

(ii) *$\overline{x}$ is the unique solution of variational inequality*

$$\langle \triangledown f(x) - \triangledown f(y), z - y\rangle \geq 0 \ \forall y \in C;$$

(iii) $\overline{x}$ *is the unique solution of inequality*

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x) \ \forall y \in C.$$

Lemma 2.7 ([34, Lemma 3.1]). *Let $f : X \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in X$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 2.8 ([27]). *Let $\{s_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_k}\}$ such that*

$$s_{n_k} \leq s_{n_k+1} \ \forall k \geq 0.$$

Define an integer sequence $\{\tau(n)\}$, where $n > n_0$, by

$$\tau(n) := \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, we have

$$\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}.$$

Lemma 2.9 ([46]). *Let $\{s_n\}$ be a sequence of nonnegative numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ and let $\{c_n\}$ be a sequence of real numbers satisfying the following two conditions*:

(i) $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n c_n$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty, \ \limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

Let X be a real reflexive Banach space. Let $T_i : X \longrightarrow X$ be Bregman strongly nonexpansive mappings which satisfy $F(T_i) = \hat{F}(T_i)$ for all $i \in \{1, 2, \ldots, N\}$ and

$$F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset.$$

Let $f : X \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X.

We consider the following problem:

$$\text{Find an element } x^* \in F. \tag{15}$$

In order to solve Problem (15), we propose the following algorithm:

Algorithm 3.1. *For an initial guess $x_0 = x \in X$ and $u \in X$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_{i,n} = T_i(x_n), \ i = 1, 2, \ldots, N, \\ \textit{chosse } i_n \textit{ such that} \\ D_f(y_{i_n,n}, x_n) = \max_{i=1,2,\ldots,N}\{D_f(y_{i,n}, x_n)\}, \ \textit{and let } y_n = y_{i_n,n}, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n)), \ n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$.

Now, we will prove the strong convergence of the above sequence $\{x_n\}$ under the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

First, we have the following propositions.

Proposition 3.2. *In Algorithm 3.1, we have that the sequence $\{x_n\}$ is bounded.*

Proof. Let $p \in F$, from (9), we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n))) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_n). \end{aligned} \tag{16}$$

From the definition of T_i and y_n, we get that

$$\begin{aligned} D_f(p, y_n) &= D_f(p, T_{i_n}(x_n)) \\ &= D_f(T_{i_n}(p), T_{i_n}(x_n)) \\ &\leq D_f(p, x_n). \end{aligned} \tag{17}$$

From (16) and (17), we obtain that

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\ &\leq \max\{D_f(p, u), D_f(p, x_n)\} \\ &\quad \vdots \\ &\leq \max\{D_f(p, u), D_f(p, x_0)\}. \end{aligned}$$

This implies that $\{D_f(p, x_n)\}$ is bounded. It follows from Lemma 2.7 that the sequence $\{x_n\}$ is bounded. □

Proposition 3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then for all $p \in F$, we have the following estimate*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n c_n, \tag{18}$$

where $s_n = D_f(p, x_n)$ and $c_n = \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle$.

Proof. For any $p \in F$, from (8), we have

$$\begin{aligned} D_f(p, x_{n+1}) &= V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n)) \\ &\leq V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(p))) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n) \nabla f(y_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &\leq \alpha_n V_f(p, \nabla f(p)) + (1-\alpha_n) V_f(p, \nabla f(y_n)) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \\ &= (1-\alpha_n) D_f(p, y_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle. \end{aligned} \tag{19}$$

Thus, combining (17) and (19), we obtain that

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n c_n.$$

This completes the proof. □

Now, strong convergence of the sequence $\{x_n\}$ in Algorithm 3.1 is given by the following theorem.

Theorem 3.4. *Let X be a real reflexive Banach space. Let $T_i : X \longrightarrow X$ be Bregman strongly nonexpansive mappings which satisfy $F(T_i) = \hat{F}(T_i)$ for all $i \in \{1, 2, \ldots, N\}$ and*

$$F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset.$$

Let $f : X \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X. If the conditions (C1) *and* (C2) *are satisfied, then the sequence $\{x_n\}$ generated by Algorithm 3.1, converges strongly to $x^* = \mathrm{proj}_F^f(u)$ as $n \to \infty$.*

Proof. Let $x^* = \mathrm{proj}_F^f(u)$ and $\{s_n\}$ be the sequence generated in Proposition 3.3 with $p = x^*$. We will show that $s_n \to 0$ by considering two possible cases.

Case 1. The sequence $\{s_n\}$ is eventually decreasing, i.e., there exists $N_0 \geq 0$ such that $\{s_n\}$ is decreasing for $n \geq N_0$ and thus $\{s_n\}$ must be convergent. This implies that

$$\begin{aligned}\lim_{n\to\infty} (D_f(x^*, x_{n+1}) - D_f(x^*, x_n)) &= \lim_{n\to\infty} (s_{n+1} - s_n)\\ &= 0. \end{aligned} \tag{20}$$

Now, we use the proof line as in [36]. From (17) and the boundedness of $\{x_n\}$, there is $M > 0$ such that

$$\begin{aligned} f(x^*) - \langle \triangledown f(y_n), x^* \rangle + f^*(\triangledown f(y_n)) &= V_f(x^*, \triangledown f(y_n))\\ &= D_f(x^*, y_n)\\ &\leq M. \end{aligned}$$

Hence, $\{\triangledown f(y_n)\}$ is contained in the set $lev_{\leq}^{\psi}(M - f(x^*))$, where $\psi = f^* - \langle \cdot, x^* \rangle$. Since f is lower semicontinuous, f^* is weak* lower semicontinuous. Hence, the function ψ is coercive by Moreau-Rockafellar Theorem (see [40, Theorem 7A] and [29]). This implies that $\{\triangledown f(y_n)\}$ is bounded. Thus, from the condition (C1), we get that

$$\begin{aligned}\lim_{n\to\infty} \| \triangledown f(x_{n+1}) - \triangledown f(y_n)\| &= \lim_{n\to\infty} \alpha_n \| \triangledown f(u) - \triangledown f(y_n)\|\\ &= 0. \end{aligned} \tag{21}$$

Since f is strongly coercive and uniformly convex on a bounded subset of X, f^* is uniformly Fréchet differentiable on a bounded subset of X^* (see [47, Proposition 3.6.2]). Moreover, f^* is bounded on a bounded subset of X^* (see

[47, Lemma 3.6.1] and [4, Theorem 3.3]). It follows from Lemma 2.3 that $\bigtriangledown f^*$ is uniformly continuous on a bounded subset of X^* and hence we get

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = \lim_{n\to\infty}\| \bigtriangledown f^*(\bigtriangledown f(x_{n+1})) - \bigtriangledown f^*(\bigtriangledown f(y_n))\| \tag{22}$$
$$= 0.$$

Thus, from Lemma 2.2, we obtain that

$$\lim_{n\to\infty}\|f(x_{n+1})-f(y_n)\| = 0. \tag{23}$$

Now, we have

$$\begin{aligned}
D_f(x^*,y_n) - D_f(x^*,x_n) &= f(x^*) - f(y_n) - \langle \bigtriangledown f(y_n), x^*-y_n\rangle - D_f(x^*,x_n)\\
&= f(x^*) - f(x_{n+1}) + f(x_{n+1}) - f(y_n)\\
&\quad - \langle \bigtriangledown f(x_{n+1}), x^* - x_{n+1}\rangle + \langle \bigtriangledown f(x_{n+1}), x^*-x_{n+1}\rangle\\
&\quad - \langle \bigtriangledown f(y_n), x^*-y_n\rangle - D_f(x^*,x_n)\\
&= D_f(x^*,x_{n+1}) + (f(x_{n+1}) - f(y_n))\\
&\quad + \langle \bigtriangledown f(x_{n+1}), x^*-x_{n+1}\rangle\\
&\quad - \langle \bigtriangledown f(y_n), x^*-y_n\rangle - D_f(x^*,x_n)\\
&= D_f(x^*,x_{n+1}) + (f(x_{n+1}) - f(y_n))\\
&\quad + \langle \bigtriangledown f(x_{n+1}) - \bigtriangledown f(y_n), x^*-x_{n+1}\rangle\\
&\quad + \langle \bigtriangledown f(y_n), y_n - x_{n+1}\rangle.
\end{aligned} \tag{24}$$

From (21)–(24), we obtain

$$\lim_{n\to\infty}(D_f(x^*,y_n) - D_f(x^*,x_n)) = \lim_{n\to\infty}(D_f(x^*,T_{i_n}(x_n)) - D_f(x^*,x_n)) \tag{25}$$
$$= 0.$$

It follows from (10)-(12) that

$$\lim_{n\to\infty} D_f(y_n,x_n) = \lim_{n\to\infty} D_f(T_{i_n}(x_n),x_n) = 0. \tag{26}$$

From the definition of y_n, we have

$$\lim_{n\to\infty} D_f(y_{i,n},x_n) = \lim_{n\to\infty} D_f(T_i(x_n),x_n) = 0 \tag{27}$$

for all $i = 1,2,\ldots,N$.

Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$ and

$$\limsup_{n\to\infty}\langle \bigtriangledown f(u) - \bigtriangledown f(x^*), x_n - x^*\rangle = \langle \bigtriangledown f(u) - \bigtriangledown f(x^*), v - x^*\rangle.$$

On the other hand, from (27), $\lim_{n\to\infty} D_f(T_i(x_n),x_n) = 0$ for all $i = 1,2,\ldots,N$, we have $v \in F$. It follows from the definition of Bregman projection and Lemma 2.6 that

$$\limsup_{n\to\infty}\langle \bigtriangledown f(u) - \bigtriangledown f(x^*), x_n - x^*\rangle \le 0,$$

that is, $\limsup_{n\to\infty} c_n \leq 0$, where c_n is in Proposition 3.3.

Applying Lemma 2.9 to (18), we get $\lim_{n\to\infty} s_n = 0$, that is,

$$\lim_{n\to\infty} D_f(x^*, x_n) = 0.$$

It follows from Lemma 2.4 that $x_n \to x^*$.

Case 2. Suppose $\{s_n\}$ is not a monotone sequence. Then, from Lemma 2.8, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}.$$

Moreover, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $s_{\tau(n)} < s_{\tau(n+1)}$ for all $n \geq n_0$.

From (19), we have

$$0 < s_{\tau(n)+1} - s_{\tau(n)} \leq 2\alpha_{\tau(n)}\langle \bigtriangledown f(u) - \bigtriangledown f(p), x_{\tau(n)+1} - p\rangle.$$

Since $\alpha_{\tau(n)} \to 0$ and the boundedness of $\{x_n\}$, we derive

$$\lim_{n\to\infty}(s_{\tau(n)+1} - s_{\tau(n)}) = 0. \tag{28}$$

By a similar argument to Case 1, we can verify $\limsup_{n\to\infty} c_{\tau(n)} \leq 0$ and

$$s_{\tau(n)+1} \leq (1 - \alpha_{\tau(n)})s_{\tau_n} + \alpha_{\tau(n)}c_{\tau(n)}.$$

Since $s_{\tau(n)+1} > s_{\tau(n)}$ and $\alpha_{\tau(n)} > 0$, we have

$$s_{\tau(n)} \leq c_{\tau(n)}.$$

Thus, from $\limsup_{n\to\infty} c_{\tau(n)} \leq 0$, we get $\lim_{n\to\infty} s_{\tau(n)} = 0$. This together with (28) implies that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. Now, we have

$$0 \leq s_n \leq \max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1} \to 0.$$

Therefore, $s_n \to 0$, that is, $\{x_n\}$ converges strongly to $x^* = \mathrm{proj}_F^f(u)$. (see, Case 1).

This completes the proof. □

We know that if X is a smooth, strictly convex and reflexive Banach space and $f(x) = \|x\|^2$ (see, Section 2), then the notion of the Bregman strongly nonexpansive mapping is reduced to the notion of the strongly relatively nonexpansive mapping. So, we have the following corollary for finding a common fixed point of a finite family of strongly relatively nonexpansive mappings [49].

Corollary 3.5. *Let X be a smooth, strictly convex and reflexive Banach space and J be the normalized duality mapping from X to 2^{X^*}. Let T_i be strongly relatively nonexpansive mappings on X for all $i \in \{1, 2, \ldots, N\}$ and*

$$F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset.$$

Suppose that $u \in X$ *and define the sequence* $\{x_n\}$ *as follows*: $x_0 \in E$ *and*

$$\begin{cases} y_{i,n} = T_i(x_n), \ i = 1, 2, \ldots, N, \\ \textit{chosse } i_n \textit{ such that} \\ \phi(y_{i_n,n}, x_n) = \max_{i=1,2,\ldots,N}\{\phi(y_{i,n}, x_n)\}, \ \textit{and let } y_n = y_{i_n,n}, \\ x_{n+1} = J^{-1}(\alpha_n J(u) + (1-\alpha_n)J(y_n)), \ n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$. *If the conditions* (C1) *and* (C2) *are satisfied, then the sequence* $\{x_n\}$ *converges strongly to* $x^* = \Pi_F(u)$ *as* $n \to \infty$.

4. Applications

4.1. Convex feasibility problems

Let C_i, $i = 1, 2, \ldots, N$ be N nonempty, closed and convex subsets of X such that

$$C = \bigcap_{i=1}^{N} C_i \neq \emptyset.$$

The convex feasibility problem (CFP) is to find an element in C. We know that $F(\mathrm{proj}^f_{C_i}) = C_i$ for all $i \in \{1, 2, \ldots, N\}$. And if the Legendre function f is uniformly Fréchet differentiable and bounded on a bounded subset of X, then the Bregman projection $\mathrm{proj}^f_{C_i}$ is BFNE and $F(\mathrm{proj}^f_{C_i}) = \hat{F}(\mathrm{proj}^f_{C_i})$ (see [36]). Thus, if we take $T_i = \mathrm{proj}^f_{C_i}$ in Theorem 3.4, then we get an algorithm for solving the convex feasibility problems.

Theorem 4.1. *Let* C_i, $i = 1, 2, \ldots, N$ *be nonempty, closed and convex subsets of* X *such that*

$$C = \bigcap_{i=1}^{N} C_i \neq \emptyset.$$

Let $f : X \longrightarrow \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of* X. *Then, for each* $x_0 \in X$, *the sequence* $\{x_n\}$ *defined by Algorithm 3.1 with* $T_i = \mathrm{proj}^f_{C_i}$ *for all* $i = 1, 2, \ldots, N$, *converges strongly to* $\mathrm{proj}^f_C(u)$, *as* $n \to +\infty$.

4.2. Common zeros of maximal monotone mappings

Let $A : X \longrightarrow 2^{X^*}$ be a maximal monotone mapping. The problem of finding an element $x \in X$ such that

$$0 \in Ax$$

is very important in optimization theory and related fields.

Recall that the resolvent of A, denoted by $\mathrm{Res}^f_A : X \longrightarrow 2^X$ is defined as follows (see [3]):

$$\mathrm{Res}^f_A(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

Bauschke et al. [3] proved that this resolvent is a single-valued BFNE mapping. In addition, if the Legendre function f is uniformly Fréchet differentiable and bounded on a bounded subset of X, then the resolvent Res_A^f is a BSNE mapping which satisfies $F(\mathrm{Res}_A^f) = \hat{F}(\mathrm{Res}_A^f)$ (see [36]). And from $F(\mathrm{Res}_A^f) = A^{-1}0$, in Theorem 3.4, if we take $T_i = \mathrm{Res}_{A_i}^f$ for all $i = 1, 2, \ldots, N$, we get an algorithm for the problem of finding a common zero of a finite family of maximal monotone mappings.

Theorem 4.2. *Let $A_i : X \longrightarrow 2^{X^*}$, $i = 1, 2, \ldots, N$ be N maximal monotone mappings such that $F = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Let $f : X \longrightarrow \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X. Then, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by Algorithm 3.1 with $T_i = \mathrm{Res}_{A_i}^f$ for all $i = 1, 2, \ldots, N$, converges strongly to $\mathrm{proj}_F^f(u)$, as $n \to +\infty$.*

4.3. System of generalized mixed equilibrium problems

Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers, $\Psi : X \longrightarrow X^*$ be a nonlinear mapping and $\varphi : C \longrightarrow \mathbb{R}$ be a real valued function. The generalized mixed equilibrium problem is to find an element $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x) \ \forall y \in C. \tag{29}$$

The set of solutions of the problem (29) is denoted by $GMEP(\Theta, \varphi, \Psi)$, that is

$$GMEP(\Theta, \varphi, \Psi) = \{x \in C : \ \Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x) \ \forall y \in C\}.$$

Let Φ_i, $i = 1, 2, \ldots, N$ be bifunctions from $C \times C$ to $\mathbb{R}$, let φ_i, $i = 1, 2, \ldots, N$ be real valued-functions from C to $\mathbb{R}$ and let Ψ_i, $i = 1, 2, \ldots, N$ be mappings from X to X^*. Solving a system of generalized mixed equilibrium problems means finding an element $x \in C$ such that

$$x \in \bigcap_{i=1}^N GMEP(\Theta_i, \varphi_i, \Psi_i).$$

In particular, if $\Psi = 0$, the problem (29) reduces to the mixed equilibrium problem (see [11]) which is to find an element $x \in C$ such that

$$\Theta(x, y) + \varphi(y) \geq \varphi(x) \ \forall y \in C. \tag{30}$$

We denote by $MEP(\Theta)$ the set of solutions of the problem (30).

If $\varphi = 0$, the problem (29) reduces to the generalized equilibrium problem (see [22, 42]), which is to find an element $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle \geq 0 \ \forall y \in C. \tag{31}$$

The set of solutions of the problem (31) is denoted by $GEP(\Theta, \Psi)$.

If $\Theta = 0$, the problem (29) reduces to the mixed variational inequality problem of Browder type (see [7]), which is to find an element $x \in C$ such that

$$\langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x)\ \forall y \in C. \tag{32}$$

The set of solutions of the problem (32) is denoted by $MVI(C, \varphi, \Psi)$.

If $\varphi = 0$ and $\Psi = 0$, the problem (29) reduces to the equilibrium problem (see [5, 21, 23–25]) which is to find an element $x \in C$ such that

$$\Theta(x, y) \geq 0\ \forall y \in C. \tag{33}$$

The set of solutions of the problem (33) is denoted by $EP(\Theta)$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$ satisfies the following conditions:

(C1) $\Theta(x, x) = 0$ for all $x \in C$;
(C2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;
(C3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(C4) for each $x \in C$, $\Theta(x, .)$ is convex and lower semi-continuous.

Let C be a nonempty, closed and convex subset of a real reflexive Banach space X and let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$ and $\Psi : C \longrightarrow X^*$ be a continuous monotone mapping. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunctional satisfying the conditions (C1)-(C4). The mixed resolvent of Θ is the mapping $\mathrm{Res}^f_{\Theta,\varphi,\Psi} : X \longrightarrow 2^C$ defined by

$$\begin{aligned}\mathrm{Res}^f_{\Theta,\varphi,\Psi}(x) = \{z \in C : \Theta(z, y) + \varphi(y) + \langle \Psi x, y - z\rangle \\ + \langle \bigtriangledown f(z) - \bigtriangledown f(x), y - z\rangle \geq \varphi(z)\ \forall y \in C\}.\end{aligned}$$

Note that, if $f : X \longrightarrow (-\infty, +\infty]$ is a coercive and Gâteaux differentiable function, then the mixed resolvent of Θ satisfies $\mathrm{dom}\,\mathrm{Res}^f_{\Theta,\varphi,\Psi} = X$ (see [16], Lemma 4.14).

Lemma 4.3 (see [16], Lemma 2.15)**.** *Let $f : X \longrightarrow (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of X. If the bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$ satisfies conditions* (C1)-(C4)*, then*

(i) $\mathrm{Res}^f_{\Theta,\varphi,\Psi}$ *is single-valued*;
(ii) $\mathrm{Res}^f_{\Theta,\varphi,\Psi}$ *is a BFNE mapping*;
(iii) *the set of fixed points of* $\mathrm{Res}^f_{\Theta,\varphi,\Psi}$ *is the solution set of the corresponding generalized mixed equilibrium problem, i.e.,* $F(\mathrm{Res}^f_{\Theta,\varphi,\Psi}) = GMEP(\Theta, \varphi, \Psi)$;
(iv) $GMEP(\Theta, \varphi, \Psi)$ *is a closed and convex subset of* C;
(v) *for all $x \in X$ and $u \in F(\mathrm{Res}^f_{\Theta,\varphi,\Psi})$, we have*

$$D_f(u, \mathrm{Res}^f_{\Theta,\varphi,\Psi}(x)) + D_f(\mathrm{Res}^f_{\Theta,\varphi,\Psi}(x), x) \leq D_f(u, x).$$

Now, by using Lemma 4.3, we have the following theorem for solving the system of generalized mixed equilibrium problems.

Theorem 4.4. *Let C_i, $i = 1, 2, \dots, N$ be nonempty, closed and convex subsets of X. Let $\Theta_i : C_i \times C_i \longrightarrow \mathbb{R}$ satisfying conditions* (C1)-(C4)*, let $\varphi_i : C_i \longrightarrow \mathbb{R}$ be lower semi-continuous and convex functions from C_i to $\mathbb{R}$ and $\Psi_i : C_i \longrightarrow X^*$ be continuous monotone mappings, for all $i = 1, 2, \dots, N$. Assume that*

$$S := \bigcap_{i=1}^{N} GMEP(\Theta_i, \varphi_i, \Psi_i) \neq \emptyset.$$

Let $f : X \longrightarrow \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X. Then, for each $x_0 \in X$, the sequence $\{x_n\}$ defined by Algorithm 3.1 with $T_i = \mathrm{Res}^f_{\Theta_i,\varphi_i,\Psi_i}$ for all $i = 1, 2, \dots, N$, converges strongly to $\mathrm{proj}^f_S(u)$, as $n \to +\infty$.

4.4. Common zeros of Bregman inverse strongly monotone mappings

The class of Bregman inverse strongly monotone mappings was introduced by Butnariu and Kassay in [9]. We assume that the Legendre function f satisfies the following range condition:

$$\text{ran } (\triangledown f - A) \subset \text{ran } (\triangledown f). \tag{34}$$

Recall that an mapping $A : X \longrightarrow 2^{X^*}$ is called Bregman inverse strongly monotone mapping (BISM for short) if $(\mathrm{dom}A) \cap (\text{int dom}f) \neq \emptyset$ and for any $x, y \in \text{int dom}f$, and if $\xi \in Ax$, $\eta \in Ay$, we have

$$\langle \xi - \eta, \triangledown f^*(\triangledown f(x) - \xi) - \triangledown f^*(\triangledown f(y) - \eta)\rangle \geq 0.$$

Recall that the anti-resolvent of A is the mapping $A^f : X \longrightarrow 2^X$, defined by

$$A^f = \triangledown f^*_o(\triangledown f - A).$$

We know that the mapping A is BISM if and only if the anti-resolvent A^f of it is a (single-valued) BFNE mapping (see [9], Lemma 3.5). Reich et al. proved that if $f : X \longrightarrow (-\infty, +\infty]$ is a Legendre function and $A : X \longrightarrow 2^{X^*}$ is a BISM mapping such that $A^{-1}(0) \neq \emptyset$, then $A^{-1}(0) = F(A^f)$ (see [34], Proposition 7). Thus, if the Legendre function f is uniformly Fréchet differentiable and totally convex on a bounded subset of X, the anti-resolvent A^f is a single-valued BSNE mapping which satisfies $F(A^f) = \hat{F}(A^f)$ (see [36], Lemma 1.3.2).

Now, let C_i, $i = 1, 2, \dots, N$ be nonempty, closed and convex subsets of X and let $A_i : X \longrightarrow 2^{X^*}$, $i = 1, 2, \dots, N$ be BISM mappings such that $C_i \subset \mathrm{dom}A_i$ for all $i \in \{1, 2, \dots, N\}$ and that $f : X \longrightarrow \mathbb{R}$. From the range condition (34), we obtain that $\mathrm{dom}A_i^f = (\mathrm{dom}A) \cap (\text{int dom}f) = \mathrm{dom}A_i$ because in this case $\text{int dom}f = X$. From Proposition 7 (i) in [36], we know that $A^{-1}(0) = F(A^f)$. So, we have the following theorem:

Theorem 4.5. *Let C_i, $i = 1, 2, \dots, N$ be nonempty, closed and convex subsets of X such that*

$$C := \bigcap_{i=1}^{N} C_i \neq \emptyset.$$

Let $A_i : X \longrightarrow 2^{X^*}$, $i = 1, 2, \dots, N$ be BISM mappings such that $C_i \subset domA_i$ and

$$S := \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset.$$

Let $f : X \longrightarrow R$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X. Suppose that the range condition (30) *is satisfied for each A_i. Then, for each $x_0 \in C$, the sequence $\{x_n\}$ defined by Algorithm 3.1 with $T_i = A_i^f$ for all $i = 1, 2, \dots, N$, converges strongly to* $\mathrm{proj}_S^f(u)$ *as $n \to +\infty$.*

4.5. System of variational inequalities

In this subsection, we consider the variational inequality problem: Find an element $x^\dagger \in C$ such that

$$\langle Ax^\dagger, y - x^\dagger \rangle \geq 0 \ \forall y \in C, \tag{35}$$

where $A : X \longrightarrow X^*$ is a BISM mapping and C is a nonempty, closed and convex subset of dom$A$. We denote by $VI(C, A)$ the set of solutions of (35).

We know that in [34] (see Proposition 8), Reich et al. proved that, if $f : X \longrightarrow (-\infty, +\infty]$ is a Legendre and totally convex function which satisfies the range condition (34) and $A : X \longrightarrow X^*$ is a BISM mapping, and if C is a nonempty, closed and convex subset of dom$A \cap$ int dom$f$, then $VI(A, C) = F(\mathrm{proj}_C^f \, oA^f)$.

Thus, if the Legendre function f is uniformly Fréchet differentiable and totally convex on bounded subsets of X, the anti-resolvent A^f is a single-valued BSNE mapping which satisfies $F(A^f) = \hat{F}(A^f)$ (see [36], Lemma 1.3.2). And, we know that the Begman projection proj_C^f is BSNE mapping which satisfies $F(\mathrm{proj}_C^f) = \hat{F}(\mathrm{proj}_C^f)$. So, by Lemma 2 in [32], $\mathrm{proj}_C^f \, {}_oA^f$ is a BSNE mapping with $F(\mathrm{proj}_C^f \, {}_oA^f) = \hat{F}(\mathrm{proj}_C^f \, oA^f)$. Hence, we have the following theorem:

Theorem 4.6. *Let C_i, $i = 1, 2, \dots, N$ be nonempty, closed and convex subsets of X such that*

$$C := \bigcap_{i=1}^{N} C_i \neq \emptyset.$$

Let $A_i : X \longrightarrow X^*$, $i = 1, 2, \dots, N$ be BISM mappings such that $C_i \subset domA_i$ and

$$S := \bigcap_{i=1}^{N} VI(C_i, A_i) \neq \emptyset.$$

Let $f : X \longrightarrow R$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of X. Suppose that the range condition (30) *is satisfied for each A_i. Then, for each $x_0 \in C$, the sequence $\{x_n\}$ defined by Algorithm 3.1 with $T_i = \mathrm{proj}^f_{C_i} \circ A^f_i$ for all $i = 1, 2, \ldots, N$, converges strongly to $\mathrm{proj}^f_S(u)$ as $n \to +\infty$.*

5. Numerical experiment

In this section, the algorithm is implemented in MATLAB 2014a running on a HP Compaq 510, Core(TM) 2 Duo CPU. T5870 with 2.0 GHz and 2GB RAM.

Example 5.1. Consider the problem of finding an element $x^* \in C := \cap_{i=1}^N C_i$, for

$$C_i = \{x \in \mathbb{R}^{\mathcal{N}} : \langle a_i^C, x\rangle \le b_i^C\},$$

where $a_i^C \in \mathbb{R}^{\mathcal{N}}$, $b_i^C, \in \mathbb{R}$ for all $i = 1, 2, \ldots, N$.

Next, we take the randomly generated values of the coordinates of a_i^C in $[1, 3]$ and b_i^C in $[2, 4]$, respectively.

Clearly, $C = \cap_{i=1}^N C_i \neq \emptyset$ because $0 \in C$.

Remark 5.2. In this example, we define the function TOL by

$$\mathrm{TOL}_n = \frac{1}{N}\sum_{i=1}^{N} \|x_n - P_{C_i}x_n\|^2$$

for all $n \geq 1$. Note that, if at the nth step, $\mathrm{TOL}_n = 0$, then $x_n \in C$, that is, x_n is a solution of this problem. So, we use the condition $\mathrm{TOL}_n <$ err to stop the iterative process.

Applying Theorem 4.1 to $\mathcal{N} = 50$, $N = 100$, $\alpha_n = 1/n$ for all $n \geq 1$, and $u = x_0$ have the elements randomly generated in $[10, 50]$, we have the following table of numerical results.

TABLE 1. Table of numerical results for Example 5.1.

Stop condition: $\mathrm{TOL}_n <$ err

err	TOL_n	n	Time (s)
10^{-2}	9.661258e − 03	664	3.287
10^{-3}	9.965119e − 04	2019	8.945
10^{-4}	9.910501e − 05	6292	26.660
10^{-5}	9.835176e − 06	19938	83.850

The behaviors of the function TOL_n in Table 1 for the case $\mathrm{TOL}_n < 10^{-4}$ is presented in the following figure.

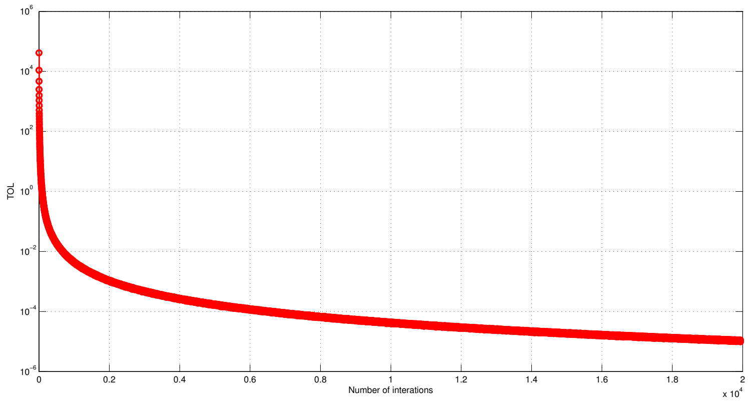

FIGURE 1. The behavior of TOL_n with the stop condition $\mathrm{TOL}_n < 10^{-4}$.

Example 5.3. Consider the following problem: Find an element

$$x^* \in S := \cap_{i=1}^{N} GMEP(\Theta_i, \varphi_i, \Psi_i),$$

where $\Theta_i(x, y) = i(\|y\|^2 - \|x\|^2)$, $\varphi_j(x) = \|x\|^2$ and $\Psi_i(x) = ix$ for all $j = 1, 2, \ldots, 50$ and for all $x, y \in \mathbb{R}^{10}$. We can see that Θ_i satisfies the conditions (A1)-(A4), φ_i is a continuous convex function, Ψ_i is a continuous monotone mapping for each $i = 1, 2, \ldots, 50$. It is easy to see that $S = \{(0, 0, 0)\}$.

Now, with $f(x) = \frac{1}{2}\|x\|^2$, from the definition of $Res^f_{\Theta_j,\varphi_j,\Psi_j}$, for each $x \in \mathbb{R}^{10}$, we have

$$Res^f_{\Theta_j,\varphi_j,\Psi_j}(x) = \{z \in \mathbb{R}^{10} : j(\|y\|^2 - \|z\|^2) + \|y\|^2 + j\langle x, y - z\rangle + \langle z - x, y - z\rangle \geq \|z\|^2, \ \forall y \in \mathbb{R}^{10}\}.$$

Hence, we obtain that

$$Res^f_{\Theta_j,\varphi_j,\Psi_j}(x) = \frac{j-1}{2j+3}x,$$

for all $x \in \mathbb{R}^{10}$ and for all $j = 1, 2, \ldots, 50$. Now, applying Theorem 4.4 with $u = x_0$ have the elements randomly generated in $[10, 50]$ and $\alpha_n = 1/n$ for all $n \geq 1$, we obtain the following table of numerical results:

The behaviors of the functions TOL_n in Table 2 for the case $\mathrm{TOL}_n < 10^{-3}$ are presented in the following figure.

TABLE 2. Table of numerical results for Example 5.3.

Stop condition: $\text{TOL}_n = \|x_n\| < \text{err}$

Algorithm 3.1			**Iterative method** (4)		
err	TOL_n	n	err	TOL_n	n
10^{-2}	$9.999425\text{e}-03$	6280	10^{-2}	$9.998964\text{e}-03$	9268
10^{-3}	$9.999902\text{e}-04$	62797	10^{-3}	$9.999935\text{e}-04$	92671
10^{-4}	$9.999998\text{e}-05$	627964	10^{-4}	$9.999999\text{e}-05$	926704

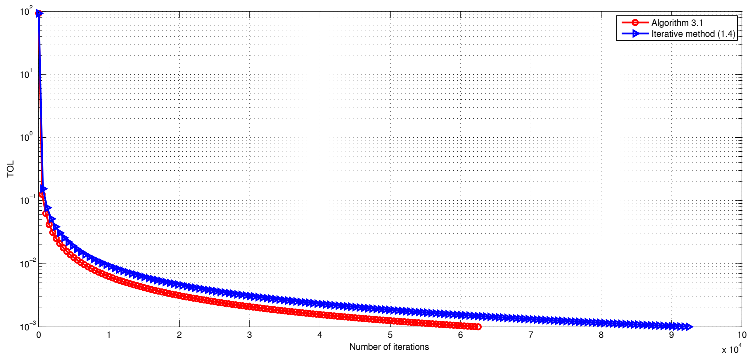

FIGURE 2. The behavior of TOL_n with the stop condition $\text{TOL}_n < 10^{-3}$.

6. Conclusions

In this paper, we have introduced and analyzed a new parallel algorithm for finding a common fixed point of a finite family of Bregman strongly nonexpansive mappings in a real reflexive Banach space (Algorithm 3.1 and Theorem 3.4). Some applications of our main results to some related problems are presented in Section 4. These problems include the convex feasibility problem (Theorem 4.1), the problem of finding a common zeros of maximal monotone mappings (Theorem 4.2), the system of generalized mixed equilibrium problems (Theorem 4.4), the problem of finding a common zeros of Bregman inverse strongly monotone mappings (Theorem 4.5) and the system of variational inequalities (Theorem 4.6). Finally, in Section 5, we exhibit two numerical examples which illustrates the effectiveness of the proposed algorithm.

References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in Theory and applications of nonlinear operators of accretive and

monotone type, 15–50, Lecture Notes in Pure and Appl. Math., **178**, Dekker, New York, 1996.

[2] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, Cambridge Studies in Advanced Mathematics, **34**, Cambridge University Press, Cambridge, 1993.

[3] H. H. Bauschke, J. M. Borwein, and P. L. Combettes, *Bregman monotone optimization algorithms*, SIAM J. Control Optim. **42** (2003), no. 2, 596–636. https://doi.org/10.1137/S0363012902407120

[4] ———, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, Commun. Contemp. Math. **3** (2001), no. 4, 615–647. https://doi.org/10.1142/S0219199701000524

[5] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), no. 1-4, 123–145.

[6] J. F. Bonnans and A. Shapiro, *Perturbation analysis of optimization problems*, Springer Series in Operations Research, Springer-Verlag, New York, 2000. https://doi.org/10.1007/978-1-4612-1394-9

[7] F. E. Browder, *Existence and approximation of solutions of nonlinear variational inequalities*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 1080–1086. https://doi.org/10.1073/pnas.56.4.1080

[8] D. Butnariu and A. N. Iusem, *Totally convex functions for fixed points computation and infinite dimensional optimization*, Applied Optimization, **40**, Kluwer Academic Publishers, Dordrecht, 2000. https://doi.org/10.1007/978-94-011-4066-9

[9] D. Butnariu and G. Kassay, *A proximal-projection method for finding zeros of set-valued operators*, SIAM J. Control Optim. **47** (2008), no. 4, 2096–2136. https://doi.org/10.1137/070682071

[10] D. Butnariu and E. Resmerita, *Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. **2006** (2006), Art. ID 84919, 39 pp. https://doi.org/10.1155/AAA/2006/84919

[11] L.-C. Ceng and J.-C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), no. 1, 186–201. https://doi.org/10.1016/j.cam.2007.02.022

[12] Y. Censor and A. Lent, *An iterative row-action method for interval convex programming*, J. Optim. Theory Appl. **34** (1981), no. 3, 321–353. https://doi.org/10.1007/BF00934676

[13] Y. Censor and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996), no. 4, 323–339. https://doi.org/10.1080/02331939608844225

[14] C. E. Chidume, A. Adamu, and L. C. Okereke, *A Krasnoselskii type algorithm for approximating solutions of variational inequality problems and convex feasibility problems*, J. Nonlinear Var. Anal. **2** (2018), 203–218.

[15] I. Cioranescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Mathematics and its Applications, **62**, Kluwer Academic Publishers Group, Dordrecht, 1990. https://doi.org/10.1007/978-94-009-2121-4

[16] V. Darvish, *Strong convergence theorem for a system of generalized mixed equilibrium problems and finite family of Bregman nonexpansive mappings in Banach spaces*, Opsearch **53** (2016), no. 3, 584–603. https://doi.org/10.1007/s12597-015-0245-2

[17] L. L. Duan, A. F. Shi, L. Wei, and R. P. Agarwal, *Construction techniques of projection sets in hybrid methods for infinite weakly relatively nonexpansive mappings with applications*, J. Nonlinear Funct. Anal. **2019** (2019), Article ID 14.

[18] G. Z. Eskandani, M. Raeisi, and J. K. Kim, *A strong convergence theorem for Bregman quasi-noexpansive mappings with applications*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 2, 353–366. https://doi.org/10.1007/s13398-017-0481-9

[19] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961. https://doi.org/10.1090/S0002-9904-1967-11864-0

[20] G. Kassay, S. Reich, and S. Sabach, *Iterative methods for solving systems of variational inequalities in reflexive Banach spaces*, SIAM J. Optim. **21** (2011), no. 4, 1319–1344. https://doi.org/10.1137/110820002

[21] J. K. Kim, *Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings*, Fixed Point Theory Appl. **2011** (2011), 10, 15 pp. https://doi.org/10.1186/1687-1812-2011-10

[22] ______, *Convergence theorems of iterative sequences for generalized equilibrium problems involving strictly pseudocontractive mappings in Hilbert spaces*, J. Comput. Anal. Appl. **18** (2015), no. 3, 454–471.

[23] J. K. Kim and N. Buong, *An iteration method for common solution of a system of equilibrium problems in Hilbert spaces*, Fixed Point Theory Appl. **2011** (2011), Art. ID 780764, 15 pp. https://doi.org/10.1155/2011/780764

[24] J. K. Kim, N. Buong, and J. Y. Sim, *A new iterative method for the set of solutions of equilibrium problems and of operator equations with inverse-strongly monotone mappings*, Abstr. Appl. Anal. **2014** (2014), Art. ID 595673, 8 pp. https://doi.org/10.1155/2014/595673

[25] J. K. Kim and W. H. Lim, *A new iterative algorithm of pseudomonotone mappings for equilibrium problems in Hilbert spaces*, J. Inequal. Appl. **2013** (2013), 128, 16 pp. https://doi.org/10.1186/1029-242X-2013-128

[26] F. Kohsaka and W. Takahashi, *Proximal point algorithms with Bregman functions in Banach spaces*, J. Nonlinear Convex Anal. **6** (2005), no. 3, 505–523.

[27] P.-E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), no. 7-8, 899–912. https://doi.org/10.1007/s11228-008-0102-z

[28] V. Martín-Márquez, S. Reich, and S. Sabach, *Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Math. Anal. Appl. **400** (2013), no. 2, 597–614. https://doi.org/10.1016/j.jmaa.2012.11.059

[29] J.-J. Moreau, *Sur la fonction polaire d'une fonction semi-continue supérieurement*, C. R. Acad. Sci. Paris **258** (1964), 1128–1130.

[30] E. Naraghirad and J.-C. Yao, *Bregman weak relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2013** (2013), 141, 43 pp. https://doi.org/10.1186/1687-1812-2013-141

[31] S. Reich, *Review of Geometry of Banach spaces, duality mappings and nonlinear problems by Ioana Cioranescu*, Bull. Amer. Math. Soc. **26** (1992), 367–370. https://doi.org/10.1090/S0273-0979-1992-00287-2

[32] ______, *A weak convergence theorem for the alternating method with Bregman distances*, in Theory and applications of nonlinear operators of accretive and monotone type, 313–318, Lecture Notes in Pure and Appl. Math., **178**, Dekker, New York, 1996.

[33] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), no. 3, 471–485.

[34] ______, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **31** (2010), no. 1-3, 22–44. https://doi.org/10.1080/01630560903499852

[35] ______, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces*, Nonlinear Anal. **73** (2010), no. 1, 122–135. https://doi.org/10.1016/j.na.2010.03.005

[36] ______, *Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces*, in Fixed-point algorithms for inverse problems in

science and engineering, 301–316, Springer Optim. Appl., **49**, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-9569-8_15

[37] ———, *A projection method for solving nonlinear problems in reflexive Banach spaces*, J. Fixed Point Theory Appl. **9** (2011), no. 1, 101–116. https://doi.org/10.1007/s11784-010-0037-5

[38] ———, *Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces*, in Optimization theory and related topics, 225–240, Contemp. Math., **568**, Israel Math. Conf. Proc, Amer. Math. Soc., Providence, RI, 2012. https://doi.org/10.1090/conm/568/11285

[39] E. Resmerita, *On total convexity, Bregman projections and stability in Banach spaces*, J. Convex Anal. **11** (2004), no. 1, 1–16.

[40] R. T. Rockafellar, *Level sets and continuity of conjugate convex functions*, Trans. Amer. Math. Soc. **123** (1966), 46–63. https://doi.org/10.2307/1994612

[41] S. Suantai, Y. J. Cho, and P. Cholamjiak, *Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces*, Comput. Math. Appl. **64** (2012), no. 4, 489–499. https://doi.org/10.1016/j.camwa.2011.12.026

[42] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), no. 2, 417–428. https://doi.org/10.1023/A:1025407607560

[43] T. M. Tuyen, *Parallel iterative methods for Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Fixed Point Theory Appl. **19** (2017), no. 3, 1695–1710. https://doi.org/10.1007/s11784-016-0325-9

[44] ———, *Parallel iterative methods for solving systems of generalized mixed equilibrium problems in reflexive Banach spaces*, Optimization **66** (2017), no. 4, 623–639. https://doi.org/10.1080/02331934.2016.1277999

[45] A. Y. Wang and Z. M. Wang, *A simple hybrid Bregman projection algorithms for a family of countable Bregman quasi-strict pseudo-contractions*, Nonlinear Funct. Anal. Appl. **22** (2017), no. 5, 1001–1011.

[46] H.-K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314** (2006), no. 2, 631–643. https://doi.org/10.1016/j.jmaa.2005.04.082

[47] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002. https://doi.org/10.1142/9789812777096

[48] H. Zegeye, *Convergence theorems for Bregman strongly nonexpansive mappings in reflexive Banach spaces*, Filomat **28** (2014), no. 7, 1525–1536. https://doi.org/10.2298/FIL1407525Z

[49] J. Zhao and S. Wang, *Strong convergence for Bregman relatively nonexpansive mapping in reflexive Banach spaces and applications*, Nonlinear Funct. Anal. Appl. **20** (2015), no. 3, 365–379.

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