GRADIENT EINSTEIN-TYPE CONTACT METRIC MANIFOLDS

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Abstract. Consider a gradient Einstein-type metric in the setting of $K$-contact manifolds and $(\kappa, \mu)$-contact manifolds. First, it is proved that, if a complete $K$-contact manifold admits a gradient Einstein-type metric, then $M$ is compact, Einstein, Sasakian and isometric to the unit sphere $S^{2n+1}$. Next, it is proved that, if a non-Sasakian $(\kappa, \mu)$-contact manifolds admits a gradient Einstein-type metric, then it is flat in dimension 3, and for higher dimension, $M$ is locally isometric to the product of a Euclidean space $E^{n+1}$ and a sphere $S^n(4)$ of constant curvature $+4$.

1. Introduction

Let $(M, g)$ be a smooth Riemannian manifold of dimension $\geq 3$. We say that $(M, g)$ is an Einstein-type manifold or that $(M, g)$ supports an Einstein-type structure if there exist a vector field $V$ on $M$ and a smooth function $\lambda : M \to \mathbb{R}$ such that

$$\alpha S + \beta \nabla^2 f + \nu df \otimes df = \gamma g = (\rho r + \lambda)g,$$

for some constants $\alpha, \beta, \nu, \rho \in \mathbb{R}$ with $(\alpha, \beta, \nu) \neq 0$. Here $\nabla$ and $V^\#(X) = g(V, X)$ stand for the Lie derivative and the 1-form metrically dual to the vector field $V$, respectively. If $V = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we say that $(M, g)$ is a gradient Einstein-type manifold. In this case, the equation (1) can be written as

$$\alpha S + \beta \nabla^2 f + \nu df \otimes df = \gamma g,$$

where $S$ is Ricci tensor and $\nabla^2$ stands for the Hessian of $f$. We refer to $f$ as the potential function.

The concept of Einstein-type manifold was studied and introduced by Catino et al. as a generalization of Einstein spaces [8]. In case $f$ is constant we say that the Einstein-type structure is trivial. Notice that, an Einstein-type structure

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on a Riemannian manifold \((M, g)\) unifies several particular cases well studied in the literature, such as Ricci solitons \([15, 22]\), Ricci almost solitons \([23]\), gradient Ricci solitons, Yamabe solitons \([6, 10]\), Yamabe quasi-solitons \([16]\), conformal gradient solitons \([25]\), \(m\)-quasi-Einstein manifolds \([7]\), \((m, \rho)\)-quasi-Einstein manifold \([17]\) and \(\rho\)-Einstein solitons \([9]\).

There has been a growing interest in the study of Einstein condition and its various generalizations in the setting of contact metric manifolds in recent years. In \([4]\), Boyer-Galicki studied Einstein and \(\eta\)-Einstein \(K\)-contact manifolds and they proved that any compact \(K\)-contact Einstein manifold is Sasakian. In \([24]\), the author generalizing Boyer-Galicki result proved that if a complete \(K\)-contact metric represents a gradient Ricci soliton, then it is compact Einstein and Sasakian. Extending these for gradient Ricci almost solitons, the author \([11]\) proved that if a compact \(K\)-contact metric represents a gradient Ricci almost soliton, then it is isometric to a unit sphere \(S^{2n+1}\).

Recently, Ghosh studied \(m\)-quasi-Einstein, generalized \(m\)-quasi-Einstein and \((m, \rho)\)-quasi-Einstein metric within the background of contact geometry respectively in \([13]\), \([14]\) and \([12]\). These works of Ghosh inspires us to study the gradient Einstein-type condition within the background of \(K\)-contact manifolds and \((\kappa, \mu)\)-contact manifolds.

In this paper, we confine our study to the gradient Einstein-type metric within the framework of \(K\)-contact and \((\kappa, \mu)\)-contact manifolds. In Section 2, we gathered some preliminary definitions and formulas on contact manifolds. In Section 3, we prove that if complete \(K\)-contact manifolds admit a gradient Einstein-type metric, then \(M\) is compact, Einstein, Sasakian and isometric to the unit sphere \(S^{2n+1}\). In Section 4, we consider \((\kappa, \mu)\)-contact manifold which admits a gradient Einstein-type metric and we prove that if a non-Sasakian \((\kappa, \mu)\)-contact manifold supports a gradient Einstein-type structure, then for \(n = 1\), \(M\) is flat, and for \(n > 1\), \(M\) is locally isometric to \(E^{n+1} \times S^n\) \((4)\).

We have borrowed some ideas and arguments from \([21]\), but our goals and main results are different from \([21]\).

2. Preliminaries

Let us recall the basic concepts and formulas of contact metric manifolds. A \((2n + 1)\)-dimensional smooth manifold \(M\) is said to be contact if it admits a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) on \(M\). This 1-form is called a contact 1-form. For a contact 1-form \(\eta\), there exists a unique vector field \(\xi\) such that \(d\eta(\xi, X) = 0\) for all vector field \(X\) and \(\eta(\xi) = 1\). Polarizing \(d\eta\) on the contact sub-bundle \(\mathcal{D}\) (defined by \(\eta = 0\)), we obtain a Riemannian metric \(g\) and a \((1, 1)\)-tensor field \(\varphi\) such that

\[
\begin{align*}
(3) & \quad d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi \\
(4) & \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).
\end{align*}
\]
The structure \((\varphi, \xi, \eta, g)\) on \(M\) is known as a contact metric structure and the metric \(g\) is called an associated metric. A Riemannian manifold \(M\) together with the structure \((\varphi, \xi, \eta, g)\) is said to be a contact metric manifold and we denote it by \((M, \varphi, \xi, \eta, g)\). On a contact metric manifold (see [1])

\[
\nabla_X \xi = -\varphi X - \varphi hX, \quad h\varphi + \varphi h = 0,
\]

\[
(\nabla_X \varphi) + (\nabla_{\varphi X} \varphi)Y = 2g(Y, X)\xi - \eta(Y)(X + hX + \eta(Z)\xi),
\]

for any vector field \(X,Y\) on \(M\) and \(\nabla\) denotes the operator of covariant differentiation of \(g\). If the vector field \(\xi\) is Killing (equivalently, \(h = 0\)) with respect to \(g\), then the contact metric manifold \(M\) is said to be \(K\)-contact. On a \(K\)-contact (Sasakian) manifold the following formulas are known [1]

\[
\nabla_X \xi = -\varphi X, \quad Q\xi = 2n\xi, \quad (\nabla_X \varphi)Y = R(\xi, X)Y,
\]

where \(Q\) and \(R\) denote the Ricci operator and the Riemann curvature tensor of \(g\), respectively. A contact metric manifold is said to be Sasakian if it satisfies

\[
(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.
\]

On a Sasakian manifold the curvature tensor satisfies

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y.
\]

Also, the contact metric structure on \(M\) is said to be Sasakian if the almost Kähler structure on the metric cone \((M \times R^+, r^2g + dr^2)\) over \(M\), is Kähler [1]. Any Sasakian manifold is \(K\)-contact, and the converse only holds when the dimension is 3. See [1] and [5] for more information about it.

3. \(K\)-contact manifold satisfying the gradient Einstein-type metrics

Here, consider a \(K\)-contact metric as a gradient Einstein-type metric. The following will be needed to prove our main result.

Lemma 3.1. If \((M, g, \alpha, \beta, \nu, \gamma)\) is a gradient Einstein-type contact metric manifold, then the curvature tensor \(R\) has the expression

\[
\beta R(X, Y)Df = \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y] + \frac{\nu\gamma}{\beta}[(Xf)Y - (Yf)X]
\]

\[
+ \frac{\nu\alpha}{\beta}[(Yf)QX - (Xf)QY] + [(X\gamma)Y - (Y\gamma)X]
\]

for any vector fields \(X, Y\) on \(M\).

Proof. The gradient Einstein-type equation (2) can be expressed as

\[
\alpha QY + \beta \nabla_Y Df + \nu g(Y, Df)Df = \gamma Y,
\]
where $D$ is the gradient operator of $g$. Differentiate (13) covariantly along $X$, we obtain
\[
\alpha(\nabla_X Q)Y + \alpha Q\nabla_X Y + \beta \nabla_X \nabla_Y Df + \nu g(\nabla_X Y, Df)Df
+ \nu g(Y, Df)\nabla_X Df = (X\gamma)Y + \gamma \nabla_X Y.
\]

Then the required result follows by applying this equation and (13) to the well known expression $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. □

**Theorem 3.2.** Let $(M, \varphi, \xi, \eta, g)$ be a complete $K$-contact manifold of dimension $2n+1$. If there exists a gradient Einstein-type structure $(f, \alpha, \beta, \nu, \gamma)$ associated with the contact metric $g$, then $M$ is compact, Einstein, Sasakian and isometric to the unit sphere $S^{2n+1}$.

**Proof.** Applying the covariant derivative to (8) and then employing (7), we obtain
\[
(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X.
\]

At this point we remember that for a $K$-contact manifold $\xi$ is Killing, and hence $L_\xi Q = 0$. In view of (7) and (8), we obtain $\nabla_\xi Q = Q\varphi - \varphi Q$. Replacing $\xi$ with $X$ in (12) and making use of $\nabla_\xi Q = Q\varphi - \varphi Q$, (15) and (9), we get
\[
-\beta g((\nabla_Y \varphi)X, Df) = \alpha g((\varphi Q - 2n\varphi Y, X) + \left[ (\gamma + \frac{\nu\gamma}{\beta})(\xi f) \right] g(Y, X)
+ \left[ \frac{\nu(2n\alpha - \gamma)}{\beta} \right] (Y f) \eta(X) - (\xi f) \frac{\nu\alpha}{\beta} g(QY, X)
- (Y\gamma)\eta(X).
\]

Replacing $X$ and $Y$ by $\varphi X$ and $\varphi Y$ respectively in relation (16), adding the resulting equation with (16) and then using (6) (where $h = 0$, as $M$ is $K$-contact) and (7), we have
\[
-2\beta(\xi f)g(Y, X) + \beta(Y f)\eta(X) + \beta(\xi f)\eta(Y)\eta(X)
= \alpha((\varphi Q + Q\varphi)Y, X) + 2 \left[ (\gamma + \frac{\nu\gamma}{\beta})(\xi f) \right] g(Y, X)
+ (\xi f) \frac{\nu\alpha}{\beta} g((\varphi Q - QY, X) - \left[ (\gamma + \frac{\nu\gamma}{\beta})(\xi f) \right] \eta(Y)\eta(X)
+ \left[ (\gamma - \frac{\nu\alpha}{\beta}) (Y f) \eta(X) - 4n\alpha g(\varphi Y, X) - (Y\gamma)\eta(X).
\]

Since $Q$ is self-adjoint, anti-symmetrizing the above equation gives
\[
\beta[(Y f)\eta(X) - (X f)\eta(Y)] = 2\alpha((\varphi Q + Q\varphi)Y, X) - 8n\alpha g(\varphi Y, X)
+ \frac{\nu(2n\alpha - \gamma)}{\beta}[(Y f)\eta(X) - (X f)\eta(Y)]
+ [(X\gamma)\eta(Y) - (Y\gamma)\eta(X)].
\]
Now replacing $X$ by $\varphi X$ and $Y$ by $\varphi Y$ in the equation (18) and applying the $K$-contact condition (8), (3), $\eta \circ \varphi = 0$ and $\varphi \xi = 0$ gives

$$g((\varphi Q + Q\varphi)Y, X) = 4ng(\varphi Y, X)$$

for all vector fields $Y, Z$ on $M$. It follows from last equation that

$$\varphi Q + Q\varphi = 4n\varphi Y.$$  

In view of above equation, it follows from (18) that

$$\nu(2n\alpha - \gamma) - \frac{\beta^2}{\beta} [(Yf)\eta(X) - (Xf)\eta(Y)] = [(Y\gamma)\eta(X) - (X\gamma)\eta(Y)].$$

Next, taking $\sigma = \nu(2n\alpha - \gamma) - \beta^2$. So $D\sigma = -\nu D\gamma$. On account of these, (20) can be exhibited as

$$\sigma Df + \frac{\beta}{\nu} D\sigma = \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} \xi.$$

Differentiating (21) in the direction of $X$ and utilizing of (7) provides

$$(X\sigma)Df + \sigma \nabla_X Df + \frac{\beta}{\nu} \nabla_X D\sigma = X \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} \xi$$

$$- \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} \varphi X.$$  

Taking inner product of (22) with $Y$ and then anti-symmetrizing the resulting equation, we obtain

$$(X\sigma)(Yf) - (Y\sigma)(Xf) = X \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} \eta(Y)$$

$$- Y \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} \eta(X)$$

$$- 2 \left\{ \sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) \right\} g(\varphi X, Y).$$

Now we can write the equation (21) as

$$\sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) = 0.$$  

Substituting (24) in (23), inserting $X$ by $\varphi X$, $Y$ by $\varphi Y$ in the resulting equation and noting that $g(\varphi X, Y) \neq 0$ for any contact metric manifold, we obtain

$$\sigma(\xi f) + \frac{\beta}{\nu}(\xi \sigma) = 0.$$  

Making use of (25) in (21), we get

$$\nu(2n\alpha - \gamma) - \beta^2)(Xf) = -\frac{\beta}{\nu}(X\sigma).$$
On the other hand, taking the trace of (12) over $X$ we obtain

\[
\begin{align*}
\left[ \frac{\beta^2 + \nu \alpha}{\beta} \right] S(Y, Df) &= \frac{\alpha}{2} (Yr) + \frac{\nu (\alpha - 2n \gamma)}{\beta} (Yf) - 2n (Y \gamma).
\end{align*}
\]

Let \( \{ e_i, \varphi e_i, \xi_i \}, i = 1, 2, 3, \ldots, n \) be an orthonormal \( \varphi \)-basis of \( M \) such that \( Qe_i = \rho_i e_i \). Thus, we have \( \varphi Qe_i = \rho_i \varphi e_i \). Substituting \( e_i \) for \( Y \) in (19), we obtain \( Q \varphi e_i = (4n - \rho_i) \varphi e_i \). Using the \( \varphi \)-basis and (8), the scalar curvature \( r \) is given by

\[
\begin{align*}
r &= g(Q \xi, \xi) + \sum_{i=1}^{n} [g(Qe_i, e_i) + g(Q \varphi e_i, \varphi e_i)] \\
&= g(Q \xi, \xi) + \sum_{i=1}^{n} [\rho_i g(e_i, e_i) + (4n - \rho_i) g(\varphi e_i, \varphi e_i)] \\
&= 2n (2n + 1).
\end{align*}
\]

Making use of the constancy of \( r \), \( D\sigma = -\nu D\gamma \) and (26) in (27), it follows that \( QDf = 2nDf \). Differentiating this along \( X \) and recalling (13) and \( QDf = 2nDf \), we obtain

\[
(\nabla_X Q)Df - \frac{\alpha}{\beta} Q^2 X + \frac{\gamma + 2n \alpha}{\beta} QX - \frac{2n \gamma}{\beta} X = 0.
\]

Contracting the foregoing equation over \( X \) and observing that \( r = 2n(2n + 1) \), we get

\[
\begin{align*}
\sum_{i=1}^{2n+1} g((\nabla_X Q)Df, e_i) - \frac{\alpha}{\beta} |Q|^2 + r \frac{\gamma + 2n \alpha}{\beta} - \frac{\gamma r}{\beta} &= 0.
\end{align*}
\]

Using that the scalar curvature is constant, the first term vanishes because \( \text{div} Q = \frac{1}{2} dr \) (this follows from the contraction of Bianchi’s second identity). From (28), we deduce \( |Q|^2 = 2nr \). Then, since \( r = 2n(2n + 1) \), we get

\[
\begin{align*}
|Q - \frac{r}{2n + 1} I|^2 &= |Q|^2 + \frac{r^2}{2n + 1} - \frac{2r^2}{2n + 1} \\
&= 2nr - \frac{r^2}{2n + 1} \\
&= 4n^2 (2n + 1) - 4n^2 (2n + 1) = 0.
\end{align*}
\]

Since the symmetric tensor \( Q - \frac{r}{2n + 1} I \) is of length zero, we get

\[
Q = \frac{r}{2n + 1} I = 2n I.
\]

This shows that \( M \) is Einstein with Einstein constant \( 2n \). Since \( M \) is complete, compactness of \( M \) follows from Myers’ theorem [19]. Applying the result of
Boyer-Galicki [4], we can conclude that $M$ is Sasakian. Consequently, (13) reduces to

$$\nabla_Y Df = \frac{(\gamma - 2n\alpha)}{\beta} Y - \frac{\nu}{\beta^2} g(Y, Df) Df.$$  \hspace{1cm} (29)

Now consider a smooth function $u = e^{\frac{\nu}{\beta} f}$ on $M$. From this we have the following relation (see Gomes [18]);

$$Du = \frac{\nu}{\beta} u Df,$$  \hspace{1cm} (30)

$$\nabla_Y Df + \frac{\nu}{\beta} g(Y, Df) Df = \frac{\beta}{\nu u} \nabla_Y Du.$$  \hspace{1cm} (31)

Comparing (29) and (31), we get

$$\nabla_Y Du = \frac{(\gamma - 2n\alpha)\nu u}{\beta^2} Y.$$  \hspace{1cm} (32)

As $M$ is Einstein with constant scalar curvature $r = 2n(2n + 1)$, the equation (27) takes the form $(\gamma\nu - 2nu\alpha + \beta^2) Df = -\beta D\gamma$. Using (30) in the foregoing equation we immediately infer that

$$(\gamma\nu - 2nu\alpha + \beta^2) Du = -\nu D\gamma.$$  

From this we can write $\gamma\nu Du + \nu D\gamma = (2nu\alpha - \beta^2) Du$, which is equivalent to $D(\gamma\nu u) = (2nu\alpha - \beta^2) Du$. In other words, $\gamma\nu u = (2nu\alpha - \beta^2) u + k$, where $k$ is a constant. This together with (32) gives

$$\nabla_Y Du = \left(-u + \frac{k}{\beta^2}\right) Y.$$  \hspace{1cm} (33)

As a result of Theorem 2 of Tashiro [25] it follows that $M$ is isometric to unit sphere $S^{2n+1}$. This completes the proof. \hspace{1cm} \Box

**Corollary 3.3.** Let $(M, g, \alpha, \beta, \nu, \gamma)$ be a complete gradient Einstein-type manifold. If $g$ represents a Sasakian metric, then it is compact, Einstein and isometric to the unit sphere $S^{2n+1}$. \hspace{1cm} \Box

**Proof.** This follows with the same proof as Corollary 3.1 in [21]. \hspace{1cm} \Box

Further, we remark that our Theorem 3.2 generalizes the results of Ghosh [11,12,14] on $K$-contact manifold admitting Ricci almost soliton, $(m, \rho)$-quasi-Einstein metric and generalized $m$-quasi-Einstein metric.

4. $(\kappa, \mu)$-contact manifold satisfying gradient Einstein-type metrics

Blair et al. [2] introduced a $(\kappa, \mu)$-contact manifold which is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}.$$  \hspace{1cm} (34)
for all vector fields \( X, Y \) on \( M \) and for some real numbers \((\kappa, \mu)\). Later on, Boeckx [3] classified these manifolds completely. This type of manifold is obtained by applying the \( D \)-homothetic deformation to a contact metric manifold that satisfies \( R(X, Y)\xi = 0 \). This class contains Sasakian manifolds (for \( \kappa = 1 \)) and the trivial sphere bundle \( E^{n+1} \times S^n(4) \) (for \( \kappa = \mu = 0 \)). Examples of non-Sasakian \((\kappa, \mu)\)-contact metric manifolds are the unit tangent bundles of Riemannian manifolds of constant curvature \( \neq 1 \). A lot of examples of \((\kappa, \mu)\)-contact structures can be constructed because of a \( D \)-homothetic deformation preserves \((\kappa, \mu)\)-contact structures (see [2]). On non-Sasakian \((\kappa, \mu)\)-contact manifolds, the following formulas are also true [2]:

\[
QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX
\]

\[
Q\xi = 2n\kappa\xi,
\]

\[
h^2 = (\kappa - 1)\varphi^2, \quad \kappa < 1.
\]

For the non-Sasakian case, i.e., \( \kappa < 1 \), the equation (34) determines the curvature of \( M \) completely. As a result of this, it is proved that a non-Sasakian \((\kappa, \mu)\)-contact manifold is locally homogeneous and hence analytic [3]. Moreover, the scalar curvature \( r \) of such manifold is given

\[
r = 2n(2(n-1) + \kappa - n\mu),
\]

which is constant. On a \((\kappa, \mu)\)-contact manifold we have

\[
(\nabla_\xi Q)X = \mu(2(n-1) + \mu)h\varphi X
\]

for any vector field \( X \) on \( M \).

Here we intend to examine the existence of gradient Einstein-type metric on \((\kappa, \mu)\)-contact manifold, and prove the following fruitful outcome.

**Theorem 4.1.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu)\)-contact manifold. If there exists a gradient Einstein-type structure \((f, \alpha, \beta, \nu, \gamma)\) associated with the metric \( g \), then for \( n = 1 \), \( M \) is flat, and for \( n > 1 \), \( M \) is locally isometric to \( E^{n+1} \times S^n(4) \).

**Proof.** First, differentiate (36) covariantly along an arbitrary vector field \( X \) and utilization of (5), we obtain

\[
(\nabla_X Q)\xi = Q(\varphi + \varphi h)X - 2nk(\varphi + \varphi h)X.
\]

Thus, taking the scalar product of (12) with \( \xi \) and using (36), the equation (40) gives

\[
g(R(X, Y)Df, \xi) = \frac{\alpha}{\beta}(g(Q\varphi Y + \varphi QY, X) + g(Q\varphi hY + h\varphi QY, X)

- 4nk g(\varphi Y, X)) + \frac{\nu(2n\kappa - \gamma)}{\beta^2} \{(Yf)\eta(X) - (Xf)\eta(Y)\}
\]
\((\nabla_{\xi} Df) = \mu h\varphi.\)

From \((13),\) we have

\[
\nabla_{\xi} Df = \frac{\gamma - 2n\kappa\alpha}{\beta} \xi - \frac{\nu}{\beta} (\xi f) Df.
\]
Differentiating (46) along $\xi$ and taking into account (39), (46)-(48) we ultimately obtain

$$\mu^2 \varphi Df = \frac{\nu b}{\beta} (\xi f) \xi + \left[ \frac{a(\gamma - 2n\kappa\alpha) + b(\gamma - 2n\kappa\alpha - \nu(\xi f)^2)}{\beta} \right] \xi$$

where we used $g(\nabla_\xi Df, \xi) = (\xi f) \xi$. Here,

$$a = \frac{4n^2\kappa\nu\alpha^2 - 2nk\beta^2 - \nu r\alpha^2}{2n} \quad \text{and} \quad b = \frac{r\nu\alpha^2 - 2nk\nu\alpha - 4n^2\kappa\nu\alpha}{2n}.$$

Applying $\varphi$ to the above equation, we obtain

$$\left\{ \mu^2 - \mu \left( \frac{\beta^2 + \nu\alpha}{2n} \right) [2(n - 1) + \mu] \right\} hDf = 0.$$

Furthermore, operating the preceding equation by $h$ and using (37), it follows that

$$\mu \left[ \mu(2n - (\beta^2 + \nu\alpha)) - 2(\beta^2 + \nu\alpha)(n - 1) \right] (\kappa - 1) \varphi^2 Df = 0.$$

Since $M$ is non-Sasakian, we have either (i) $\mu = 0$ or (ii) $\varphi^2 Df = 0$ or (iii) $\mu = \frac{2(\beta^2 + \nu\alpha)(n - 1)}{2n - (\beta^2 + \nu\alpha)}$.

**Case (i).** Here, it follows from (43) that $\kappa = 0$ because of $\mu = 0$. Hence $R(X, Y)\xi = 0$, according to the result of Blair [1] we obtain that $M$ is flat in dimension 3 and in higher dimensions it is locally isometric to the trivial bundle $\mathbb{E}^{(n+1)} \times \mathbb{S}^n(4)$.

**Case (ii).** Making use of (3) in $\varphi^2 Df = 0$ yields $Df = (\xi f)\xi$. Differentiating this along $X$, employing (3) gives that $\nabla_X Df = X(\xi f)\xi - (\xi f)(\varphi X + \varphi h X)$. As a result of Poincare lemma $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, the last equation provides

$$X(\xi f)\eta(Y) - Y(\xi f)\eta(X) + 2(\xi f)g(X, \varphi Y) = 0.$$

Replacing $X$ and $Y$ with $\varphi X$ and $\varphi Y$, respectively, in the above equation furnishes $\xi f = 0$, where we applied $g(X, \varphi Y) \neq 0$ for any contact metric structure. By virtue of this, we have $Df = 0$, i.e., $f$ is constant and consequently (13) shows that $M$ is Einstein, i.e., $QX = \frac{2}{n}X = 2n\kappa X$ by (48). Contracting this over $X$ we find that the scalar curvature $r = 2n\kappa(2n + 1)$. It shows $\eta\mu = 2(n - 1) - 2n\kappa$ in combination with (38). On the other hand, we can easily find $[2(n - 1) + \mu]h = 0$ from (35) on the basis of last equation and $QX = 2n\kappa X$. Since $M$ is non-Sasakian, we must have $2(n - 1) + \mu = 0$. So it follows for dimension 3 that $\mu = 0 = \kappa$, and by applying Blair’s result [1] we obtain that $M$ is flat. Again, for higher dimension it follows from $\mu = 2(1 - n)$ and (43) that $\kappa = n - \frac{1}{n} > 1$, contradicting our assumption.
Case (iii). Since $\mu = \frac{2(\beta^2 + \nu\alpha)(n-1)}{2n-(\beta^2 + \nu\alpha)}$, it follows from (43) that

$$\kappa = \frac{(\beta^2 + \nu\alpha)(n^2 - 1)}{n(\beta^2 + \nu\alpha) - 2n}.$$ 

For $n = 1$, it follows that $\mu = \kappa = 0$ and hence flat. For $n > 1$, making use of (35) in (46) provides

$$\left[4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu\alpha + (\beta^2 + \nu\alpha)(2(n-1) - n\mu)\right]\{Df - (\xi f)\xi\} + [2(\beta^2 + \nu\alpha)(n-1) + \mu((\beta^2 + \nu\alpha) - 2n)]hDf = 0.$$

By virtue of $\mu = \frac{2(\beta^2 + \nu\alpha)(n-1)}{2n-(\beta^2 + \nu\alpha)}$, the above equation entails that

$$\left[4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu\alpha + (\beta^2 + \nu\alpha)(2(n-1) - n\mu)\right]\{Df - (\xi f)\xi\} = 0.$$

If $Df - (\xi f)\xi = 0$, then proceeding as in Case (ii) it follows that, for $n > 1$, a contraction. Therefore, we only have $4n^2\kappa\nu\alpha - 2n\kappa\beta^2 - \nu\alpha + (\beta^2 + \nu\alpha)(2(n-1) - n\mu) = 0$. This together with (38) entails that

$$((2n-1)\nu\alpha - \beta^2)[2(1-n) + n(2\kappa + \mu)] = 0,$$

which implies that either $\beta^2 = (2n-1)\nu\alpha$, or $2(1-n) + n(2\kappa + \mu) = 0$. The former case shows that $\kappa > 1$, a contradiction. For later case, utilization of $\mu = \frac{2(\beta^2 + \nu\alpha)(n-1)}{2n-(\beta^2 + \nu\alpha)}$ and $\kappa = \frac{(\beta^2 + \nu\alpha)(n^2 - 1)}{n(\beta^2 + \nu\alpha) - 2n}$, the last equation transforms into

$$\beta^2 + \nu\alpha = \frac{2n - 2n^2}{n^3 - 2n^2 + 1}.$$

Making use of this in $\kappa = \frac{(\beta^2 + \nu\alpha)(n^2 - 1)}{n(\beta^2 + \nu\alpha) - 2n}$, we obtain $\kappa = 1$, and this leads to a contradiction as $M$ is non-Sasakian. This establishes the proof. \hfill $\square$

It is known [18] that a compact Riemannian manifold admitting a nontrivial gradient Einstein-type metric with constant scalar curvature is isometric to the standard sphere. But a contact metric manifold of constant curvature is a Sasakian manifold of constant curvature in dimension $> 3$ [20]. On the other hand, in dimension 3, it is either flat or Sasakian manifold of constant curvature 1 (see Blair [1]). From (38) we see that the scalar curvature of a $(\kappa, \mu)$-space is constant. Thus, for a compact $(\kappa, \mu)$-contact manifold we have the following:

**Corollary 4.2.** If a compact $(\kappa, \mu)$-contact manifold admits a gradient Einstein-type metric, then in dimension 3 it is either flat or Sasakian and for higher dimensions it is isometric to a unit sphere $S^{2n+1}$.

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