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GENERALIZED YANG'S CONJECTURE ON THE PERIODICITY OF ENTIRE FUNCTIONS

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Abstract. On the periodicity of transcendental entire functions, Yang's Conjecture is proposed in [6, 13]. In the paper, we mainly consider and obtain partial results on a general version of Yang's Conjecture, namely, if $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function. We also prove that if $f(z)^n + f^{(k)}(z)$ is a periodic function with additional assumptions, then $f(z)$ is also a periodic function, where n, k are positive integers.

1. Introduction and main results

Titchmarsh [12, p. 267] considered the real transcendental entire solutions of the differential equation

$$f(z)f^{(k)}(z) = p(z)\sin^2 z, \tag{1}$$

where $p(z)$ is a non-zero polynomial and obtained the following result. The real entire function $f(z)$ means that $f : \mathbb{R} \to \mathbb{R}$.

Theorem A. *The differential equation $f(z)f''(z) = -\sin^2 z$ has no real entire functions of finite order other than $f(z) = \pm \sin z$.*

Recently, Li, Lü and Yang [6, Theorem 1] considered Theorem A by removing the assumption that $f(z)$ is real and of finite order as follows.

Theorem B. *If $f(z)$ is an entire function satisfying $f(z)f''(z) = p(z)\sin^2 z$, where $p(z)$ is a non-zero polynomial with real coefficients and real zeros, then $p(z)$ must be a non-zero constant p and $f(z) = a\sin z$, where a is a constant satisfying $a^2 = -p$.*

Obviously, Theorem B generalizes Theorem A, that is $f(z)f''(z) = -\sin^2 z$ has entire solutions $f(z) = \pm \sin z$ and no other solutions exist. Remark that $-\sin^2 z$ is a periodic function and $\pm \sin z$ are also periodic functions with the

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different periods. An interesting conjecture below related to the periodicity of transcendental entire functions is proposed in [6], which is also mentioned in a former paper and called Yang's Conjecture [13, Conjecture 1.1].

Yang's Conjecture. *Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Some results on the periodicity of transcendental meromoprhic functions can be found in [1, 3–5, 11, 15]. Obviously, Yang's Conjecture is also related to the properties of transcendental entire solutions of complex differential-difference equation

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c),$$

provided that $f(z)f^{(k)}(z)$ is a periodic function with period c, where c is non-zero constant. However, it is well known that the complex differential-difference equations are difficult to solve, the simplest complex differential-difference equation $f'(z) = f(z+c)$ is not solved completely, where c is a non-zero constant, the partial results on the above equation can be found in [2, 7]. Until now, Yang's Conjecture has also not been proved completely. We summarize the partial results in the below remark.

Remark 1.1. (1) Yang's Conjecture is true for $k = 1$; it is proved by Wang and Hu [13, Theorem 1.1]. We recall an example given by Zhang and Yi [17, Corollary 1.7]. They obtained that all the entire solutions of

$$f(z)f'(z) = \frac{1}{2}\sin 2z$$

are $f(z) = \pm \mathrm{i} \cos z$ and $f(z) = \pm \sin z$. Obviously, $\pm\mathrm{i} \cos z$, $\pm \sin z$ and $\frac{1}{2}\sin 2z$ are periodic functions. Of course, $ff^{(k)} = \frac{1}{2}\sin 2z$ also admits entire solutions $f(z) = \pm\mathrm{i} \cos z$ and $f(z) = \pm \sin z$ when $k = 4m+1$ and m is a positive integer.

(2) Yang's Conjecture is true for the transcendental entire functions $f(z)$ with a non-zero Picard exceptional value, see Liu and Yu [9, Theorem 1.1].

(3) Yang's Conjecture is true for the finite order transcendental entire functions $f(z)$ with 0 as the Picard exceptional value, see Liu and Korhonen [8]. Actually, $f(z) = e^{h(z)}$ in this case, where $h(z)$ is a non-constant polynomial. The basic computations imply that $h(z)$ is a linear polynomial, and thus $f(z)$ is periodic.

(4) Recall that Rényi and Rényi [11] proved that if $f(z)$ is a non-constant entire function and $P(z)$ is a polynomial with $\deg(P(z)) \geq 3$, then $f(P(z))$ cannot be a periodic function. Hence, if $e^{h(z)}$ is a periodic function, then $\deg(h(z)) \leq 2$ follows immediately. In addition,

$$f(z) = e^{iz^3} + e^{z^3} = (e^{iz} + e^{z}) \circ (z^3).$$

Thus $e^{iz^3} + e^{z^3}$ is not a periodic function by Rényi and Rényi's result. However, if $f(z) = e^{H(z)} + Q(z)$ is a prime function, then Rényi and Rényi's result cannot be used, where $H(z)$ is an entire function and $Q(z)$ is a polynomial. Here, the prime function means that one of h and s must be linear for every factorization of $F(z) = h(s(z))$, h, s are entire functions. Ozawa [10, Theorem 1] showed that for any $\rho \in [1, +\infty)$ there exists a prime periodic entire function h of the order $\rho(h) = \rho$.

(5) Difference and differential-difference versions of Yang's Conjecture are also considered in Liu and Korhonen [8].

In the paper, the generalized Yang's Conjecture can be stated as follows. We give the first result based on the corresponding discussions in Remark 1.1.

Generalized Yang's Conjecture. *Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Theorem 1.2. *Let $f(z)$ be a transcendental entire function and n, k be positive integers. If one of the following conditions is satisfied*

(i) $k = 1$;

(ii) $f(z) = e^{h(z)}$, *where $h(z)$ is a non-constant polynomial*;

(iii) *$f(z)$ has a non-zero Picard exceptional value and $f(z)$ is of finite order, and if $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Remark 1.3. Remark that an assumption that $f(z)$ is of finite order is added in (iii). It remains open for us to remove this assumption.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f(z)^n f^{(k)}(z)$ and $f(z)^n f^{(k+1)}(z)$ are periodic functions with the same principal period c, then $f(z)$ is also a periodic function with period c, $2c$ or $(n+1)c$.*

Remark 1.5. If $f^{(k)}(z)$ and $f^{(l)}(z)$ are periodic entire functions, then the principal periods are the same, where k, l are positive integers. In fact, we assume that $f^{(k)}(z) = f^{(k)}(z+c)$ and $f^{(l)}(z) = f^{(l)}(z+b)$ for all $z \in \mathbb{C}$, where c, b are the principal periods for $f^{(k)}(z)$ and $f^{(l)}(z)$. Without loss of generalization, we assume that $k < l$. In this case,

$$f^{(l)}(z) = f^{(l)}(z+c) = f^{(l)}(z+b)$$

for all $z \in \mathbb{C}$. So $c = b$. However, it remains open for us that $f(z)^n f^{(k)}(z)$ and $f(z)^n f^{(l)}(z)$ have the same principal periods or not if they are periodic functions.

Finally, we consider the periodicity of $f(z)$ if the differential polynomial $f(z)^n + f^{(k)}(z)$ is a periodic function and obtain the following result.

Theorem 1.6. *Let $f(z)$ be a transcendental entire function and $n \geq 2$, k be a positive integer. If $f(z)^n + f^{(k)}(z)$ is a periodic function with period c and one of the following conditions is satisfied*

(i) $k = 1$;

(ii) *$f(z+c)-f(z)$ has no zeros*;

(iii) *the zeros multiplicity of $f(z+c)-f(z)$ is great than or equal to k; then $f(z)$ is also a periodic function with period c or $2c$.*

Remark 1.7. Theorem 1.6 is not true provided that $n=1$. We see that $f(z)=ze^{-z}$ is not a periodic function, but

$$f(z)+f^{(k)}(z)=(-1)^{k-1}ke^{-z}+(-1)^k ze^{-z}+ze^{-z}$$

is a periodic function provided that k is odd. Here, it is open for us that Theorem 1.6 is true for any positive integers k.

2. Lemmas

Lemma 2.1 ([14, Theorem 1])**.** *Let m,n be positive integers satisfying $\frac{1}{m}+\frac{1}{n}<1$. Then there are no non-constant entire solutions $f(z)$ and $g(z)$ that satisfy*

$$a(z)f(z)^n+b(z)g(z)^m=1,$$

where $a(z),b(z)$ are small functions with respect to $f(z)$.

Lemma 2.2 ([16, Theorem 1.62])**.** *Suppose that $f_j (j=1,2,\dots,n)$ $(n\geq 3)$ are meromorphic functions which are not constants except for f_n. Furthermore, let*

$$\sum_{j=1}^{n} f_j=1.$$

If $f_n\neq 0$ and

$$\sum_{j=1}^{n} N(r,\frac{1}{f_j})+(n-1)\sum_{j=1}^{n}\overline{N}(r,f_j)<(\lambda+o(1))T(r,f_k),$$

where $r\in I$, I is a set whose linear measure is infinite, $k\in\{1,2,\dots,n-1\}$ and $\lambda<1$, then $f_n\equiv 1$.

3. Proofs of Theorems

Proof of Theorem 1.2. We assume that $n\geq 2$ in the following proof. The case of $n=1$ has been showed in Remark 1.1.

Case (i). If $k=1$ and $f^n(z)f'(z)$ is a periodic function with period c, then

$$f^n(z)f'(z)=f^n(z+c)f'(z+c).$$

Integrating the above equation, we have

$$f(z+c)^{n+1}-f(z)^{n+1}=A, \tag{2}$$

where A is a constant. If $n\geq 2$, then $A\equiv 0$ from Lemma 2.1. Hence, $f(z+c)=tf(z)$ and $t^{n+1}=1$, thus $f(z)$ is a periodic function with period $(n+1)c$.

Case (ii). Since $f(z)^n f^{(k)}(z)$ is a periodic function with period c, we have

$$f(z)^n f^{(k)}(z)=f(z+c)^n f^{(k)}(z+c). \tag{3}$$

Substituting $f(z) = e^{h(z)}$ into (3), where $h(z)$ is a non-constant polynomial. We have

$$e^{(n+1)[h(z+c)-h(z)]} = \frac{H(z)}{H(z+c)},$$

where $H(z)$ is a differential polynomial of $h(z)$, hence $H(z)$ is also a polynomial in z. Since the rational function $\frac{H(z)}{H(z+c)}$ has neither zeros nor poles, we have $\frac{H(z)}{H(z+c)} \equiv 1$. Thus, we have $e^{(n+1)[h(z+c)-h(z)]} \equiv 1$, that is, $f(z)$ is a periodic function with period c or $(n+1)c$.

Case (iii). Assume that d is the non-zero Picard exceptional value of $f(z)$. Then $f(z) = e^{p(z)} + d$ follows by the Hadamard factorization theorem, where $p(z)$ is a non-constant polynomial. Substituting $f(z) = e^{p(z)} + d$ into (3), a basic computation implies that

$$P_1(z)(e^{p(z)} + d)^n e^{p(z)} = P_1(z+c)(e^{p(z+c)} + d)^n e^{p(z+c)}, \tag{4}$$

where $P_1(z)$ is a differential polynomial of $p(z)$. Furthermore, we obtain

$$\begin{aligned} &e^{(n+1)p(z)} + C_n^1 d e^{np(z)} + \cdots + C_n^{n-1} d^{n-1} e^{2p(z)} + d^n e^{p(z)} \\ &= H_1(z)\left[e^{(n+1)p(z+c)} + C_n^1 d e^{np(z+c)} + \cdots + d^n e^{p(z+c)}\right], \end{aligned} \tag{5}$$

where $H_1(z) = \frac{P_1(z+c)}{P_1(z)}$ and $T(r, H_1(z)) = S(r, f)$. We also obtain

$$\begin{aligned} &\frac{H_1(z)}{d^n} e^{(n+1)p(z+c)-p(z)} + \cdots + H_1(z) e^{p(z+c)-p(z)} \\ &- \frac{e^{np(z)}}{d^n} - \cdots - \frac{ne^{p(z)}}{d} = 1. \end{aligned} \tag{6}$$

Since $p(z)$ is a non-constant polynomial, then $mp(z+c) - p(z)$ $(m = 2, \ldots, n+1)$ cannot be constants other than $p(z+c) - p(z)$. From Lemma 2.2 and (6), we have

$$\frac{P_1(z+c)e^{p(z+c)-p(z)}}{P_1(z)} \equiv 1.$$

Thus, we have $p(z)$ is a linear polynomial. Furthermore, we have $P_1(z)$ and $P_1(z+c)$ are the same constants. Hence, we have $e^{p(z+c)} = e^{p(z)}$, so $f(z) = f(z+c)$. Thus, $f(z)$ is a periodic function with period c. □

Proof of Theorem 1.4. We assume that $f(z)^n f^{(k)}(z)$ and $f(z)^n f^{(k+1)}(z)$ are periodic functions with period c. Then

$$\begin{cases} f(z)^n f^{(k)}(z) = f(z+c)^n f^{(k)}(z+c), \\ f(z)^n f^{(k+1)}(z) = f(z+c)^n f^{(k+1)}(z+c). \end{cases} \tag{7}$$

Let $F(z) = f^{(k)}(z)$. Thus $F'(z) = f^{(k+1)}(z)$, $F(z+c) = f^{(k)}(z+c)$, $F'(z+c) = f^{(k+1)}(z+c)$. By (7), we have

$$\frac{F'(z+c)}{F(z+c)} = \frac{F'(z)}{F(z)}.$$

Integrating the above equation, we have

$$F(z+c) = e^A F(z), \tag{8}$$

that is

$$f^{(k)}(z+c) = e^A f^{(k)}(z), \tag{9}$$

where A is a constant. From the first equation of (7) and (9), we have

$$f(z+c)^n e^A = f(z)^n. \tag{10}$$

Case 1. If $n = 1$, then (10) means

$$f(z+c)e^A = f(z). \tag{11}$$

Differentiating the equation (11) k times, we have

$$f^{(k)}(z+c)e^A = f^{(k)}(z),$$

that is

$$F(z+c)e^A = F(z). \tag{12}$$

Thus $e^{2A} = 1$ follows by (8) and (12). The equation (11) implies that $f(z)$ is a periodic function with period c or $2c$.

Case 2. In $n \geq 2$, by integrating (9) k times, then

$$f(z+c) = e^A(f(z) + P(z)),$$

where $P(z)$ is a polynomial with degree less than k. Thus,

$$f(z+c)^n = e^{nA}(f(z) + P(z))^n. \tag{13}$$

We have $P(z) \equiv 0$ and $e^{(n+1)A} = 1$ by comparing (10) and (13). So $f(z)$ must be a periodic function with period $(n+1)c$ from (11). □

Proof of Theorem 1.6. We assume that $f(z)^n + f^{(k)}(z)$ is a periodic function with period c, then

$$f(z+c)^n + f^{(k)}(z+c) = f(z)^n + f^{(k)}(z).$$

Thus,

$$f(z+c)^n - f(z)^n = f^{(k)}(z) - f^{(k)}(z+c). \tag{14}$$

The above equation implies that either $f(z+c) - f(z) \equiv 0$, i.e., $f(z)$ is a periodic function with period c or $f(z+c) - f(z)$ has no zeros under one of the conditions (i), (ii), (iii) in Theorem 1.6. We assume that $f(z+c) - f(z) = e^{h(z)}$ from the Hadamard factorization theorem, where $h(z)$ is an entire function.

Case 1. If $n = 2$, then $f(z+c) + f(z) = H(z)$ follows (14), where $H(z)$ is a differential polynomial of $h(z)$ and $T(r, H(z)) = S(r, e^{h(z)})$. Hence, we have

$$\begin{cases} f(z) = \dfrac{H(z) - e^{h(z)}}{2}, \\ f(z+c) = \dfrac{H(z) + e^{h(z)}}{2} = \dfrac{H(z+c) - e^{h(z+c)}}{2}. \end{cases} \tag{15}$$

The above system implies that

$$\begin{aligned}T(r,f(z)) &= T(r,f(z+c)) + S(r,e^{h(z)})\\ &= T(r,e^{h(z)}) + S(r,e^{h(z)}) = T(r,e^{h(z+c)}) + S(r,e^{h(z+c)}).\end{aligned}$$

Thus, we have $T(r,e^{h(z)}) = O(T(r,e^{h(z+c)}))$. Hence,

$$T(r,H(z+c)) = S(r,e^{h(z)}).$$

Furthermore, (15) also implies that $H(z+c) - H(z) = e^{h(z+c)} + e^{h(z)}$ and

$$T(r,e^{h(z+c)}) = T(r,e^{h(z)}) + S(r,e^{h(z)}).$$

If $H(z+c) - H(z) \not\equiv 0$, then

$$\begin{aligned}T(r,e^{h(z)}) &\le N\left(r,\frac{1}{e^{h(z)}}\right) + N\left(r,\frac{1}{e^{h(z)} - H(z+c) + H(z)}\right) + S(r,e^{h(z)})\\ &\le N\left(r,\frac{1}{e^{h(z+c)}}\right) + S(r,e^{h(z)})\\ &\le S(r,e^{h(z)}),\end{aligned}$$

from the second main theorem of Nevanlinna theory. Hence, $H(z+c) \equiv H(z)$ and $e^{h(z+c)} = -e^{h(z)}$ follows. Thus, we have $f(z)$ is a periodic function with period $2c$ from (15).

Case 2. If $n \ge 3$, from (14), then we also obtain

$$\begin{cases} f(z+c) - f(z) = e^{h(z)},\\ f(z+c)^{n-1} + f(z+c)^{n-2}f(z) + \cdots + f(z)^{n-1} = H(z), \end{cases} \tag{16}$$

where $H(z)$ is a differential polynomial of $h(z)$ and $T(r,H(z)) = S(r,e^{h(z)})$. The equation (16) means

$$\begin{cases} f(z)\left[\dfrac{f(z+c)}{f(z)} - 1\right] = e^{h(z)},\\ f(z)^{n-1}\left[\dfrac{f(z+c)^{n-1}}{f(z)^{n-1}} + \dfrac{f(z+c)^{n-2}}{f(z)^{n-2}} + \cdots + \dfrac{f(z+c)}{f(z)} + 1\right] = H(z). \end{cases} \tag{17}$$

Let $M(z) = \frac{f(z+c)}{f(z)}$. Obviously, if $M(z) \equiv 1$, then $f(z)$ is a periodic function with period c. Suppose that $M(z) \not\equiv 1$ and $M(z)$ is not a constant. The equation (17) implies that

$$\frac{(M(z)-1)^{n-1}}{M(z)^{n-1} + M(z)^{n-2} + \cdots + M(z) + 1} = \frac{e^{(n-1)h(z)}}{H(z)}. \tag{18}$$

From (18), we obtain

$$(n-1)T(r,M(z)) = (n-1)T(r,e^{h(z)}) + S(r,e^{h(z)}),$$

$$N\left(r,\frac{1}{M(z)^{n-1} + M(z)^{n-2} + \cdots + M(z) + 1}\right) = S(r,e^{h(z)}),$$

and $M-1$ has no zeros. Using the second main theorem of Nevanlinna theory, we have

$$(n-2)T(r,M) \leq N\left(r, \frac{1}{M-1}\right) + N\left(r, \frac{1}{M^{n-1}+M^{n-2}+\cdots+M+1}\right) + S(r,M)$$
$$\leq S(r,e^{h(z)}) + S(r,M) = S(r,M),$$

which is impossible for $n \geq 3$. Hence, we have $M(z)$ must be a constant and $M(z) \not\equiv 1$. The first equation of (17) means

$$T(r,f) = T(r,e^{h(z)}),$$

the second equation of (17) means

$$(n-1)T(r,f) = T(r,H(z)) = S(r,e^{h(z)}) = S(r,f),$$

which leads to a contradiction. □

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