Abstract. In this paper we consider $L_k$-conjecture introduced in [5, 6] for hypersurface $M^n$ in space form $\mathbb{R}^{n+1}(c)$ with three principal curvatures. When $c = 0, -1$, we show that every $L_1$-biharmonic hypersurface with three principal curvatures and $H_1$ is constant, has $H_2 = 0$ and at least one of the multiplicities of principal curvatures is one, where $H_1$ and $H_2$ are first and second mean curvature of $M$ and we show that there is not $L_2$-biharmonic hypersurface with three disjoint principal curvatures and, $H_1$ and $H_2$ is constant. For $c = 1$, by considering having three principal curvatures, we classify $L_1$-biharmonic hypersurfaces with multiplicities greater than one, $H_1$ is constant and $H_2 = 0$, proper $L_1$-biharmonic hypersurfaces which $H_1$ is constant, and $L_2$-biharmonic hypersurfaces which $H_1$ and $H_2$ is constant.

1. Introduction and statement of result

B. Y. Chen in [12] made the conjecture: Any biharmonic submanifold of a Euclidean space is minimal. Several authors have proved it under some conditions, see for example, [1, 14–17, 20]. Also this conjecture has been generalized in [9]: Any biharmonic submanifold of a Riemannian manifold of non-positive sectional curvature is minimal. This generalized conjecture has been proved for constant sectional curvature ambient spaces in numerous cases as in [2, 7, 9, 19, 24, 25]. The Generalized Chen conjecture has been shown to be false by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space of non-constant negative sectional curvature in [26]. In case of positive sectional curvature ambient spaces, there are several families of biharmonic submanifolds which are not minimal. For example in [8], the authors classified proper biharmonic hypersurfaces in the unit Euclidean sphere with at most two distinct principal curvatures.

Let $\varphi : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion from a connected oriented Riemannian manifold into the Euclidean space $\mathbb{R}^{n+1}$ with $N$ as the unit normal.
direction. We have, [3],

\[ L_k \varphi = (k + 1) \left( \binom{n}{k+1} H_{k+1} N \right), \]

where \( k = 0, \ldots, n - 1 \) and \( H_{k+1} \) is \((k + 1)\)-th mean curvature of \( M \). When \( k = 0 \), the above equation reduces to \( \Delta \varphi = nH_1 N = n\tilde{H} \) which is the Beltrami equation. In [5], we proposed the \( L_k \)-conjecture: Every Euclidean hypersurface \( \varphi : M^n \to \mathbb{R}^{n+1} \) satisfying the condition \( L_k^2 \varphi = 0 \) for some \( k \), \( 0 \leq k \leq n - 1 \), has zero \((k + 1)\)-th mean curvature, namely it is \( k \)-minimal. We have proved that the \( L_k \)-conjecture is true for Euclidean hypersurfaces with at most two principal curvatures, [5]. Hereafter in [6], we have generalized the notions of tension and bitension fields to introduce \( L_k \)-harmonic and \( L_k \)-biharmonic maps.

Let \( M \) be a connected, oriented isometrically immersed Riemannian hypersurface in a simply connected space form \( \mathbb{R}^{n+1}(c) \), \( c = 0, \pm 1 \). Then \( M \) is called an \( L_k \)-biharmonic hypersurface if the following equations are satisfied:

1. \[ \left( \binom{n}{k+1} H_{k+1} \right) \nabla H_{k+1} + 2(S \circ P_k)(\nabla H_{k+1}) = 0, \]
2. \[ L_k H_{k+1} - \left( \binom{n}{k+1} H_{k+1} \right) (nH_1 H_{k+1} - (n - k - 1)H_{k+2} - c(k+1)H_k) = 0. \]

In addition \( M \) is called a proper \( L_k \)-biharmonic hypersurface if \( M \) is an \( L_k \)-biharmonic hypersurface and \( H_{k+1} \neq 0 \).

**L\_k-conjecture 1.1 ([6]).** Let \( \varphi : M^n \to \mathbb{R}^{n+1}(c) \), \( c = 0, \pm 1 \), be a connected oriented hypersurface immersed into a simply connected space form \( \mathbb{R}^{n+1}(c) \). If \( M \) is an \( L_k \)-biharmonic hypersurface, then \( H_{k+1} \) is zero.

The \( L_k \)-conjecture has been proved in some cases. For \( c = 0, -1 \), the \( L_k \)-conjecture is proved as hypersurface \( M \) has two principal curvatures, or \( M \) is weakly convex, or \( M \) is complete with some constraint on it and on \( L_k \), and it is shown that there is not any \( L_k \)-biharmonic hypersurface \( M^n \) in \( \mathbb{H}^{n+1} \) with two principal curvatures of multiplicities greater than one, [6].

In this paper we consider \( L_k \)-conjecture for hypersurface \( M^n \) in space form \( \mathbb{R}^{n+1}(c) \) with three principal curvatures. When \( c = 0, -1 \), in Theorem 1.2, we show that every \( L_1 \)-biharmonic hypersurface with three principal curvatures and \( H_1 \) is constant, has \( H_2 = 0 \) and at least one of the multiplicities of principal curvatures is one, and we show that there is not \( L_2 \)-biharmonic hypersurface with three disjoint principal curvatures and \( H_1 \) and \( H_2 \) is constant. Recently, in [22] for the case \( c = 0 \), the authors prove that the \( L_1 \)-conjecture is true for \( L_1 \)-biharmonic hypersurfaces with three distinct principal curvatures and constant mean curvature of a Euclidean space, meanwhile in our paper we give more result in this case and also we consider \( L_2 \)-conjecture and we give some classification for cases \( c = 0, 1, -1 \) which are completely different.

For the case \( c = 1 \), the \( L_k \)-conjecture is false by considering hypersurface \( \mathbb{S}^n(\frac{n}{2}) \) in the \( n \)-dimensional unit Euclidean sphere \( \mathbb{S}^n \), so \( \mathbb{S}^n(\frac{n}{2}) \) is a proper
$L_4$-biharmonic hypersurface. This result has been extended to hypersurfaces having two distinct principal curvatures and it’s shown that they are open pieces of the standard products of spheres, [4].

For $c = 1$, in Theorem 1.2, by considering hypersurfaces having three principal curvatures in the unit Euclidean sphere, we classify $L_1$-biharmonic hypersurfaces with multiplicities greater than one, $H_1$ is constant and $H_2 = 0$, proper $L_1$-biharmonic hypersurfaces which $H_1$ is constant, and $L_2$-biharmonic hypersurfaces which $H_1$ and $H_2$ is constants.

**Theorem 1.2.** Let $M^n$ be a connected, oriented isometrically immersed hypersurface in space form $\mathbb{R}^{n+1}(c)$. Suppose that $M$ has three distinct principal curvatures and $H_1, \ldots, H_k$ are constant. Let $c = 0, -1$. If $k = 1$ and $M$ is $L_1$-biharmonic, then $H_2 = 0$ and at least one of the multiplicities of principal curvatures is one. If $k = 2$, then $M$ is not $L_2$-biharmonic. Let $c = 1$. If $k = 1$ and $M$ is $L_1$-biharmonic, then $H_2$ is constant, and if $H_2 = 0$ and multiplicities of principal curvatures are greater than one, and or $M$ is proper $L_1$-biharmonic, then $M$ is an isoparametric hypersurface. If $k = 2$ and $M$ is $L_2$-biharmonic, then $M$ is an isoparametric hypersurface.

Assume that $k_1 > k_2 > k_3$ denote the principal curvatures of an isoparametric hypersurface in the unit Euclidean Sphere $S^{n+1}$. Then multiplicities of principal curvatures is equal, say $m$, $m$ is either $1, 2, 4$ or $8$, and $k_2 = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3} k_1}, k_3 = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3} k_1}$, and there is a homogeneous polynomial $F$ of degree $3$ over $\mathbb{R}^{n+2}$ where for any $a \in (-1, 1)$, $f^{-1}(a) = F^{-1}_{|\mathbb{R}^{n+1}}(a)$ is an isoparametric hypersurface (see Theorem 2.1).

(a) Let $k = 1$ and $M$ be $L_1$-biharmonic, $H_2 = 0$ and the multiplicities of principal curvatures be greater than one. Then we have the followings:

- If $m = 2$, then $k_1, k_2, k_3$ approximately are $k_1 \approx 3.286, k_2 \approx 0.232, k_3 \approx -1.069$ or $k_1 \approx 1.069, k_2 \approx -0.232, k_3 \approx -3.286$. So $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a complex projective plane $\mathbb{C}P^2$ into $S^7$ where $a \approx 0.632$ and $\theta \approx \pi/10.634$.

- If $m = 4$, then $k_1, k_2, k_3$ approximately are $k_1 \approx 2.527, k_2 \approx 0.147, k_3 \approx -1.261$, or $k_1 \approx 1.261, k_2 \approx -0.147, k_3 \approx -2.527$. So $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a quaternionic projective plane $\mathbb{H}P^2$ into $S^{13}$ where $a \approx 0.426$ and $\theta \approx \pi/8.337$.

(b) Let $k = 1$ and $M$ be proper $L_1$-biharmonic. Then we have the followings:
• If $m = 1$, then $k_1, k_2, k_3$ satisfy the following equation

$$3H_1H_2 - H_3 - 2H_1 = 0,$$

so that $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. Therefore $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a real projective plane $\mathbb{RP}^2$ into $S^4$. Also $M$ is a Cartan minimal hypersurface of dimension 3.

• If $m = 2$, then $k_1, k_2, k_3$ satisfy the following equation

$$6H_1H_2 - 4H_3 - 2H_1 = 0,$$

so that either $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or approximately $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ or $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$. If $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$, then $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a complex projective plane $\mathbb{CP}^2$ into $S^7$. Also $M$ is a Cartan minimal hypersurface of dimension 6. If $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ or $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$, then $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a complex projective plane $\mathbb{CP}^2$ into $S^7$ where $a \approx 0.316$ and $\theta \approx \pi/7.544$.

• If $m = 4$, then $k_1, k_2, k_3$ satisfy the following equation

$$12H_1H_2 - 10H_3 - 2H_1 = 0,$$

so that $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. Therefore $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a quaternionic projective plane $\mathbb{HP}^2$ into $S^{13}$. Also $M$ is a Cartan minimal hypersurface of dimension 12.

• If $m = 8$, then $k_1, k_2, k_3$ satisfy the following equation

$$24H_1H_2 - 22H_3 - 2H_1 = 0,$$

so that $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. Therefore $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a Cayley projective plane $\mathbb{OP}^2$ into $S^{25}$. Also $M$ is a Cartan minimal hypersurface of dimension 24.

(c) Let $k = 2$ and $M$ be $L_2$-biharmonic and $H_3 = 0$. Then we have the following:

• If $m = 1$, then $k_1, k_2, k_3$ are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. So $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a real projective plane $\mathbb{RP}^2$ into $S^4$. Also $M$ is a Cartan minimal hypersurface of dimension 3.

• If $m = 2$, then $k_1, k_2, k_3$ are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. So $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a complex projective plane $\mathbb{CP}^2$ into $S^7$. Therefore $M$ is a Cartan minimal hypersurface of dimension 6. If $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ or $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$, then $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a complex projective plane $\mathbb{CP}^2$ into $S^7$ where $a \approx 0.316$ and $\theta \approx \pi/7.544$.
into $S^7$. Also $M$ is a Cartan minimal hypersurface of dimension 6.

- If $m = 4$, then $k_1, k_2, k_3$ are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or approximately $k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777$ or $k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993$. If $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$, then $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a quaternionic projective plane $\mathbb{HP}^2$ into $S^{13}$. Also $M$ is a Cartan minimal hypersurface of dimension 12. If $k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777$ or $k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993$, then $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a quaternionic projective plane $\mathbb{HP}^2$ into $S^{13}$ where $a \approx 0.713$ and $\theta \approx \pi/12.138$.

- If $m = 8$, then $k_1, k_2, k_3$ are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or approximately $k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757$ or $k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189$. If $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$, then $M$ is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a Cayley projective plane $\mathbb{OP}^2$ into $S^{25}$. Also $M$ is a Cartan minimal hypersurface of dimension 24. If $k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757$ or $k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189$, then $M$ is congruent to an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a Cayley projective plane $\mathbb{OP}^2$ into $S^{25}$ where $a \approx 0.502$ and $\theta \approx \pi/9.028$.

(d) Let $k = 2$ and $M$ be proper $L_2$-biharmonic. Then we have the followings:

- If $m = 1$, then $k_1, k_2, k_3$ satisfy the equation

$$H_1 H_3 - H_2 = 0,$$

so that either $k_1 = 1, k_2 = \sqrt{3} - 2, k_3 = -\sqrt{3} - 2$ or $k_1 = 2 + \sqrt{3}, k_2 = 2 - \sqrt{3}, k_3 = -1$. Therefore $M$ is congruent to an open part of $f^{-1}(\frac{\sqrt{2}}{2})$ and a tube of radius $\pi/12$ around the standard embedding of a real projective plane $\mathbb{RP}^2$ into $S^4$.

- If $m = 2$, then $k_1, k_2, k_3$ satisfy the following equation

$$2H_1 H_3 - H_4 - H_2 = 0,$$

so that there is no real solution for all $k_1, k_2, k_3$. Therefore there is no proper $L_2$-biharmonic hypersurface in $S^7$ with three disjoint principal curvatures, and $H_1$ and $H_2$ are constants.

- If $m = 4$, then $k_1, k_2, k_3$ satisfy the equation

$$4H_1 H_3 - 3H_4 - H_2 = 0,$$

so that approximately either $k_1 \approx 1.083, k_2 \approx -0.225, k_3 \approx -3.213$ or $k_1 \approx 3.213, k_2 \approx 0.225, k_3 \approx -1.083$. Then $M$ is congruent to
an open part of $f^{-1}(a)$ and a tube of radius $\theta$ around the standard embedding of a quaternionic projective plane $\mathbb{H}P^2$ into $S^{13}$ where $a \approx 0.617$ and $\theta \approx \pi/10.411$.

- If $m = 8$, then $k_1, k_2, k_3$ satisfy the following equation
  \[ 8H_1H_3 - 7H_4 - H_2 = 0, \]
  so that there is no real solution for all $k_1, k_2, k_3$. Therefore there is no proper $L_2$-biharmonic hypersurface in $S^{25}$ with three disjoint principal curvatures, and $H_1$ and $H_2$ are constants.

An immediate result of Theorem 1.2, we get the following classification of proper $L_2$-biharmonic hypersurfaces in space form $R^{n+1}(c)$ with three distinct principal curvatures and $H_2$ is constant.

**Theorem 1.3.** Let $M^3$ be a connected, oriented isometrically immersed hypersurface in space form $R^{n+1}(c)$ with three distinct principal curvatures. If $M$ is proper $L_2$-biharmonic and $H_2$ is constant, then $c = 1$ and $M$ is congruent to an open part of $f^{-1}(\frac{\sqrt{3}}{2})$ and a tube of radius $\pi/12$ around the standard embedding of a real projective plane $\mathbb{RP}^2$ into $S^4$ and principal curvatures of $M$ are $2 + \sqrt{3}, 2 - \sqrt{3}, -1$.

### 2. Preliminaries

We recall the prerequisites from [3, 10, 11, 13, 23, 27]. Let $R^{n+1}(c)$ be the simply connected Riemannian space form of constant sectional curvature $c$ which is the Euclidean space $R^{n+1}$ for $c = 0$, and the Hyperbolic space $H^{n+1}$, for $c = -1$, and the Euclidean sphere $S^{n+1}$ for $c = +1$. Let $\varphi : M^n \rightarrow R^{n+1}(c)$ be a connected oriented hypersurface isometrically immersed into $R^{n+1}(c)$ with $N$ as a unit normal vector field, $\nabla$ and $\overline{\nabla}$ the Levi-Civita connections on $M$ and $R^{n+1}(c)$, respectively. For simplicity we also denote the induced connection on the pullback bundle $\varphi^{*}TR^{n+1}(c)$ by $\overline{\nabla}$. Let $X, Y$ be vector fields on $M$. We have the following formula for the shape operator of $M$,

\[
\overline{\nabla}_X d\varphi(Y) = d\varphi(\nabla_X Y) + (SX, Y) N, \\
d\varphi(SX) = -\overline{\nabla}_X N.
\]

As it is known, the shape operator is a self-adjoint linear operator. Let $k_1, \ldots, k_n$ be its eigenvalues which are called principal curvatures of $M$. Define $s_0 = 1$ and

\[
s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} k_{i_1} \cdots k_{i_k}.
\]

The $k$-th mean curvature of $M$ is defined by

\[
\left(\begin{array}{c} n \\ k \end{array} \right) H_k = s_k.
\]
For $k = 1$, $H_1 = \frac{1}{n} \text{tr}(S) = H$ is the mean curvature of $M$. For $k = 2$, the scalar curvature of $M$ is $s = n(n-1)H_2$. In general, when $k$ is odd, the sign of $H_k$ depends on the chosen orientation and when $k$ is even, $H_k$ is an intrinsic geometric quantity.

Let $M^n$ have three principal curvatures, $k_1, k_2, k_3$ with respective multiplicities $m_1, m_2, m_3$, $n = m_1 + m_2 + m_3$. Therefore we get by Equation (3),

$$s_k = \sum_{i,j} \binom{m_1}{i} \binom{m_2}{j} \binom{m_3}{k-i-j} k_1^{i} k_2^{j} k_3^{k-i-j}. $$

The Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - S \circ P_{k-1}, \quad 1 \leq k \leq n.$$ 

Therefore

$$P_k = \sum_{l=0}^{k} (-1)^l s_{k-l} S^l. $$

From the Cayley-Hamilton theorem, one gets that $P_n = 0$. Each $P_k$ is a self adjoint linear operator which commutes with $S$ and the eigenvalues of $P_k$ are given by

$$\mu_{k,i} = \sum_{1 \leq i_1 < \cdots < i_k \leq n, i_j \neq i} k_{i_1} \cdots k_{i_k}. $$

For $0 \leq k \leq n-1$, the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ as the natural generalization of the Laplace operator for Euclidean hypersurfaces $M$, is defined by

$$L_k f = \text{tr}(P_k \circ \nabla^2 f),$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of $f$ and is defined by $
abla^2 f = (\langle \nabla^2 f, Y \rangle \cdot X) = \langle \nabla_X (\nabla f), Y \rangle$ for all vector fields $X, Y$ on $M$, and $\nabla f$ is the gradient vector field of $f$. When $k = 0$, $L_0 = \Delta$.

We have the following properties of shape operator, curvature tensor and Newton transformation which they are used to prove other results of the paper.

If $X, Y, Z$ are tangent vector fields on $M$, then we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$= c (\langle Z, Y \rangle X - \langle Z, X \rangle Y) + \langle SY, Z \rangle SX - \langle SX, Z \rangle SY, $$

$$\langle \nabla_X S \rangle Y = \langle \nabla_Y S \rangle X, \quad \text{(Codazzi equation)}$$

$$\text{tr}(P_k) = (n-k)s_k. $$

We recall that a hypersurface $M^n$ in $R^{n+1} \cdot c$ is said to be isoparametric if it has constant principal curvatures $k_1 > k_2 > \cdots > k_l$ with respective constant multiplicities $m_1, m_2, \ldots, m_l$, $n = m_1 + m_2 + \cdots + m_l$. It is known for $c = 0, -1$, isoparametric hypersurfaces has at most two principal curvatures. For $l = 3$
we have the following classification of isoparametric hypersurfaces in Euclidean sphere.

**Theorem 2.1** (cf. [10, 11, 23]). Let \( M^n \) be an isoparametric hypersurface in \( S^{n+1} \) with three constant principal curvatures \( k_1 > k_2 > k_3 \) and respective multiplicities \( m_1, m_2, m_3 \). Then we have the followings:

I. \( m = m_1 = m_2 = m_3 = 2^q, n = 3 \cdot 2^q, q = 0, 1, 2, 3, \) and there exists an angle \( \theta, 0 < \theta < \pi/3 \) such that

\[
(10) \quad k_1 = \cot \theta, \quad k_2 = \cot(\theta + \pi/3) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \quad k_3 = \cot(\theta + 2\pi/3) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.
\]

II. In the ambient Euclidean space \( \mathbb{R}^{n+2} \supset S^{n+1} \), there is a homogeneous polynomial \( F \) of degree 3 over \( \mathbb{R}^{n+2} \) whose the range of \( f = F|_{S^{n+1}} \) is \([-1, 1] \), the only critical values of \( f \) are \( \pm 1 \) and for any \( a \in (-1, 1) \), \( f^{-1}(a) \) is an isoparametric hypersurface and is a tube around the two focal submanifolds \( f^{-1}(1) \) and \( f^{-1}(-1) \). For \( a = \cos(3\theta) \), \( M \) is up to congruency an open part of \( f^{-1}(a) \) and a tube of radius \( \theta \) around the two focal submanifolds.

III. The two focal submanifolds are standard embedding of a projective plane \( \mathbb{P}^2 \) into \( S^{n+1} \) where \( F \) is the division algebra \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) (quaternions), \( \mathbb{O} \) (Cayley numbers) corresponding to the principal multiplicity \( m = 1, 2, 4, \) or \( 8 \).

IV. Let \( F \) be one of the division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \). Let \( X, Y, Z \in F \) and \( a, b \in \mathbb{R} \). Then

\[
F = a^3 - 3ab^2 + \frac{3a}{2}(X \overline{X} + Y \overline{Y} - 2Z \overline{Z}) + \frac{3\sqrt{3}b}{2}(X \overline{Y} - Y \overline{X}) + \frac{3\sqrt{3}}{2}(X Y Z + Z \overline{X} Y).
\]

Isoparametric hypersurfaces with three distinct principal curvatures are usually called Cartan hypersurfaces. When a Cartan hypersurface in \( S^{n+1} \) is minimal, it is congruent to one of the following hypersurfaces:

\[
\begin{align*}
M^3 &= SO(3)/(Z_2 + Z_2) \to S^4 \\
M^6 &= SU(3)/T^2 \to S^7 \\
M^{12} &= Sp(3)/(Sp(1) \times Sp(1) \times Sp(1)) \to S^{13} \\
M^{24} &= F_4/Spin(8) \to S^{25}
\end{align*}
\]

Principal curvatures of a Cartan minimal hypersurface are \( \sqrt{3}, 0, -\sqrt{3} \).

3. Proof of main result

Before proving Theorem 1.2, we give an auxiliary Lemma for \( L_k \)-biharmonic hypersurface \( M \) in space form \( R^{n+1}(c) \) which has three distinct principal curvatures and we show \( H_k+c \) is constant when \( k = 1 \) or 2 and \( H_1, \ldots, H_k \) are constant. In its proof, we benefit from the techniques of [17–19, 21] but adapt them to our context. So our proof is much involved and quite different.
Lemma 3.1. Let $M^n$ be a connected, oriented isometrically immersed $L_k$-biharmonic hypersurface in space form $R^{n+1}(c)$. Suppose that $M$ has three distinct principal curvatures and $k = 1$ or $2$. If $H_1, \ldots, H_k$ are constant, then $H_{k+1}$ is constant.

**Proof.** We have $P_{k+1} = s_{k+1}I - S \circ P_k$. So by Equation (1) we get

$$P_{k+1} \nabla s_{k+1} = \frac{3}{2}s_{k+1} \nabla s_{k+1}. \tag{11}$$

Let $s_{k+1}$ be non constant. We consider $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on $M$ which diagonalize $S$ and $P_{k+1}$ simultaneously and $e_1 = \frac{\nabla s_{k+1}}{|\nabla s_{k+1}|}$. We put

$$Se_i = \lambda_i e_i \text{ and } P_{k+1}e_i = \mu_{k+1,i} e_i, \quad i = 1, \ldots, n. \tag{12}$$

Then we have by Equations (11) and (12),

$$\mu_{k+1,1} = \frac{3}{2}s_{k+1}. \tag{13}$$

So we get by Equations (5) and (13),

$$\frac{3}{2}s_{k+1} = \sum_{l=0}^{k+1} (-1)^l s_{k+1-l}\lambda_1^l = s_{k+1} + \sum_{l=1}^{k+1} (-1)^l s_{k+1-l}\lambda_1^l. \tag{14}$$

Therefore

$$s_{k+1} = 2 \sum_{l=1}^{k+1} (-1)^l s_{k+1-l}\lambda_1^l. \tag{15}$$

We have $\nabla s_{k+1} = \sum_{i=1}^n e_i(s_{k+1}) e_i = |\nabla s_{k+1}| e_1$. Thus

$$e_1(s_{k+1}) \neq 0 \quad \text{and} \quad \forall i \neq 1 e_i(s_{k+1}) = 0. \tag{16}$$

By assumption $s_1, \ldots, s_k$ are constant, so by Equation (14) we get for every $i$,

$$e_i(s_{k+1}) = 2e_i(\lambda_1) \sum_{l=1}^{k+1} (-1)^l s_{k+1-l}\lambda_1^{l-1}. \tag{17}$$

Since $e_1(s_{k+1}) \neq 0$, by Equation (16) we have $e_1(\lambda_1) \neq 0$. If $e_1(\lambda_1) \neq 0$ for some $i \neq 1$, then $e_i(s_{k+1}) = 0$ and Equation (16) imply that $\sum_{l=1}^{k+1} (-1)^l s_{k+1-l}\lambda_1^{l-1} = 0$. So this polynomial shows that $\lambda_1$ is constant which is a contradiction with $e_1(\lambda_1) \neq 0$. Thus $\lambda_1$ is non constant,

$$e_1(\lambda_1) \neq 0 \quad \text{and} \quad \forall i \neq 1 e_i(\lambda_1) = 0. \tag{18}$$

Now we show that multiplicity of $\lambda_1$ is one. Let’s $\nabla_{e_i} e_j = \sum_{l} \omega^l_{ij} e_l$. Then $\nabla_{e_i}(e_i, e_j) = 0$ and the Codazzi equation $(\nabla_{e_i} S) e_j = (\nabla_{e_j} S) e_i$ give that

$$\omega^l_{ij} = -\omega^l_{ji}, \tag{19}$$

$$e_1(\lambda_j) = (\lambda_i - \lambda_j) \omega^l_{ij} \quad i \neq j, \tag{20}$$

$$(\lambda_i - \lambda_j) \omega^l_{ii} = (\lambda_i - \lambda_j) \omega^l_{ll} \quad i \neq j \neq l.$$
If $\lambda_1 = \lambda_j$ for some $j \neq 1$, then by Equation (19) we get $e_1(\lambda_1) = e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^1 = 0$ which is a contradiction with Equation (17). By assumption $M$ has three distinct principal curvatures. Without loss of generality, We denote them by

(21) \[ \lambda_1, \lambda_2 = \ldots = \lambda_p = \alpha_2, \lambda_{p+1} = \ldots = \lambda_n = \beta_n. \]

Let's $n \geq 4$ and $p = n - 1$ (for $n = 3$ or $p \leq n - 2$, the proof is in similar way).

By Equations (19) and (21) we have

(22) \[ e_2(\alpha_2) = \ldots = e_{n-1}(\alpha_2) = 0. \]

In the following we show that $e_n(\alpha_2) = 0$. We have by Equation (19), for $i \neq 1$, $e_i(\lambda_1) = (\lambda_i - \lambda_1)\omega_{1i}^1 = 0$. So

(23) \[ \omega_{1i}^1 = 0, \quad i = 1, \ldots, n. \]

We know by Equation (21),

(24) \[ \beta_n = s_1 = \lambda_1 - (n - 2)\alpha_2. \]

Thus by Equations (17), (22) and (24) for $i = 2, \ldots, n - 1$, $e_1(\beta_n) = 0$, and by Equations (19) and (21), $e_i(\beta_n) = e_i(\lambda_n) = (\lambda_i - \lambda_n)\omega_{ni}^n = 0$. So

(25) \[ \omega_{ni}^n = 0, \quad i = 2, \ldots, n. \]

By Equations (19) and (21), $\omega_{ni}^n = \frac{e_i(\lambda_n)}{\lambda_i - \lambda_n} = \frac{e_i(\beta_n)}{\lambda_1 - \lambda_n}$ and so by Equation (24)

(26) \[ \omega_{ni}^n = -\frac{e_1(\lambda_1) + (n - 2)\alpha_2)}{2\lambda_1 + (n - 2)\alpha_2 - s_1}. \]

By Equation (19), for $j = 2, \ldots, n - 1$, we have $\omega_{j1}^j = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}$ and $\omega_{jn}^j = \frac{e_n(\lambda_j)}{\lambda_n - \lambda_j}$. So by Equation (21) we get

(27) \[ \omega_{j1}^j = \frac{e_1(\alpha_2)}{\lambda_1 - \alpha_2}, \quad j = 2, \ldots, n - 1, \]

(28) \[ \omega_{jn}^j = \frac{e_n(\alpha_2)}{s_1 - \lambda_1 - (n - 2)\alpha_2}, \quad j = 2, \ldots, n - 1. \]

For $j \neq l$ and $j, l = 2, \ldots, n - 1$, we have by Equation (20), $(\lambda_1 - \lambda_j)\omega_{j1}^j = (\lambda_1 - \lambda_j)\omega_{11}^j = 0$ and $(\lambda_n - \lambda_j)\omega_{jn}^j = (\lambda_n - \lambda_j)\omega_{n1}^n = 0$. Thus

(29) \[ \omega_{j1}^j = \omega_{lj}^l = 0, \quad j \neq l \text{ and } j, l = 2, \ldots, n - 1. \]

For $i, j = 2, \ldots, n$, by Equation (17) we get, $[e_i, e_j](\lambda_1) = e_i e_j (\lambda_1) - e_j e_i (\lambda_1) = 0$ and so $[e_i, e_j](\lambda_1) = \sum_l (\omega_{ij}^l - \omega_{ji}^l)e_l(\lambda_1) = (\omega_{ij}^j - \omega_{ji}^j)e_1(\lambda_1) = 0$. Therefore

(30) \[ \omega_{ij}^1 = \omega_{ji}^1, \quad i, j = 2, \ldots, n. \]

For $l = 2, \ldots, n - 1$, by Equation (20), $(\lambda_n - \lambda_1)\omega_{ln}^n = (\lambda_n - \lambda_1)\omega_{nl}^n$ and $(\lambda_1 - \lambda_n)\omega_{ln}^l = (\lambda_1 - \lambda_n)\omega_{nl}^n$. Therefore by Equations (18) and (30) we get

(31) \[ \omega_{ln}^l = \omega_{nl}^n = \omega_{ln}^n = 0, \quad l = 2, \ldots, n - 1. \]
By Equations (18), (23) and (31) we get

\[ \nabla e_i e_1 = \nabla e_1 e_n = 0. \]

We have \( \nabla e_i e_1 = \sum_j \omega_{ij}^1 e_1 = -\sum_j \omega_{ij}^1 e_1 \), so Equations (26), (27), (29) and (31) imply that

\[ \nabla e_i e_1 = -\frac{e_1 (\alpha_2)}{\lambda_1 - \alpha_2} e_i, \quad i = 2, \ldots, n - 1, \]

\[ \nabla e_n e_1 = -\frac{e_1 (\lambda_1 + (n - 2) \alpha_2)}{2 \lambda_1 + (n - 2) \alpha_2 - s_1} e_1, \]

and we get by Equations (25) and (26),

\[ \nabla e_n e_1 = \frac{e_1 (\lambda_1 + (n - 2) \alpha_2)}{2 \lambda_1 + (n - 2) \alpha_2 - s_1} - e_1. \]

By Equations (28), (29) and (31), we get

\[ \nabla e_i e_n = -\frac{e_n (\alpha_2)}{s_1 - \lambda_1 - (n - 2) \alpha_2} e_i, \quad i = 2, \ldots, n - 1. \]

Let’s put

\[ \alpha = -\frac{e_1 (\alpha_2)}{\lambda_1 - \alpha_2}, \quad \beta = -\frac{e_1 (\lambda_1 + (n - 2) \alpha_2)}{2 \lambda_1 + (n - 2) \alpha_2 - s_1}, \quad \gamma = \frac{e_n (\alpha_2)}{s_1 - \lambda_1 - (n - 2) \alpha_2}. \]

Now by Equations (27), (28) and (37),

\[ \nabla e_i e_1 = \alpha e_1 + \sum_{i=2,\ldots,n-1} \omega_{ij}^1 e_i - \gamma e_n. \]

Then by Equations (8), (12), (21), (23), (25), (31), (32), (33), (34), (35), (36) and (37) we get that

\[ R(e_1, e_2) e_1 = (-e_1 (\alpha) + \alpha^2) e_2 = -(c + \lambda_1 \alpha_2) e_2. \]

Therefore

\[ e_1 (\alpha) = c + \lambda_1 \alpha_2 + \alpha^2. \]

We have

\[ R(e_1, e_n) e_1 = (e_1 (\beta) + \beta^2) e_n = -(c + \lambda_1 \beta_n) e_n. \]

Therefore by Equations (24) and (40),

\[ e_1 (\beta) = -(c + \lambda_1 \beta_n + \beta^2) = -(c + \lambda_1 (s_1 - \lambda_1 - (n - 2) \alpha_2) + \beta^2). \]

We have

\[ R(e_3, e_n) e_1 = \left( e_n (\alpha) + \frac{(\alpha + \beta) e_n (\alpha_2)}{s_1 - \lambda_1 - (n - 2) \alpha_2} \right) e_3 + e_3 (\beta) e_n = 0. \]

So

\[ e_n (\alpha) = -\frac{(\alpha + \beta) e_n (\alpha_2)}{s_1 - \lambda_1 - (n - 2) \alpha_2}. \]
We have
\[ R(e_n, e_2)e_n = (e_n(\gamma) - \alpha \beta + \gamma^2) e_2 = -(c + \beta_n \alpha_2) e_2. \]
Therefore by Equations (24) and (43),
\[ e_n(\gamma) - \alpha \beta + \gamma^2 = -(c + (s_1 - \lambda_1 - (n - 2) \alpha_2) \alpha_2). \]
We have by Equations (6) and (5) we have
\[ \mu_{k,1} = \binom{n - 2}{k} \alpha_k^2 + \binom{n - 2}{k - 1} \beta_n \alpha_2^{k - 1}, \]
\[ \mu_{k,1} = \sum_{l=0}^{k} (-1)^l s_{k-l} \lambda_1^l, \]
\[ \mu_{k,n} = \binom{n - 2}{k} \alpha_k^2 + \binom{n - 2}{k - 1} \lambda_1 \alpha_2^{k - 1}, \]
\[ \mu_{k,n} = \sum_{l=0}^{k} (-1)^l s_{k-l} \beta_n^l = \sum_{l=0}^{k} (-1)^l s_{k-l}(s_1 - \lambda_1 - (n - 2) \alpha_2)^l. \]
Also by Equation (4), we have
\[ s_r = \binom{n - 2}{r - 1} \lambda_1 \alpha_2^{r - 1} + \binom{n - 2}{r} \alpha_2^r + \binom{n - 2}{r - 1} \alpha_2^{r - 1} \beta_n + \binom{n - 2}{r - 2} \lambda_1 \alpha_2^{r - 2} \beta_n. \]
We have by Equation (7),
\[ L_{k} s_{k+1} = \sum_{i=0}^{n} \mu_{k,i} (e_i e_i(s_{k+1}) - (\nabla e_i e_i)(s_{k+1})). \]
Thus we get by Equations (15), (32), (35), (38) and (50),
\[ L_{k} s_{k+1} = \mu_{k,1} e_1 e_1(s_{k+1}) - \left( \sum_{i=2}^{n-1} \mu_{k,i}(\alpha e_1 + \sum_{l=2, \ldots, n-1; i \neq l} \omega^i l e_l - \gamma e_n)(s_{k+1}) \right) + \beta \mu_{k,n} e_1(s_{k+1}). \]
Then Equations (15) and (51) imply that
\[ L_{k} s_{k+1} = \mu_{k,1} e_1 e_1(s_{k+1}) - \alpha \left( \sum_{i=2}^{n-1} \mu_{k,i} e_1(s_{k+1}) \right) + \beta \mu_{k,n} e_1(s_{k+1}). \]
We know \( \sum_{i=2}^{n-1} \mu_{k,i} = \text{tr}(P_k) - \mu_{k,1} - \mu_{k,n} \) and by Equation (9), \( \sum_{i=2}^{n-1} \mu_{k,i} = (n - k)s_k - \mu_{k,1} - \mu_{k,n} \). So by Equations (2) and (52) we get that
\[ \mu_{k,1} = (e_1 e_1(s_{k+1}) + \alpha e_1(s_{k+1})) + ((\alpha + \beta) \mu_{k,n} - \alpha(n - k)s_k) e_1(s_{k+1}) = s_{k+1}(s_1 s_{k+1} - (k + 2)s_{k+2} - c(n - k)s_k). \]
Now we show that \( e_i e_1(s_{k+1}) = 0 \) and \( e_i e_1(s_{k+1}) = 0 \) for every \( i = 2, \ldots, n \).
We know by Equation (15) for every \( i = 2, \ldots, n, [e_i, e_1](s_{k+1}) = e_i e_1(s_{k+1}) - e_1 e_i(s_{k+1}) = e_i e_1(s_{k+1}). \) On the other hand by Equations (15), (23) and (29),
By Equation (37), we have $(\lambda_1 - \alpha_2)\alpha = -e_1(\alpha_2)$ and $(2\lambda_1 + (n - 2)\alpha_2 - s_1)\beta = -e_1(\lambda_1 + (n - 2)\alpha_2)$. Differentiating these equations in direction of $e_n$ and by Equations (17) and (55), and constancy of $s_1$ we get

\[(56) \quad -e_n(\alpha_2)\alpha + (\lambda_1 - \alpha_2)e_n(\alpha) = -e_n e_1(\alpha_2),
\]

\[(57) \quad \beta(n - 2)e_n(\alpha_2) + (2\lambda_1 + (n - 2)\alpha_2 - s_1)e_n(\beta) = -(n - 2)e_n e_1(\alpha_2).
\]

So by eliminating $e_ne_1(\alpha_2)$ from Equations (56) and (57) we get

\[(58) \quad (n - 2)(-e_n(\alpha_2)\alpha + (\lambda_1 - \alpha_2)e_n(\alpha)) = (n - 2)\beta e_n(\alpha_2) + (2\lambda_1 + (n - 2)\alpha_2 - s_1)e_n(\beta).
\]

By substituting $e_n(\alpha)$ of Equation (42) in Equation (58) we get

\[(59) \quad e_n(\beta) = \frac{(n - 2)(\alpha + \beta)(n\alpha_2 - s_1)e_n(\alpha_2)}{(2\lambda_1 + (n - 2)\alpha_2 - s_1)(\alpha_1 - (n - 1)\alpha_2)}.
\]

By Equation (24),

\[(60) \quad e_n(\beta) = -(n - 2)e_n(\alpha_2).
\]

Thus by Equations (45) and (60), we have

\[(61) \quad e_n(\mu_{k,1}) = (k - 1)(s_1 - \lambda_1 - (n - 1)\alpha_2)\left(\frac{n - 2}{k - 1}\right)\alpha_1^{k - 2}e_n(\alpha_2).
\]

Differentiating of Equation (53) in direction of $e_n$ and use of Equation (54) we get

\[(62) \quad e_n(\mu_{k,1})(e_1 e_1(s_{k+1}) + \alpha e_1(s_{k+1})) + \mu_{k,1}e_n(\alpha)e_1(s_{k+1}) + e_1(s_{k+1})(e_n(\mu_{k,n})(\alpha + \beta) + \mu_{k,n}(e_n(\beta) + e_n(\alpha))) = -(k + 2) s_{k+1} e_n(s_{k+2}).
\]

Differentiating of Equation (47) in direction of $e_n$ we get

\[(63) \quad e_n(\mu_{k,n}) = ((n - k - 1)\alpha_2 + (k - 1)\lambda_1)\left(\frac{n - 2}{k - 1}\right)\alpha_1^{k - 2}e_n(\alpha_2).
\]

By Equations (17) and (46) we get

\[(64) \quad e_i(\mu_{k,1}) = 0, \quad i = 2, \ldots, n.
\]

Now for showing that $e_n(\alpha_2) = 0$ we consider two cases:

Case 1: If $k = 1$, then by Equations (62) and (64) we have

\[(65) \quad e_1(s_2)(e_n(\mu_{1,n})(\beta + \alpha) + \mu_{1,n}(e_n(\beta) + e_n(\alpha))) + e_{1,1}(e_n(\alpha)) = -3s_2 e_n(s_3).
\]
By Equation (63) we have

\[ e_n(\mu_{1,n}) = (n-2)e_n(\alpha_2). \]

By Equation (6) we have

\[ \mu_{1,1} = s_1 - \lambda_1, \]
\[ \mu_{1,n} = \lambda_1 + (n-2)\alpha_2. \]

Now by Equations (42), (59), (65), (66), (67) and (68) we get

\[ e_1(s_2)e_n(\alpha_2) \left[ \frac{(\beta+\alpha)}{(n-2) + (\lambda_1 + (n-2)\alpha_2)} \right] \]
\[ \times \left[ \frac{1}{(2\lambda_1 + (n-2)\alpha_2 - s_1)(s_1 - \lambda_1 - (n-1)\alpha_2)} \right] \]
\[ + \frac{s_1 - \lambda_1}{s_1 - \lambda_1 - (n-1)\alpha_2} \]
\[ = -3s_2e_n(s_3). \]

We have by Equation (49),

\[ s_3 = \left( \frac{n-2}{2} \right) \lambda_1 \alpha_2 + \left( \frac{n-2}{3} \right) \alpha_2^3 + \frac{n-2}{2} \alpha_2^2 \beta_n + (n-2)\lambda_1 \alpha_2 \beta_n. \]

Differentiating of Equation (70) in direction of \( e_n \) and using Equation (60) we get

\[ e_n(s_3) = e_n(\alpha_2) \left[ 2\left( \frac{n-2}{2} \right) \lambda_1 \alpha_2 + 3\left( \frac{n-2}{3} \right) \alpha_2^2 \right] \]
\[ + 2\left( \frac{n-2}{2} \right) (s_1 - \lambda_1 - (n-2)\alpha_2) \alpha_2 - (n-2)\left( \frac{n-2}{2} \right) \alpha_2^2 \]
\[ + (n-2)(s_1 - \lambda_1 - (n-2)\alpha_2) \lambda_1 - (n-2)^2 \lambda_1 \alpha_2 \].

Let \( e_n(\alpha_2) \neq 0 \). So using Equation (71) and dividing Equation (69) by \( e_n(\alpha_2) \), we have

\[ e_1(s_2)(\beta+\alpha) \left[ (n-2) + (\lambda_1 + (n-2)\alpha_2) \right] \frac{(n-2)(\alpha_2 - s_1)}{(2\lambda_1 + (n-2)\alpha_2 - s_1)(s_1 - \lambda_1 - (n-1)\alpha_2)} \]
\[ = -3s_2 \left[ 2\left( \frac{n-2}{2} \right) \lambda_1 \alpha_2 + 3\left( \frac{n-2}{3} \right) \alpha_2^2 + 2\left( \frac{n-2}{2} \right) (s_1 - \lambda_1 - (n-2)\alpha_2) \alpha_2 \right] \]
\[ - (n-2) \left( \frac{n-2}{2} \right) \alpha_2^2 + (n-2)(s_1 - \lambda_1 - (n-2)\alpha_2) \lambda_1 - (n-2)^2 \lambda_1 \alpha_2 \].
Then differentiating of Equation (72) in direction of $e_n$ and using Equations (42) and (59) we get

(73)

$$e_1(s_2) \left[ \frac{(\beta + \alpha)e_n(\alpha_2)}{s_1 - \lambda_1 - (n-1)\alpha_2} (\lambda_1 - (n-1)\alpha_2) \right] [n - 2 + (\lambda_1 + (n-2)\alpha_2)]$$

$\times \left[ \frac{(n-2)(n\alpha_2 - s_1)}{(s_1 - \lambda_1 - (n-1)\alpha_2)^2} - \frac{1}{s_1 - \lambda_1 - (n-1)\alpha_2} \right] + (\beta + \alpha) \left( (n-2)e_n(\alpha_2) \right)$

$\times \left[ \frac{(n-2)(n\alpha_2 - s_1)}{(s_1 - \lambda_1 - (n-1)\alpha_2)^2} - \frac{1}{s_1 - \lambda_1 - (n-1)\alpha_2} \right]

+ (\lambda_1 + (n-2)\alpha_2) \left[ (n-2)e_n(\alpha_2) \right]

= \left[ \frac{(n-2)(n\alpha_2 - s_1)}{(s_1 - \lambda_1 - (n-1)\alpha_2)^2} - \frac{1}{s_1 - \lambda_1 - (n-1)\alpha_2} \right]

+ (\alpha\beta - c + (s_1 - \lambda_1 - (n-2)\alpha_2)\alpha_2)

= -3s_2e_n(\alpha_2) \left[ \frac{2}{n-2} \right] s_1 - 2(n-2)^2\lambda_1 + \alpha_2 \left[ 6 \left( \frac{n-2}{3} \right) - 6(n-2) \left( \frac{n-2}{2} \right) \right].

Let’s divide Equation (73) by $e_n(\alpha_2)$ and then substitute $e_1(s_2)$ of Equation (72). So coefficients $\beta + \alpha$ and $s_2$ are eliminated. Thus we get that $\alpha_2$ should satisfy of a polynomial of degree 7 which its coefficients of functions of $\lambda_1$. So $\alpha_2$ is a function of $\lambda_1$. Then by Equation (17), we get $e_n(\alpha_2) = 0$ which is contradiction.

Case 2: If $k = 2$, then by Equations (61), (64) and $\beta_n - \alpha_2 = s_1 - \lambda_1 - (n-1)\alpha_2 \neq 0$ we have $e_n(\alpha_2) = 0$.

Therefore by Case 1 and Case 2, we have

(74) $e_2(\alpha_2) = \cdots = e_n(\alpha_2) = 0.$

Now by Equations (37) and (44),

(75) $\alpha\beta = c + (s_1 - \lambda_1 - (n-2)\alpha_2)\alpha_2.$

By Equation (37) we have

(76) $e_1(\lambda_1) = -\beta(2\lambda_1 + (n-2)\alpha_2 - s_1) - (n-2)\alpha(\alpha_2 - \lambda_1).$

Differentiating Equation (76) in direction $e_1$ and by use of Equations (16), (37), (39), (41) and (75) we get

(77) $e_1e_1(\lambda_1) = (c + \lambda_1(s_1 - \lambda_1 - (n-2)\alpha_2))(2\lambda_1 + (n-2)\alpha_2 - s_1)$

$\times \left[ \frac{(n-2)(s_1 - \lambda_1 - (n-1)\alpha_2)}{(s_1 - \lambda_1 - (n-1)\alpha_2)^2} - \frac{1}{s_1 - \lambda_1 - (n-1)\alpha_2} \right] + (\beta + \alpha) \left( \frac{(n-2)(s_1 - \lambda_1 - (n-1)\alpha_2)}{(s_1 - \lambda_1 - (n-1)\alpha_2)^2} - \frac{1}{s_1 - \lambda_1 - (n-1)\alpha_2} \right)

- (n-2)(c + (s_1 - \lambda_1 - (n-2)\alpha_2)\alpha_2)(\alpha_2 - \lambda_1)$

$- (n-2)(c + \lambda_1\alpha_2)(\alpha_2 - \lambda_1)$.
So substituting Equation (78) in equation (77), we get

\[ + \beta^2(2\lambda_1 + (n - 2)\alpha_2 - s_1) - 2(n - 2)\alpha^2(\alpha_2 - \lambda_1). \]

We rewrite the last term of Equation (77). We have by Equations (16), (37), (75) and (76),

\[ \beta^2(2\lambda_1 + (n - 2)\alpha_2 - s_1) - 2(n - 2)\alpha^2(\alpha_2 - \lambda_1) \]
\[ = - \beta(e_1(\lambda_1) + (n - 2)e_1(\alpha_2) - 2(n - 2)\alpha e_1(\alpha_2) \]
\[ = (2\alpha - \beta) \frac{e_1(s_{k+1})}{2^{k+1}} \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ - (n - 2)(c + \alpha_2(\lambda_1 - (n - 2)\alpha_2))(\alpha_2 - \lambda_1) \]
\[ + 2(c + \alpha_2(\lambda_1 - (n - 2)\alpha_2))(2\lambda_1 + (n - 2)\alpha_2 - s_1). \]

So substituting Equation (78) in equation (77), we get

\[ e_1e_1(\lambda_1) = (n\alpha - 3\beta) \frac{e_1(s_{k+1})}{2^{k+1}} \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ + (c + \alpha_2(\lambda_1 - (n - 2)\alpha_2))(\alpha_2 - \lambda_1) \]
\[ + (n - 2)(c + \alpha_1\alpha_2)(\alpha_2 - \lambda_1). \]

Differentiating of Equation (16) in direction \( e_1 \) and use of Equations (76) and (79) we obtain

\[ e_1e_1(s_{k+1}) = 2e_1e_1(\lambda_1) \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ + 2(e_1(\lambda_1))^2 \sum_{l=2}^{k+1}(-1)^l s_{k+1-l}(l - 1) \lambda_1^{l-2} \]
\[ = 2e_1e_1(\lambda_1) \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ + (\frac{(-\beta(2\lambda_1 + (n - 2)\alpha_2 - s_1) - (n - 2)\alpha(\alpha_2 - \lambda_1))}{(n\alpha - 3\beta)} \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ \times \frac{e_1(s_{k+1})}{\sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1}} \sum_{l=2}^{k+1}(-1)^l s_{k+1-l}(l - 1) \lambda_1^{l-2} \]
\[ = \frac{e_1(s_{k+1})}{\sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1}} ((n\alpha - 3\beta) \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ + (\frac{-\beta(2\lambda_1 + (n - 2)\alpha_2 - s_1) - (n - 2)\alpha(\alpha_2 - \lambda_1))}{(n\alpha - 3\beta)} \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} \lambda_1^{l-1} \]
\[ \times \sum_{l=2}^{k+1}(-1)^l s_{k+1-l}(l - 1) \lambda_1^{l-2} \]
\[ + 2((c + \lambda_1 s_1 - (n - 2)\alpha_2))(n\lambda_1 - s_1) \]
and by use of Equations (37), (39), (41), (75), (76) and (83) we get

\[ (84) \]

\[- (n - 2)(c + \lambda_1 \alpha_2)(\alpha_2 - \lambda_1) \sum_{i=1}^{k+1} (-1)^i s_{k+1-i} \lambda_1^{i-1}. \]

Let \( F_i = F_i(\lambda_1^{\max}, \alpha_1^{\min}, \lambda_1^{\min}, \alpha_2^{\max})'s \) be polynomials in term of \( \lambda_1 \) and \( \alpha_2 \) of degree \( \max_1 + \min_2 = \min_1 + \max_2 \) where \( \max_1 \) and \( \min_1 \) show the maximum and minimum power of its base. So by use of this notation and by Equation (80) we get

\[ e_1 e_1(s_{k+1}) \]

\[ = e_1(s_{k+1}) \left\{ (n \alpha - 3 \beta) F_2(\lambda_1^k) \right. \]

\[ + (-\beta F_2(\lambda_1, \alpha_2) - (n - 2) \alpha F_3(\lambda_1, \alpha_2)) F_4(\lambda_1^{k-1}) \]

\[ + [2 F_5(\lambda_1^2, \lambda_1 \alpha_2) - 2(n - 2) F_6(\lambda_1^2 \alpha_2, \lambda_1 \alpha_2^2)] \]

\[ = e_1(s_{k+1}) \frac{e_1(s_{k+1})}{F_1(\lambda_1^k)} \left[ \frac{e_1(s_{k+1})}{F_1(\lambda_1^k)} \left[ \alpha F_7(\lambda_1^k, \lambda_1^{k-1} \alpha_2) + \beta F_8(\lambda_1^k, \lambda_1^{k-1} \alpha_2) \right] \right. \]

\[ + F_9(\lambda_1^2 \alpha_2, \lambda_1^2 \alpha_2^2) \right\}. \]

Now by Equations (46), (48), (49), (53) and (81) we have

\[ F_{10}(\lambda_1^k) \left[ \frac{e_1(s_{k+1})}{F_1(\lambda_1^k)} \left[ \alpha F_7(\lambda_1^k, \lambda_1^{k-1} \alpha_2) + \beta F_8(\lambda_1^k, \lambda_1^{k-1} \alpha_2) \right] \right. \]

\[ + F_9(\lambda_1^2 \alpha_2, \lambda_1^2 \alpha_2^2) + e_1(s_{k+1}) \left[ F_{11}(\lambda_1^k, \alpha_2)(\alpha + \beta) - \alpha(n - k)s_{k+1} \right] \]

\[ = F_{12}(\lambda_1^k \alpha_2^{k-1}, \alpha_2^{k+1}) \left[ F_{13}(\lambda_1^2 \alpha_2^{k-1}, \alpha_2^{k+1}) + F_{14}(\lambda_1^2 \alpha_2^2, \alpha_2^{k+2}) \right]. \]

Therefore

\[ e_1(s_{k+1}) \left[ \alpha F_{15}(\lambda_1^2 \alpha_2^2, \lambda_1^2 \alpha_2^2) + \beta F_{16}(\lambda_1^2 \alpha_2^2, \lambda_1^2 \alpha_2^2) \right] \]

\[ = F_{17}(\lambda_1^k \alpha_2^{k-1}, \lambda_1^k \alpha_2^{k+3}) + F_{18}(\lambda_1^2 \alpha_2^3, \lambda_1^2 \alpha_2^2). \]

Differentiating of Equation (82) in direction \( e_1 \),

\[ e_1 e_1(s_{k+1}) [\alpha F_{15} + \beta F_{16}] \]

\[ + e_1(s_{k+1}) \left[ e_1(\alpha) F_{15} + e_1(\beta) F_{16} + \alpha e_1(\lambda_1) \frac{\partial F_{15}}{\partial \lambda_1} + e_1(\alpha_2) \frac{\partial F_{15}}{\partial \alpha_2} \right] \]

\[ + \beta \left[ e_1(\lambda_1) \frac{\partial F_{16}}{\partial \lambda_1} + e_1(\alpha_2) \frac{\partial F_{16}}{\partial \alpha_2} \right] \]

\[ = e_1(\lambda_1) \frac{\partial F_{17}}{\partial \lambda_1} + e_1(\alpha_2) \frac{\partial F_{17}}{\partial \alpha_2} + e_1(\lambda_1) \frac{\partial F_{18}}{\partial \lambda_1} + e_1(\alpha_2) \frac{\partial F_{18}}{\partial \alpha_2} \]

and by use of Equations (37), (39), (41), (75), (76) and (83) we get

\[ e_1 e_1(s_{k+1}) \left[ \alpha F_{15}(\lambda_1^2 \alpha_2^2, \lambda_1^2 \alpha_2^2) + \beta F_{16}(\lambda_1^2 \alpha_2^2, \lambda_1^2 \alpha_2^2) \right] \]

\[ + e_1(s_{k+1}) \left[ \alpha^2 F_{19}(\lambda_1^2 \alpha_2^{k-1}, \alpha_2^{k+1}) + \beta^2 F_{20}(\lambda_1^2 \alpha_2^{k-1}, \alpha_2^{k+1}) \right. \]

\[ + \beta^2 F_{21}(\lambda_1^2 \alpha_2^{k+2}, \alpha_2^{k-1} \alpha_2^{k+3}) \right\].
We compute two terms (87)

Substituting Equations (86) and (87) in Equation (85) we get

Now by Equations (75), (82) and (89),

By Equations (16) and (76) we have

Substituting Equations (86) and (87) in Equation (85) we get

By Equations (88) and (89) we have

Now by Equations (75), (82) and (89),

\[ \alpha^2 F_{48}(\lambda_1^{3k+1}, \lambda_1^k \alpha_2^{k+1}) + \beta^2 F_{49}(\lambda_1^{3k+1}, \lambda_1^k \alpha_2^{k+1}) = F_{50}(\lambda_1^{3k+2}, \lambda_1^k \alpha_2^{k+3}). \]
By multiplying Equation (90) in $\alpha$ and $\beta$ and by use of Equation (75) we get

$$\alpha^2 = \frac{F_{51}(\lambda_1^{4k+4} \alpha_2, \lambda_1^{1k-1} \alpha_2^{4k+6})}{F_{46}(\lambda_1^{4k+4} \alpha_2, \lambda_1^{1k-1} \alpha_2^{4k+4})},$$

$$\beta^2 = \frac{F_{52}(\lambda_1^{4k+4} \alpha_2, \lambda_1^{1k-1} \alpha_2^{4k+6})}{F_{47}(\lambda_1^{4k+4} \alpha_2, \lambda_1^{1k-1} \alpha_2^{4k+4})}.$$  

Now by substituting Equations (92) and (93) in Equation (91) we get

$$F_{53}(\lambda_1^{10k+8} \alpha_2, \lambda_1^{10k-2} \alpha_2^{10k+11}) = 0.$$  

In the following we show that $e_1(\alpha_2) \neq 0$. Let $e_1(\alpha_2) = 0$ then by Equation (74), $\alpha_2$ is constant and by Equation (37), $\alpha = 0$. Therefore by Equation (39), $c + \lambda_1 \alpha_2 = 0$ and by differentiating we get $e_1(\lambda_1)\alpha_2 = 0$ and so by Equation (17), $\alpha_2 = 0$. If $k = 2$, then by hypothesis $s_1$ and $s_2$ is constant. Since $\alpha_2 = 0$, by Equations (4) and (21), $s_1 = \lambda_1 + \beta_n$, $s_2 = \lambda_1 \beta_n$. Then differentiating in direction of $e_1$ we get $e_1(\lambda_1) + e_1(\beta_n) = 0$ and $e_1(\lambda_1)\beta_n + \lambda_1 e_1(\beta_n) = 0$, so $e_1(\lambda_1)(\beta_n - \lambda_1) = 0$. Therefore $\beta_n = \lambda_1$ which is a contradiction. If $k = 1$, we have $s_1$ is constant. Since $\alpha = \alpha_2 = 0$, by Equation (90), $\beta F_{47}(\lambda_1^{14}) = 0$. If $\beta \neq 0$, then $F_{47}(\lambda_1^{14}) = 0$, so $\lambda_1$ is constant which contradicts Equation (17). Thus $\beta = 0$ and by Equation (76), $e_1(\lambda_1) = 0$ which contradicts Equation (17). Finally by Equation (74) we have

$$e_1(\alpha_2) \neq 0 \quad \text{and} \quad e_2(\alpha_2) = \cdots = e_\alpha(\alpha_2) = 0.$$  

Now assume that $\gamma(t)$ be integral curve of $e_1$ that $\gamma(t_0) = p$ which $p \in M$ and $t_0 \in I$. By Equations (17) and (95), we have in some neighborhood of $t_0$, $\lambda_1 = \lambda_1(t)$ and $\alpha_2 = \alpha_2(t)$, and so $t = t(\alpha_2)$ and $\lambda_1 = \lambda_1(\alpha_2)$. Therefore by Equations (37), (76) and (90) we have

$$\frac{d\lambda_1}{d\alpha_2} = \frac{d\lambda_1}{dt} \frac{dt}{d\alpha_2} = \frac{e_1(\lambda_1)}{e_1(\alpha_2)} = \frac{F_{54}(\lambda_1^{10k+4} \alpha_2^{10k-4} \alpha_2^{4k+5})}{F_{55}(\lambda_1^{10k+4} \alpha_2^{10k-4} \alpha_2^{4k+5})}.$$  

Now differentiating of Equation (94) relative to $\alpha_2$ and using Equation (96) we get

$$F_{56}(\lambda_1^{27k+12}, \lambda_1^{13k-4} \alpha_2^{14k+16}) = 0.$$  

Now rewriting polynomials (94) and (97) in term of $\alpha_2$ we get

$$\sum_{i=0}^{10k+11} f_i(\lambda_1)\alpha_2^i = 0,$$

$$\sum_{i=0}^{14k+16} g_i(\lambda_1)\alpha_2^i = 0,$$

where $f_i(\lambda_1)$ and $g_i(\lambda_1)$ are polynomials in term of $\lambda_1$. By multiplying equation (98) in $g_{14k+16}(\lambda_1)\alpha_2^{4k+5}$ and Equation (99) in $f_{10k+11}(\lambda_1)$ and subtracting them we get a polynomial in term of $\alpha_2$ of degree $14k + 15$. Then by this new polynomial and Equation (98), similarly we get a polynomial of degree $14k + 14$. 


By continuing this method, finally we omit \( \alpha_2 \) and we earn a polynomial in term of \( \lambda_1 \) with constant coefficients. So \( \lambda_1 \) should be constant which is a contradiction. Therefore \( s_{k+1} \) is constant.

**Proof of Theorem 1.2.** Let \( k_1, k_2, k_3 \) be principal curvatures of \( M \), respectively with multiplicities \( m_1, m_2, m_3, n = m_1 + m_2 + m_3 \). Suppose that \( \{e_i\}_{i=1}^n \) is a local orthonormal frame field on \( M \) which are the eigenvectors of the shape operator \( S \) of \( M \) with respect to the globally chosen unit normal vector field \( N \) and \( Se_i = k_1 e_i \, i \leq m_1, \, Se_i = k_2 e_i \, m_1 < i \leq m_1 + m_2, \, Se_i = k_3 e_i \, m_1 + m_2 < i \leq n \) together with the Codazzi equation, \( (\nabla_{e_i} S)e_j = (\nabla_{e_j} S)e_i \), imply that

\[
\begin{align*}
(100) \quad & \nabla_{e_i} k_1 = 0, \quad i \leq m_1, \\
(101) \quad & \nabla_{e_i} k_2 = 0, \quad m_1 < i \leq m_1 + m_2, \\
(102) \quad & \nabla_{e_i} k_3 = 0, \quad m_1 + m_2 < i \leq n.
\end{align*}
\]

Since \( s_1 \) is constant and \( s_2 = 0 \), by Equation (4) we get that \( k_2 = g_1(k_1) \) and \( k_3 = g_2(k_1) \) where \( g_1 \) and \( g_2 \) are some smooth functions. So for every \( i \) we have

\[
(103) \quad \nabla_{e_i} k_2 = g_1'(k_1) \nabla_{e_i} k_1.
\]

We have

\[
(104) \quad s_1 = m_1 k_1 + m_2 k_2 + m_3 k_3.
\]

Thus we have by Equations (102) and (103),

\[
(105) \quad (m_1 + m_2 g_1'(k_1)) \nabla_{e_i} k_1 = 0 \quad m_1 + m_2 < i \leq n.
\]

If for some \( i, m_1 + m_2 < i \leq n \), \( \nabla_{e_i} k_1 \neq 0 \), then by Equation (105), \( g_1'(k_1) = -\frac{m_2}{m_1} \). So \( k_2 = g_1(k_1) = -\frac{m_2}{m_1} k_1 + C \) where \( C \) is a constant. Therefore by Equation (104), \( k_3 \) is constant. Now by equation \( s_2 = 0 \), we get that \( k_1 \) should satisfy a polynomial. Therefore \( k_1 \) is constant which is a contradiction. Thus for every \( i, m_1 + m_2 < i \leq n, \nabla_{e_i} k_1 = 0 \), and together with Equations (100), (101) and (103), we get \( k_2 \) is constant. In a similar way we get \( k_3 \) and so \( k_1 \) is constant. Therefore \( M \) is an isoparametric hypersurface. If \( s_2 \neq 0 \), By Equation (2), we have

\[
(106) \quad s_1 s_2 - 3s_3 - c(n - 1) s_1 = 0.
\]

Since \( s_1 \) and \( s_2 \) are constant, Equation (106) implies that \( s_3 \) is constant. Because \( M \) has three principal curvatures, we get that all principal curvatures are constant. So \( M \) is an isoparametric hypersurface.

**Case 2.** Let \( k = 2 \). By hypothesis \( s_1 \) and \( s_2 \) is constant and by Lemma 3.1, \( s_3 \) is constant. Because \( M \) has three principal curvatures, we get that all principal
curvatures are constant. So $M$ is an isoparametric hypersurface. We know for $c = 0, -1$, isoparametric hypersurfaces has at most two principal curvatures, So by Case 1, we get $s_2 = 0$ and at least one of the multiplicities of principal curvatures is one, and by Case 2, there is not $L_2$-biharmonic hypersurface with three disjoint principal curvatures, and $s_1$ and $s_2$ is constant. In the rest, we assume that $c = 1$. By Theorem 2.1, an isoparametric hypersurface with three constant principal curvature $k_1 > k_2 > k_3$ in $\mathbb{S}^{n+1}$ have the multiplicities: $m = 1, 2, 4$ and 8. Therefore we have the following equations:

If $m = 1$, then by Equation (4), we have

$$s_1 = k_1 + k_2 + k_3, \quad s_2 = k_1 k_2 + k_1 k_3 + k_2 k_3, \quad s_3 = k_1 k_2 k_3.$$ (107)

If $m = 2$, then by Equation (4), we have

$$s_1 = 2(k_1 + k_2 + k_3),$$ (108)
$$s_2 = 4k_1 k_2 + 4k_1 k_3 + k_1^2 + k_2^2 + 4k_2 k_3 + k_3^2,$$ (109)
$$s_3 = 8k_1 k_2 k_3 + 2k_1^2 k_2 + 2k_1 k_3^2 + 2k_2^2 k_3 + 2k_2 k_3^2 + 2k_3^2 k_3 + 2k_1 k_2^2,$$ (110)
$$s_4 = k_2^2 k_3^2 + 4k_1 k_2 k_3^3 + 4k_1 k_3^2 k_3^3 + k_2^2 k_3^4 + 4k_1^2 k_2 k_3 + k_1^2 k_2 k_3^2.$$ (111)

If $m = 4$, then by Equation (4), we have

$$s_1 = 4(k_1 + k_2 + k_3),$$ (112)
$$s_2 = 6k_1^2 + 16k_2 k_3 + 6k_2^2 + 16k_1 k_3 + 16k_1 k_2 + 6k_1^2,$$ (113)
$$s_3 = 4k_1^3 + 24k_2 k_3 k_1 + 4k_2^3 + 24k_1 k_3^2 + 24k_1 k_2 k_3 + 64k_1 k_2 k_3$$
$$+ 24k_1 k_2^2 k_3 + 24k_2^2 k_3 + 24k_2 k_3 + k_3^2,$$ (114)
$$s_4 = k_2^4 + 16k_3 k_3 + 36k_2^2 k_3^2 + 16k_2 k_3 + k_2^4 + 16k_3^4$$
$$+ 96k_3 k_3 k_2 + 96k_1 k_2^2 k_3 + 16k_1 k_2^3 + 36k_2^2 k_3$$
$$+ 96k_1^2 k_3 + 36k_3^2 k_2 + 16k_1 k_3 + 16k_1 k_2 + k_1^4.$$ (115)

If $m = 8$, then by Equation (4), we have

$$s_1 = 8(k_1 + k_2 + k_3),$$ (116)
$$s_2 = 28k_1^2 + 64k_2 k_3 + 28k_2^2 + 64k_1 k_3 + 64k_1 k_2 + 28k_1^2,$$ (117)
$$s_3 = 56k_1^3 + 224k_2 k_3 k_1 + 224k_2^2 k_3 + 56k_2^3 + 224k_1 k_3^2 + 512k_1 k_2 k_3$$
$$+ 224k_1 k_2^2 k_3 + 56k_1^3 + 224k_2^2 k_3 + 224k_2 k_3,$$ (118)
$$s_4 = 170k_1^4 + 448k_2^3 k_2 + 448k_2^3 k_3 + 784k_2^3 k_2 + 1792k_2^2 k_3 k_2$$
$$+ 784k_2^2 k_3 + 1792k_1 k_3 k_2^3 + 448k_1 k_3^2 + 1792k_1 k_2 k_3^2 + 448k_1 k_3^3$$
$$+ 170k_2^2 + 448k_2^3 k_3 + 784k_2^3 k_3 + 448k_3^2 + 170k_3^4.$$ (119)

Let $k = 1$. If $s_2 = 0$ and multiplicities of principal curvatures are greater than one, and or $s_2 \neq 0$, by Case1, $M$ is an isoparametric hypersurface with three constant principal curvature $k_1 > k_2 > k_3$. 

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If \( s_2 = 0 \) and multiplicities of principal curvatures are greater than one, we have the following:

If \( m = 2 \), by Equations (10) and (109), we get \( k_1 \approx 3.286, k_2 \approx 0.232, k_3 \approx -1.069 \) or \( k_1 \approx 1.069, k_2 \approx -0.232, k_3 \approx -3.286 \).

If \( m = 4 \), by Equations (10) and (113), we get \( k_1 \approx 2.527, k_2 \approx 0.147, k_3 \approx -1.261 \), or \( k_1 \approx 1.261, k_2 \approx -0.147, k_3 \approx -2.527 \).

If \( m = 8 \), by Equations (10) and (117), we get \( k_1 \approx 2.216, k_2 \approx 0.1, k_3 \approx -1.39 \) or \( k_1 \approx 1.39, k_2 \approx -0.1, k_3 \approx -2.216 \).

If \( s_2 \neq 0 \), by Case 1, \( M \) is an isoparametric hypersurface with three constant principal curvature \( k_1 > k_2 > k_3 \) and so the multiplicities: \( m = 1, 2, 4 \) and 8.

So we have the following:

If \( m = 1 \) by Equations (10), (106) and (107), we get \( k_1 = \sqrt{3}, k_2 = 0 \) and \( k_3 = -\sqrt{3} \).

If \( m = 2 \) by Equations (10), (106), (108), (109) and (110), we get either \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \) or \( k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261 \) or \( k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx 1.369 \).

If \( m = 4 \) by Equations (10), (106), (112), (113) and (114), we get \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \).

If \( m = 8 \) by Equations (10), (106), (116), (117) and (118), we get \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \).

Let \( k = 2 \). By Case 2, \( M \) is an isoparametric hypersurface. So the multiplicities of constant principal curvatures \( k_1 > k_2 > k_3 \) is \( m = 1, 2, 4 \) and 8.

If \( s_3 = 0 \), then we have the following:

If \( m = 1 \), by Equations (10) and (107), we get \( k_1 = \sqrt{3}, k_2 = 0 \) and \( k_3 = -\sqrt{3} \).

If \( m = 2 \) by Equations (10) and (110), we get \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \).

If \( m = 4 \) by Equations (10) and (114), we get \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \) or \( k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777 \) or \( k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993 \).

If \( m = 8 \) by Equations (10) and (118), we get \( k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3} \) or \( k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757 \) or \( k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189 \).

If \( s_3 \neq 0 \), then by Equation (2), we have

\[
(120) \quad s_1 s_3 - 4 s_4 - (n - 2) s_2 = 0.
\]

If \( m = 1 \), by Equations (10), (107) and (120), we get either \( k_1 = 1, k_2 = \sqrt{3} - 2, k_3 = -\sqrt{3} - 2 \) or \( k_1 = 2 + \sqrt{3}, k_2 = 2 - \sqrt{3}, k_3 = -1 \).

If \( m = 2 \), by Equations (10), (108), (109), (110), (111) and (120), we get that there is not real solution for all \( k_1, k_2, k_3 \). So there is not proper \( L_2 \)-biharmonic hypersurface in \( S^7 \) with three disjoint principal curvatures, and \( s_1 \) and \( s_2 \) is constant.

If \( m = 4 \), by Equations (10), (112), (113), (114), (115) and (120), we get either \( k_1 \approx 1.083, k_2 \approx -0.225, k_3 \approx -3.213 \) or \( k_1 \approx 3.213, k_2 \approx 0.225, k_3 \approx -1.083 \).
If \( m = 8 \), by Equations (10), (116), (117), (118), (119) and (120), we get that there is not real solution for all \( k_1, k_2, k_3 \). So there is not proper \( L_2 \)-biharmonic hypersurface in \( S^{25} \) with three disjoint principal curvatures, and \( s_1 \) and \( s_2 \) is constant. Summarizing all of above and Theorem 2.1, we get the result. \( \square \)

Proof of Theorem 1.3. We have \( P_3 = s_3 I - S \circ P_2 \). Since \( P_3 = 0 \), Equation (1) implies that \( 3s_3 \nabla s_3 = 0 \). Thus \( \nabla s_3 = 0 \), and so \( s_3 \) is constant. By assumption \( s_2 \) is constant and \( s_3 \neq 0 \), and so by Equation (2), \( s_1 \) is constant. Now by Theorem 1.2, we get the result. \( \square \)

References


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