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CLASSIFICATION OF (k, μ)-ALMOST CO-KÄHLER MANIFOLDS WITH VANISHING BACH TENSOR AND DIVERGENCE FREE COTTON TENSOR

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Abstract. The object of the present paper is to characterize Bach flat (k, μ)-almost co-Kähler manifolds. It is proved that a Bach flat (k, μ)-almost co-Kähler manifold is K-almost co-Kähler manifold under certain restriction on μ and k. We also characterize (k, μ)-almost co-Kähler manifolds with divergence free Cotton tensor.

1. Introduction

In 1921, Bach introduced a tensor [1] to study the conformal relativity in the context of conformally Einstein spaces. This tensor is known as the Bach tensor and is a symmetric (0,2)-tensor $\mathcal{B}$ on a pseudo-Riemannian manifold(M, g), defined by

$$\mathcal{B}(U,V) = \frac{1}{2n-2}\sum_{i,j=1}^{2n+1}(\nabla_{e_i}\nabla_{e_j}W)(U,e_i,e_j,V) + \frac{1}{2n-1}\sum_{i,j=1}^{2n+1}S(e_i,e_j)W(U,e_i,e_j,V), \tag{1}$$

where $\{e_i\}, i = 1, \ldots, 2n+1$, is a local orthonormal frame on (M, g), S is the Ricci tensor of type (0,2), W denotes the Weyl tensor of type (0,3) defined by

$$W(U,V)Z = R(U,V)Z - \frac{1}{2n-1}\{S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV\} + \frac{r}{2n(2n-1)}\{g(V,Z)U - g(U,Z)V\}. \tag{2}$$

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We recall the Cotton tensor C which is a (0,3)-tensor defined by

$$C(U,V)Z = (\nabla_U S)(V,Z) - (\nabla_V S)(U,Z) - \frac{1}{4n}[(Ur)g(V,Z) - (Vr)g(U,Z)]. \tag{3}$$

In view of (1) and (2), the Bach tensor can be expressed as [12]

$$\mathcal{B}(U,V) = \frac{1}{2n-1}[\sum_{i=1}^{2n+1}(\nabla_{e_i}C)(e_i,U,V) + \sum_{i,j=1}^{2n+1}S(e_i,e_j)W(U,e_i,e_j,V)]. \tag{4}$$

In dimension 3, the Weyl tensor W vanishes and hence the Bach tensor reduces to

$$\mathcal{B}(U,V) = \sum_{i=1}^{3}(\nabla_{e_i}C)(e_i,U,V). \tag{5}$$

If (M,g) is locally conformally related to an Einstein space, $\mathcal{B}$ has to vanish, but there are Riemannian manifolds with $\mathcal{B} = 0$, which are not conformally related to Einstein spaces [30]. By (3), it is easy to see that Bach flatness is natural generalization of Einstein and conformal flatness. For more details about Bach tensor, we refer to reader ([3, 18, 29, 30]) and references therein.

Recently, Ghosh and Sharma [19] studied Sasakian manifolds with purely transversal Bach tensor. In particular, they proved that if a Sasakian manifold admits a purely transversal Bach tensor, then g has a constant scalar curvature $\geq 2n(2n-1)$, with equality holds if and only if g is Einstein and the Ricci tensor g has a constant norm. Also, they studied (k,μ)-constant manifolds with divergence free Cotton tensor and vanishing Bach tensor in [20]. These works of Ghosh and Sharma ([19, 20]) turn our attention to study Bach tensor in the framework of (k,μ)-almost co-Kähler manifolds.

The paper is organized as follows: After introduction in Section 2, we give a brief description on (k,μ)-almost co-Kähler manifolds and state definition of K-almost co-Kähler manifold. In Section 3, we consider Bach flat (k,μ)-almost co-Kähler manifolds and prove that an (k,μ)-almost co-Kähler manifold with vanishing Bach tensor is a K-almost co-Kähler manifold. Finally, we characterize (k,μ)-almost co-Kähler manifolds with divergence free Cotton tensor.

2. Preliminaries

Let a $(2n+1)$-dimensional smooth manifold $M^{2n+1}(n \geq 1)$ admits a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η such that

$$\phi^2 + I = \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{6}$$

where I denotes the identity transformation. Then the structure (ϕ,ξ,η) is called an almost contact structure of M^{2n+1}. The manifold M^{2n+1} equipped with structure (ϕ,ξ,η) is known as an almost contact manifold ([5, 6]). From (6), we can easily prove that

$$\phi\xi = 0, \qquad \eta \circ \phi = 0 \quad \text{and} \quad rank(\phi) = 2n. \tag{7}$$

A $(2n+1)$-dimensional almost contact metric manifold is said to be normal if the induced almost complex structure J defined on $M^{2n+1} \times \mathbb{R}$ by

$$J(U, f\tfrac{d}{dt}) = (\phi U - f\xi, \eta(U)\tfrac{d}{dt}),$$

where U is any vector field on M^{2n+1}, f is a smooth function defined on $M^{2n+1} \times \mathbb{R}$ and t is the coordinate of $\mathbb{R}$, is normal, that is, J is a complex structure. The normality of an almost contact structure (ϕ, ξ, η) is expressed as $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where d denotes the exterior derivative and $[\phi, \phi]$ represents the Nijenhuis tensor of ϕ [6] defined as

$$[\phi, \phi](U, V) = \phi^2[U, V] + [\phi U, \phi V] - \phi[\phi U, V] - \phi[U, \phi V]$$

for any vector fields U and V on M^{2n+1}.

If the associated Riemannian metric g of an almost contact manifold M^{2n+1} satisfies

$$g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V) \tag{8}$$

for all vector fields U and U on M^{2n+1}, then an almost contact manifold endowed with g is known as almost contact metric manifold. In consequence of (6), (7) and (8), we have

$$g(\phi U, V) + g(U, \phi V) = 0 \quad \text{and} \quad \eta(U) = g(\xi, U) \tag{9}$$

for all vector fields U and V on M^{2n+1}.

A $(2n+1)$-dimensional almost contact metric manifold is said to be a contact metric manifold if

$$d\eta(U, V) = \Phi(U, V) = g(U, \phi V)$$

for all vector fields U and V on M^{2n+1}, where Φ denotes the fundamental 2-form on an almost contact metric manifold.

According to Cappelletti-Montano et al. [10], an almost co-Kähler manifold is an almost contact metric manifold with closed contact 1-form η and fundamental 2-form Φ. If the associated almost contact structures of an almost co-Kähler manifold M^{2n+1} is normal, that is, $\nabla\phi = 0$, (or equivalently $\nabla\Phi = 0$), then M^{2n+1} is called co-Kähler manifold [22]. The simplest example of (almost) co-Kähler manifold is the Riemannian product of a real line or a circle and a (almost) co-Kähler manifold. However, there do exist some examples of (almost) co-Kähler manifolds which are not globally the product of a real line or a circle and a (almost) co-Kähler manifold ([13, 23–27], [31–35]).

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [21]. The simplest examples of such manifolds are those being the products of almost Kählerian manifolds and the real line $\mathbb{R}$ or the circle S^1. In particular, cosymplectic manifolds in the sense of Blair [4] are of this type. However, the class of almost cosymplectic manifolds is much more wider. In early literature, (almost) co-Kähler manifolds were usually referred to as (almost) cosymplectic manifolds. For example, in 1967, Blair and Goldberg in [7] obtained that the Betti numbers of any compact cosymplectic manifold are non-zero.

If we set $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on an almost co-Kähler manifold, then we recall the following results proved by Dacko et al. ([14–16]), Endo [17] and Olszak ([24, 26]):

$$h\xi = h'\xi = 0, \quad trh = trh' = 0, \quad h\phi + \phi h = 0, \tag{10}$$

where tr stand for trace, $\mathcal{L}$ represents the Lie derivative operator and R is the Riemannian curvature tensor of the Riemannian connection ∇ defined as

$$R(U,V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z$$

for all vector fields U, V and Z on M^{2n+1}. Notice that the vector fields h and h' are symmetric operators with respect to the Riemannian metric g. In an almost co-Kähler manifold, the 1-form η is closed, that is,

$$(\nabla_U \eta)(V) - (\nabla_V \eta)(U) = 0 \Longleftrightarrow g(h\phi U, V) - g(h\phi V, U) = 0 \tag{11}$$

for all vector fields U and V on M^{2n+1}. In almost co-Kähler manifolds

$$\nabla_U \xi = h'U = h\phi U \Longleftrightarrow (\nabla_U \eta)(V) = g(h\phi U, V) \tag{12}$$

for all vector fields U and V on M^{2n+1}.

The Lie derivative of Riemannian metric g along the Reeb vector ξ and (12) infer that the relation $(\mathcal{L}_\xi g)(U,V) = 2g(h'U,V)$ holds on an almost co-Kähler manifold. This reflects that the Reeb vector field ξ of M^{2n+1} is Killing if and only if the tensor field h vanishes on M^{2n+1}. Thus we have the following:

Definition. An almost co-Kähler manifold is said to be a K-almost co-Kähler manifold if the Reeb vector field ξ is Killing.

If the distribution $\mathcal{D}$ on an almost co-Kähler manifold M^{2n+1} is defined by $\mathcal{D} = ker\eta$. Then by using (6)-(8) and relation $d\Phi = 0$, one can define an almost K-Kähler structure $(g\mathcal{D}, \phi\mathcal{D})$ on $\mathcal{D}$. In 1987, Olszak [26] proved that an associated almost Kähler structure is integrable if and only if

$$(\nabla_U \phi)(V) = g(hU,V)\xi - \eta(V)hU \tag{13}$$

for all vector fields U and V on M^{2n+1}. This reflects that an almost co-Kähler manifold is co-Kähler if and only if it is K-almost co-Kähler and the associated almost Kähler structure is integrable. It is noticed that a 3-dimensional almost co-Kähler manifold is co-Kähler if and only if it is K-almost co-Kähler.

Blair et al. [8] introduced the notion of (k,μ)-contact metric manifolds, where k and μ are real numbers. The full classification of these manifolds was given by Boeckx [9]. By a (k,μ)-almost co-Kähler manifold we mean an almost co-Kähler manifold such that the Reeb vector field ξ belongs to the (k,μ)-nullity distribution, that is,

$$R(U,V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV] \tag{14}$$

for any vector fields U, V in $\chi(M)$ and $k, \mu \in \mathbb{R}$. In this paper, a (k,μ)-almost co-Kähler manifold with $k < 0$ will be called a proper (k,μ)-almost coKähler manifold or a non-coKähler (k,μ)-almost co-Kähler manifold. This was first introduced by Endo [17] and was generalized to (k,μ,v)-spaces by Dacko and Olszak in [16] (see also Carriazo and Martin-Monina [11] and [28]).

It is well known that an almost co-Kähler (k,μ)-manifold of dimension $(2n+1)$, $n>1$ is a K-almost co-Kahler manifold if and only if $k=0$.

A (k,μ)-almost co-Kähler manifold satisfies the curvature properties [28]

$$h^2U = k\phi^2U,\ k\leq 0. \tag{15}$$

Balkan et al. proved the following lemmas [2].

Lemma 2.1. *If the almost cosymplectic manifold M^{2n+1} with ξ belonging to the (k,μ)-nullity distribution, then the following relations hold*:

(i) $(\nabla_U h)V-(\nabla_V h)U = k(\eta(V)\phi U-\eta(U)\phi V+2g(\phi U,V)\xi)+\mu(\eta(V)\phi hU -\eta(U)\phi hV)$,

(ii) $R(\xi,U)V = k[g(U,V)\xi-\eta(V)U]+\mu[g(hU,V)\xi-\eta(V)hU]$,

(iii) $S(U,\xi)=2nk\eta(U)$,

where U and V are vector fields on M^{2n+1}, k,$\mu\in\mathbb{R}$ and S is the Ricci tensor of M^{2n+1} [8].

Lemma 2.2. *Let M be an almost cosymplectic manifold with ξ belonging to the (k,μ)-nullity distribution. For any vector field U, the Ricci operator Q is given by*

(i) $QU=\mu hU+2nk\eta(U)\xi$ *and*

(ii) $r=2nk$.

Also putting $U=\xi$ in Lemma 2.2(i), we get

$$Q\xi = 2nk\xi. \tag{16}$$

3. Bach flat (k,μ)-almost co-Kähler manifold

Lemma 2.2(i) implies

$$S(V,Z)=\mu g(hV,Z)+2nk\eta(V)\eta(Z). \tag{17}$$

Differentiating (17) along U and using (10) and (12), we have

$$\begin{aligned}(\nabla_U S)(V,Z) = \mu g((\nabla_U h)V,Z) &- 2nkg(V,\phi hU)\eta(Z)\\ &- 2nkg(Z,\phi hU)\eta(V).\end{aligned} \tag{18}$$

In (k,μ)-almost co-Kähler manifold r is constant, then (3) implies

$$C(U,V)Z = (\nabla_U S)(V,Z)-(\nabla_V S)(U,Z).$$

Using (18), (9), (10) and Lemma 2.1(i) in the above equation yields

$$\begin{aligned}C(U,V)Z = k\mu\{&g(\phi U,Z)\eta(V)-g(\phi V,Z)\eta(U)+2g(\phi U,V)\eta(Z)\}\\ &-(\mu^2-2nk)\{g(\phi hV,Z)\eta(U)-g(\phi hU,Z)\eta(V)\}.\end{aligned} \tag{19}$$

Substituting $Z=\xi$ in the above equation provides

$$C(U,V)\xi = 2k\mu g(\phi U,V). \tag{20}$$

Differentiating $C(U,V)\xi$ along X and using (10) and (12), we get

$$\begin{aligned}(\nabla_X C)(U,V)\xi &= 2k\mu g((\nabla_X\phi)U,V)\\ &\quad - k\mu\{g(\phi V,\phi hX)\eta(U) - g(\phi U,\phi hX)\eta(V)\}\\ &\quad - (\mu^2-2nk)\{g(\phi hU,\phi hX)\eta(V) - g(\phi hV,\phi hX)\eta(U)\}. \qquad (21)\end{aligned}$$

Using (8), (10) and (13) in the above equation yields

$$\begin{aligned}(\nabla_X C)(U,V)\xi &= 2k\mu\{g(hX,U)\eta(V) - g(hX,V)\eta(U)\}\\ &\quad - k\mu\{g(V,hX)\eta(U) - g(U,hX)\eta(V)\}\\ &\quad - (\mu^2-k)\{g(hU,hX)\eta(V) - g(hV,hX)\eta(U)\}. \qquad (22)\end{aligned}$$

Putting $X = U = e_i$ in (22), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \le i \le n)$, we get

$$\begin{aligned}(\nabla_{e_i} C)(e_i,V)\xi &= 2k\mu\{g(he_i,e_i)\eta(V) - g(he_i,V)\eta(e_i)\}\\ &\quad - k\mu\{g(V,he_i)\eta(e_i) - g(e_i,he_i)\eta(V)\}\\ &\quad - (\mu^2-2nk)\{g(he_i,he_i)\eta(V) - g(hV,he_i)\eta(e_i)\}. \qquad (23)\end{aligned}$$

Using (6), (10) and (15) in the above equation infers

$$(\nabla_{e_i} C)(e_i,V)\xi = -k(\mu^2-2nk)(1-n)\eta(V). \qquad (24)$$

Substituting ξ for Z in (2) and operating by the Ricci operator Q, we get

$$\begin{aligned}QW(U,V)\xi &= R(U,V)Q\xi - \frac{1}{2n-1}\{S(V,\xi)QU\\ &\quad - S(U,\xi)QV + g(V,\xi)Q^2U - g(U,\xi)Q^2V\}\\ &\quad + \frac{r}{2n(2n-1)}\{g(V,\xi)QU - g(U,\xi)QV\}. \qquad (25)\end{aligned}$$

Substituting e_i for V in the above equation, taking inner product with e_i, summation over i, and using (6), (9), (14), Lemma 2.1(iii) and (16), we get

$$\begin{aligned}\sum_{i=1}^{2n+1} g(QW(U,e_i)\xi,e_i) &= -4n^2k^2\eta(U)\\ &\quad - \frac{1}{2n-1}\{4n^2k^2 - 2nkr + 4n^2k^2 - |Q|^2\}\eta(U)\\ &\quad + \frac{r}{2n(2n-1)}\{2nk - r\}\eta(U). \qquad (26)\end{aligned}$$

Putting $U = V = e_i$ in (17) infer that

$$|Q| = 2nk. \qquad (27)$$

Using (27) and Lemma 2.2(ii) in (26) provides

$$\sum_{i=1}^{2n+1} g(QW(U,e_i)\xi,e_i) = -4n^2k^2\eta(U). \qquad (28)$$

Now we notice that the last term of the Bach tensor in (4) can be written as

$$g(Qe_i, e_j)g(W(U, e_i)e_j, V) = -g(W(U, e_i)V, Qe_i) = -g(QW(U, e_i)V, e_i). \quad (29)$$

Using (29) in (4), we have

$$B(U, V) = \frac{1}{2n-1}[\sum_i (\nabla_{e_i} C)(e_i, U, V) - \sum_i g(QW(U, e_i)V, e_i)]. \quad (30)$$

Substituting ξ for V in the above equation, using the hypothesis $B(U, \xi) = 0$ and with the help of (24) and (28) yields

$$k\{\mu^2(1-n) - 2nk(1+n)\}\eta(U) = 0. \quad (31)$$

Taking $U = \xi$ in the above equation, we get

$$k\{\mu^2(1-n) - 2nk(1+n)\} = 0. \quad (32)$$

This implies either $k = 0$ or, $\mu^2(1-n) - 2nk(1+n) = 0$.

When $k = 0$ we say from definition the manifold is K-almost co-Kähler manifold, provided $\mu^2(1-n) - 2nk(1+n) \neq 0$.

Thus, in view of the above result, we can state the following theorem.

Theorem 3.1. *A Bach flat (k, μ)-almost co-Kähler manifold is K-almost co-Kähler manifold, provided $\mu^2(1-n) - 2nk(1+n) \neq 0$.*

Now we consider 3-dimensional Bach flat (k, μ)-almost co-Kähler manifolds.

Then from (32), we get either $k = 0$ or $4k = 0$.

From these two cases, we get $k = 0$.

This implies that the manifold is K-almost co-Kähler manifold.

It is known that any 3-dimensional almost co-Kähler manifold is co-Kähler manifold if and only if it is K-almost co-Kähler manifold [24].

Thus we conclude that:

Corollary 3.2. *A 3-dimensional Bach flat (k, μ)-almost co-Kähler manifold is co-Kähler manifold.*

4. (k, μ)-almost co-Kähler manifolds satisfying $divC = 0$

By hypothesis $divC = 0$ and this implies that

$$(\nabla_U S)(V, Z) - (\nabla_V S)(U, Z) = \frac{1}{4n}[dr(U)g(V, Z) - dr(V)g(U, Z)]. \quad (33)$$

In an almost co-Kähler manifold r is constant. Hence the above equation implies

$$(\nabla_U S)(V, Z) - (\nabla_V S)(U, Z) = 0. \quad (34)$$

Using (18), (9) and (10) in the above equation yields

$$\mu[g((\nabla_V h)U - (\nabla_U h)V, Z)] - 2nkg(\phi hU, Z)\eta(V) + 2nkg(\phi hV, Z)\eta(U) = 0. \quad (35)$$

Using Lemma 2.1(i) in (35), we get

$$k\mu\{g(\phi U, Z)\eta(V) - g(\phi V, Z)\eta(U) + 2g(\phi U, V)\eta(Z)\}$$

$$-(\mu^2 - 2nk)\{g(\phi hV, Z)\eta(U) - g(\phi hU, Z)\eta(V)\} = 0. \tag{36}$$

Substituting ξ for Z in the above equation, we get

$$2k\mu g(\phi U, V) = 0. \tag{37}$$

This implies either $k = 0$ or, $\mu = 0$ or, both are zero.

If $\mu = 0$, then the manifold is $N(k)$-almost co-Kähler manifold.

Again, if $k = 0$, then the manifold is K-almost co-Kähler manifold and if μ and k both are zero, then $R(U, V)\xi = 0$.

Dacko [14] proved that in an almost co-Kähler manifold $R(U, V)\xi = 0$ if and only if the manifold is locally a product of an open interval and an almost Kähler manifold. Hence we can state the following theorem.

Theorem 4.1. *In a (k, μ)-almost co-Kähler manifold with divergence free Cotton tensor, one of the following cases occur*:

(i) *the manifold is K-almost co-Kähler manifold,*

(ii) *the manifold is $N(k)$-almost co-Kähler manifold,*

(iii) *the manifold is locally a product of an open interval and an almost Kähler manifold.*

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